Twistor Spaces and Special Holonomy

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Plan

1. Twistor spaces of 4-manifolds

Some background, and their role in creating special geometries.

2. A circle action on the cone $\mathbb{R}^+ \times \mathbb{CP}^3$ with G_2 holonomy

Illustrating its projection to a singular space $\mathbb{R}^6 = \mathbb{R}^3 \times \mathbb{R}^3$ à la Atiyah-Witten.

3. $\mathrm{SU}(3)$ structure invariant by $\mathrm{SO}(3)$

Diagnosing the geometry induced by G_2 on $\mathbb{R}^6$ in the same example.

Parts 2 and 3 represent joint work with Bobby Acharya and Robert Bryant, naïvely inspired by the successful use by Foscolo-Haskins-Nordström of circle bundles to construct new manifolds with G_2 holonomy.

1.1 Penrose's twistor theory

is based on choosing a real form of the Klein correspondence between lines in complex projective 3-space $\mathbb{CP}^3$ and points in the quadric Q^4:

$$\begin{array}{ccc} & \mathbb{F}_{1,2}(\mathbb{C}^4) & \\ \swarrow & & \searrow \\ \mathbb{CP}^3 & & \mathrm{Gr}_2(\mathbb{C}^4) = Q^4 \subset \mathbb{CP}^5 \end{array} \qquad \begin{array}{l} \pi \in \Lambda^2\mathbb{C}^4 \\ \pi \wedge \pi = 0 \end{array}$$

The real form can be defined by a reduction

$$\mathrm{SL}(4,\mathbb{C}) \supset \left\{ \begin{array}{llll} \mathrm{SU}(2,2) \simeq \mathrm{SO}(4,2) & \text{acting on} & S^3 \times S^1 & \supset \mathbb{R}^{3,1} \\ \mathrm{SL}(2,\mathbb{H}) \simeq \mathrm{SO}(5,1) & \quad '' & \mathbb{HP}^1 = S^4 & \supset \mathbb{R}^4 \\ \mathrm{SL}(4,\mathbb{R}) \simeq \mathrm{SO}(3,3) & \quad '' & \mathrm{Gr}_2(\mathbb{R}^4) & \supset \mathbb{R}^{2,2}. \end{array} \right.$$

In the first case, a point $p \in \mathbb{R}^{3,1}$ defines a line in a real hypersurface $\mathcal{PN}$ of $\mathbb{CP}^3$. Two such $\mathbb{CP}^1$'s intersect in $x \in \mathcal{PN}$ iff the points in $\mathbb{R}^{3,1}$ lie on a light ray, while a point $y \in \mathbb{CP}^3 \setminus \mathcal{PN}$ defines a 3-parameter family of light rays in $\mathbb{R}^{3,1}$.

1.2 Hopf geometry

We shall concentrate on the second case in which the fibres of

$$\begin{array}{c} \mathbb{CP}^3 \\ \pi \Big\downarrow \\ \mathbb{HP}^1 = S^4 \subset \mathbb{G}\mathrm{r}_2(\mathbb{C}^4) \end{array}$$

parametrize 'real' lines relative to the antilinear involution $j\colon \mathbb{CP}^3 \to \mathbb{CP}^3$, which acts as the antipodal map on each fibre $S^2 \cong \mathbb{CP}^1$. The fibres are complex, but they have non-trivial normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$.

Any point of $z \in \mathbb{CP}^3$ will induce an almost complex structure on $T_{\pi(z)}S^4$, and a complex surface will define an integrable orthogonal complex structure (OCS) on an open subset of S^4.

Example. A plane $\mathbb{CP}^2$ determines a (conformally) constant OCS on $\mathbb{R}^4 = S^4 \setminus \infty$.

1.3 Associated bundles

In place of S^4, start with any oriented Riemannian 4-manifold M. Its $SO(4)$ holonomy defines a splitting

$$\Lambda^2 T^*M = \Lambda^+ \oplus \Lambda^-$$

and one can construct the 2-sphere bundle

$$\begin{array}{ccc} Z & \subset & \Lambda^- \\ \pi \downarrow & & \\ M. & & \end{array}$$

Each point $z \in \pi^{-1}(m)$ defines an almost complex structure on its horizontal space $H_z \cong T_m M$ (relative to the Levi Civita connection), which can be combined with (1) the natural complex structure on $V_z \cong T_z S^2$, or (2) its negative. This equips Z with almost complex structures J_1, J_2; the latter is never integrable but $c_1(J_2) = 0$.

The $\mathrm{SO}(4)$-structure on the 7-dimensional total space Λ^- is compatible with the representation $\mathrm{SO}(4) \subset \mathrm{G}_2 \subset SO(7)$.

1.4 Self-duality in 4 dimensions

A local section $s\colon U \to Z$ determines an almost Hermitian structure $(g, \mathrm{J}_s, \omega_s)$ on M. The map s is

- J_1-holomorphic iff J_s is integrable
- J_2-holomorphic iff $(d\omega_s)^{1,2} = 0$
- horizontal iff J_s is Kähler.

$$\begin{array}{ccc} & & Z \subset \Lambda^- \\ & \pi & \downarrow \\ U & \subseteq & M \end{array} .$$

Theorems.

J_1 is integrable iff M is self-dual, i.e. its Weyl tensor satisfies $W_- \equiv 0$ [Atiyah-Hitchin-Singer].

If (Z, J_1) is (compact) Kähler then M is isometric to S^4 or $\mathbb{CP}^2$ [Besse, Hitchin]. In the latter case $Z \cong \mathrm{SU}(3)/T^2$.

Given M^4 (compact, oriented) there exists n such that $M \# n\mathbb{CP}^2$ admits a self-dual metric [Poon, LeBrun, Joyce; Floer, Donaldson-Friedman, Taubes].

1.5 Twistor lifts

Let Σ be a Riemann surface. An immersion

$$\phi\colon \Sigma \longrightarrow M$$

can be lifted to $\psi\colon \Sigma \to Z$, so as to render $\phi_*(T_\sigma\Sigma)$ complex.

Proposition [Eells-S, Lichnerowicz]. ϕ is harmonic iff ψ is J_2-holomorphic.

Examples. If ψ is also J_1-holomorphic then ϕ is 'superminimal'. Such immersions can be constructed for any genus using Bryant's formula $\left[1,\ f - \frac{1}{2}g\frac{df}{dg},\ g,\ \frac{1}{2}\frac{df}{dg}\right]$.

In the case $M = S^4$ or $\mathbb{CP}^2$, the twistor space (Z, J_2) admits a compatible 'nearly Kähler' (non-standard Einstein) metric h. It follows that the cone $\mathbb{R}^+ \times Z$ admits a Ricci-flat metric $dr^2 + r^2 h$ with holonomy G_2 [Bär].

1.6 Symplectic Calabi-Yau spaces

can be constructed from twistor spaces [Fine-Panov].

Take M to be real hyperbolic 4-space. Then Z admits a symplectic form taming J_2 and $c_1(J_2) = 0$.

Z is symplectomorphic to $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{CP}^1$, a small resolution of the conifold $\{wx - yz = 0\}$ in $\mathbb{C}^4$.

$$\begin{array}{c} Z \cong \dfrac{\mathrm{SO}(4,1)}{\mathrm{U}(2)} \\ \pi \big\downarrow \\ \mathcal{H}^4. \end{array}$$

This construction can be applied to hyperbolic orbifolds $\mathcal{H}^4/\Gamma$ by resolving the twistor space to obtain simply-connected examples with $b_3 = 0$ (so non-Kähler) and arbitrary b_2. Higher dimensional twistor spaces (of even-dimensional Riemannian manifolds) provide further results.

1.7 G_2 holonomy

Theorem [Bryant-S]. If M is S^4 or $\mathbb{CP}^2$ then the 7-manifold Λ^- admits a complete metric g with holonomy equal to G_2 and asymptotic to the cone over Z.

If r denotes norm in the fibres of Λ^-, one can express

$$g = (r^2+1)^{-1/2} g_{\mathrm{ver}} + (r^2+1)^{1/2} g_{\mathrm{hor}}.$$

Over S^4, the associated 3-form φ on $\mathbb{R}^+ \times \mathbb{CP}^3$ can be pulled back to $\mathbb{C}^4$. Then $r = R^2$ where $R = \sum_{i=0}^3 |z_i|^2$ is the Euclidean norm squared, and

$$\varphi = d\big((R^4+1)^{1/4}\tau\big), \qquad \tau = -dR \wedge \alpha_1 + \alpha_2 \wedge \alpha_3.$$

Together with a G_2 metric over S^3, these were the only such AC metrics known until Foscolo-Haskins-Nordström established the existence of complete G_2 metrics on circle bundles $M_{m,n} \to K_{\mathbb{CP}^1 \times \mathbb{CP}^1}$ invariant by $\mathrm{SU}(2)^2 \times \mathrm{U}(1)$. These include ones that are asymptotic to cones over finite quotients of $S^3 \times S^3$.

2.1 M-theory

Regard $\mathbb{CP}^3$ as the quotient of S^7 inside $\mathbb{C}^4 = \mathbb{H}^2$ by $\mathrm{U}(1)_R$, and $S^4 = S^7/\mathrm{Sp}(1)_R$. We shall focus on $\mathscr{C}^7 = \mathbb{R}^+ \times \mathbb{CP}^3$, its conical metric with G_2 holonomy, and the quotient

$$\frac{\mathscr{C}^7}{\mathrm{U}(1)_L} \stackrel{*}{=} \frac{\mathbb{C}^4}{T^2} \cong \frac{\mathbb{C}^2}{\mathrm{U}(1)} \times \frac{\mathbb{C}^2}{\mathrm{U}(1)} \cong \mathbb{R}^3 \times \mathbb{R}^3.$$

We use hyperkähler moment maps to describe the resulting projection

$$\mathscr{C}^7 \ni [z_0, z_1, z_2, z_3] \longmapsto (\mathbf{u}, \mathbf{v}) \in \mathbb{R}^6,$$

whose circle fibres collapse over $\mathbb{R}^3 \cup \mathbb{R}^3$. The G_2 metric on the cone relates to Type IIA string theory of $\mathbb{R}^6$ with a singular locus $\mathbb{R}^3 \cup \mathbb{R}^3$ as described by Atiyah-Witten, who state:

“details of the induced metric are unimportant”.

Nonetheless, we [Acharya-Bryant-S] set out describe the $\mathrm{SU}(3)$ structure induced on $\mathbb{R}^6$, using the bivector formalism.

2.2 Gibbons-Hawking coordinates

The action of $\mathrm{U}(1)_L$ on $\mathbb{C}^4$ covers a rotation in 2 coordinates of S^4:

$$\begin{array}{ccccc} \mathrm{U}(1)_{\mathrm{L}} & \subset & \mathrm{U}(2)_L & \subset & \mathrm{Sp}(2)_L \\ & & & & \downarrow \\ \mathrm{SO}(2) & \subset & \mathrm{SO}(3)\times\mathrm{SO}(2) & \subset & \mathrm{SO}(5). \end{array}$$

To make $\mu\colon \mathscr{C}^7 \to \mathbb{R}^6$ explicit, observe that

$$\begin{array}{rcccc} \mathrm{U}(1)_L & \text{acts on} & \mathbb{C}^4_{0123} & \text{with weights} & (1,\ 1,\ 1,\ 1) \\ \mathrm{U}(1)_R & \text{"} & \text{"} & \text{"} & (1,-1,1,-1) \\ \Longrightarrow\ T^2 & \text{acts on} & \mathbb{C}^2_{02}\times\mathbb{C}^2_{13} & \text{with weights} & (1,1)\times(1,1). \end{array}$$

Then $\mathbf{u}=(u_1,u_2,u_3)$ and $\mathbf{v}=(v_1,v_2,v_3)$, where

$$u_1=|z_0|^2-|z_2|^2,\quad u_2-iu_3=z_0\overline{z}_2;\qquad v_1=|z_1|^2-|z_3|^2,\quad v_2-iv_3=z_1\overline{z}_3.$$

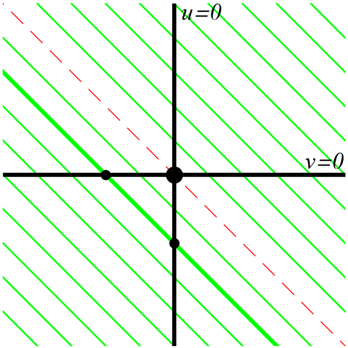
Moreover $R=\sum_{i=0}^3|z_i|^2$ equals $u+v$, where $u=|\mathbf{u}|$ and $v=|\mathbf{v}|$.

2.3 Eguchi-Hanson sheets

Provided $\mathbf{m} \in \mathbb{R}^3$ is non-zero, the hyperkähler quotient

$$\frac{\{\mathbf{z} \in \mathbb{C}^4 : \mathbf{u} - \mathbf{v} = \mathbf{m}\}}{\mathrm{U}(1)_R} \subset \mathscr{C}^7$$

can be identified with T^*S^2 endowed with a metric k of holonomy $\mathrm{SU}(2)$. It has a triholomorphic action by $\mathrm{U}(1)_\mathrm{L}$ with moment map $\mathbf{u}$.



Its image in $\mathbb{R}^6$ acquires the harmonic function

$$V = \frac{1}{|\mathbf{u}|} + \frac{1}{|\mathbf{u} - \mathbf{m}|},$$

used to recover $k = V^{-1}\Theta^2 + V g_{\mathrm{euc}}$. Each diagonal represents $\{(\mathbf{u}, \mathbf{v}) \in \mathbb{R}^6 : \mathbf{u} - \mathbf{v} = \mathbf{m}\}$.

2.4 Rotation of the 4-sphere

Consider again the reduction to $\mathrm{SO}(3) \times \mathrm{SO}(2)$:

$$\begin{array}{ccc} \mathscr{C}^7 & = & \mathbb{R}^+ \times \mathbb{CP}^3 \\ \pi \big\downarrow & & \\ S^4 & \subset & \mathbb{R}^2 \oplus \mathbb{R}^3 \end{array}$$

Let

- $\mathbb{S}^1 = S^4 \cap \mathbb{R}^2$ be the fixed point set for the action of $\mathrm{SO}(3)$
- $\mathbb{S}^2 = S^4 \cap \mathbb{R}^3$ be the fixed point set for the action of $\mathrm{SO}(2)$.

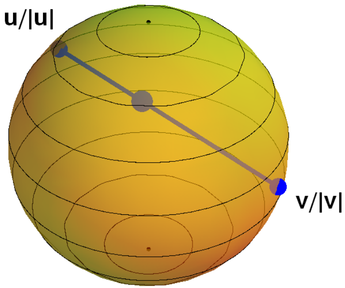
View $S^4 \setminus \mathbb{S}^1$ as a trivial disk bundle over $\mathbb{S}^2$, whose boundaries are fused into $\mathbb{S}^1$.
If X is the Killing field generated by $\mathrm{SO}(2)$, then

$$X^\flat = (1 - s^2)dt,$$

where $t\colon S^4 \setminus \mathbb{S}^2 \to [0, 2\pi)$ is 'longitude' and $s\colon S^4 \to [0, 1]$ is sine of 'latitude'.
In fact, s represents the radius in $\mathbb{R}^3$ under the projection $S^4 \subset \mathbb{R}^5 \to \mathbb{R}^3$:

2.5 Reduced twistor fibration

Orthogonal projection $\mathbb{R}^5 \longrightarrow \mathbb{R}^3$ identifies $S^4/\mathrm{SO}(2)$ with the closed unit ball $\overline{D}^3$ whose boundary is effectively $\mathbb{S}^2$.



Proposition. The projection $\mathscr{C}^7/\mathrm{U}(1)_L \to \overline{D}^3$ is given by

$$(\mathbf{u}, \mathbf{v}) \longmapsto \frac{\mathbf{u}+\mathbf{v}}{u+v} = \frac{1}{R}(\mathbf{u}+\mathbf{v}),$$

and $s = |\mathbf{u}+\mathbf{v}|/R$ equals the radius in $\overline{D}^3$.

Examples.

- $(\mathbf{u}, \mathbf{v})$ arises from a point of $\mathbb{S}^1$ iff $\mathbf{u}+\mathbf{v} = \mathbf{0}$ (implying $u = v$).
- $(\mathbf{u}, \mathbf{v})$ maps into $\mathbb{S}^2$ (equivalently $s = 1$) iff $\mathbf{u}$ and $\mathbf{v}$ are aligned.

2.6 Two quadrics

help to interpret the preceding geometry. Set

$$\begin{aligned} Q_+ &= \{[z_0, z_1, z_2, z_3] \in \mathbb{CP}^3 : z_0\overline{z}_3 - \overline{z}_1 z_2 = 0\} \\ Q_- &= \{[z_0, z_1, z_2, z_3] \in \mathbb{CP}^3 : z_0 z_1 + z_2 z_3 = 0\}. \end{aligned}$$

These subvarieties are both $\mathrm{SU}(2)$ invariant, and arise from points of $\mathbb{R}^6$ where $\mathbf{u}, \mathbf{v}$ are aligned (respectively, anti-aligned):

$$\mu(\mathbb{R}^+ \times Q_\pm) = \{(\mathbf{u}, \mathbf{v}) : \mathbf{u}\cdot\mathbf{v} = \pm uv\}.$$

While $\pi(Q_+) = \mathbb{S}^2$ (making it obvious that $Q_+ \approx S^2 \times S^2$), the holomorphic one Q_- double covers

$$S^4 \setminus \mathbb{S}^1 \;\cong\; \mathbb{R}^4 \setminus \mathbb{R} \;\cong\; S^2 \times \mathcal{H}^2,$$

encoding the scalar flat Kähler metric [Pontecorvo, S-Viaclovsky]. It is the locus of points in $\mathbb{CP}^3$ for which the $\mathrm{U}(1)_L$ orbits are horizontal over S^4.

2.7 Coassociative subvarieties

The defining function for Q_+ equals

$$z_0\bar{z}_3 - \bar{z}_1 z_2 = ae^{it},$$

where $a = 2\sqrt{uv - \mathbf{u}\cdot\mathbf{v}}$ so that $(a/R)^2 = 1 - s^2$. Both a and t are invariants for the action of $SO(3)$, as are u and v because $\mathrm{SO}(3)$ acts diagonally on $(\mathbf{u}, \mathbf{v}) \in \mathbb{R}^6$.

By general principles, any 3-dimensional $SO(3)$ orbit is contained in a unique coassociative subvariety V of the G_2 manifold $\mathscr{C}^7$ (so $\varphi|_V \equiv 0$). For the chosen $\mathrm{SO}(3)$, the resulting family has been described by Karigiannis-Lotay. Our task was to interpret this using $\mathbf{u}$ and $\mathbf{v}$.

An $SO(3)$ orbit will intersect a twistor fibre S^2 of fixed radius over $p \in S^4 \setminus \mathbb{S}^1$ in a parallel circle at 'height' $h \in [-1, 1]$ relative to the poles defined by Q_-. Define another $SO(3)$ invariant

$$b = u^2 - v^2 = R(u - v) = shR^2.$$

2.8 Coassociatives (continued)

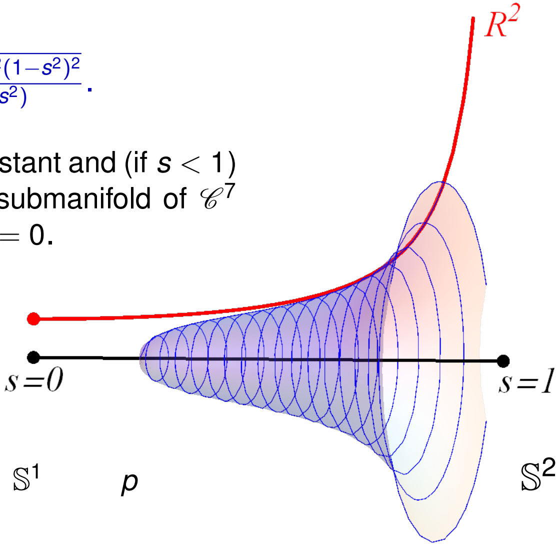
Sphere radius, $R^2 = \frac{a^2}{1-s^2}$

Circle radius, $R^2\sqrt{1-h^2} = \frac{\sqrt{a^4s^2-b^2(1-s^2)^2}}{s(1-s^2)}$.

Theorem [KL, ABS]. Setting a, b constant and (if $s < 1$) t constant defines a coassociative submanifold of $\mathscr{C}^7$ diffeomorphic to T^*S^2 unless $a = b = 0$.

A fibre over $p \in S^2$ is shown for $(a, b, t) = (\frac{1}{2}, \frac{1}{4}, 0)$:

$a = 0$ gives coassociatives over $\mathbb{S}^2$, while $b = 0$ $(a \neq 0)$ gives equators all the way to the twistor fibres over $\mathbb{S}^1$.



3.1 The induced metric

Let h be the conical metric on $\mathscr{C}^7$ with holonomy G_2. We seek the metric g induced on $\mathbb{R}^6 \setminus (\mathbb{R}^3 \cup \mathbb{R}^3)$ by setting

$$h = \mu^* g + N\Theta^2,$$

where $\Theta = (X \lrcorner\, h)/N$ is the connection 1-form, and $N = h(X, X) = 6uv - 2\mathbf{u}\cdot\mathbf{v}$ measures the size of the circle fibres. This makes μ a Riemannian submersion.

Theorem [ABS].

$$g = \tfrac{1}{2}dR^2 + \tfrac{1}{2}\big|d\mathbf{u} + d\mathbf{v}\big|^2 + \tfrac{2}{N}\big|u\,d\mathbf{v} - v\,d\mathbf{u}\big|^2 + \tfrac{1}{2N}\Gamma_+^2 - \tfrac{1}{4N}\Gamma_-^2,$$

where

$$\begin{aligned}\Gamma_+ &= u\,dv + v\,du - \mathbf{u}\cdot d\mathbf{v} - \mathbf{v}\cdot d\mathbf{u},\\ \Gamma_- &= u\,dv - v\,du + \mathbf{u}\cdot d\mathbf{v} - \mathbf{v}\cdot d\mathbf{u}.\end{aligned}$$

Example. If $u\mathbf{v} = \pm v\mathbf{u}$ then $\Gamma_\pm = 0$ (and $N = 4uv$ or $8uv$).

3.2 Two-dimensional quadrants

The formula for g simplifies on certain subvarieties of $\mathbb{R}^6$. Consider the negative quadrant

$$\mathscr{L}^2 = \{(\mathbf{u},\mathbf{v}) = (0,0,u;0,0,-v),\ u,v>0\} \subset \mathbb{R}^2.$$

Corollary. The restriction of g to $\mathscr{L}^2$ equals

$$\left(1+\frac{v}{2u}\right)du^2 + du\,dv + \left(1+\frac{u}{2v}\right)dv^2$$

and is locally Euclidean, i.e. $K \equiv 0$.

$\mathscr{L}^2$ is in fact superminimal, being the projection of (a cone over an open subset of) a horizontal projective line $\mathbb{CP}^1$ inside $Q_- \subset \mathbb{CP}^3$. We shall see that it is also $\mathbb{J}$-holomorphic, where $\mathbb{J}$ is the induced almost complex structure on $\mathbb{R}^6$.

3.3 Three-dimensional slices

Extend $\mathscr{L}^2$ to

$$\mathscr{L}^3 = \{(0, u\sin\theta, u\cos\theta;\ 0,\ -v\sin\theta, v\cos\theta)\}.$$

so that $\mathbf{u}\cdot\mathbf{v} = uv\cos 2\theta$, and set

$$u = R\cos^2(\tfrac{1}{2}\phi), \quad v = R\cos^2(\tfrac{1}{2}\phi)$$

so that $u+v=R$ and $b = R^2\cos\phi$. The orbits of $\mathrm{SO}(3)$ on $\mathbb{R}^6$ are parametrized by u, v, θ, so $\mathscr{L}^3$ is a slice to the orbits (expressed symmetrically in $\mathbf{u}, \mathbf{v}$).

Corollary. The restriction of g to $\mathscr{L}^3$ equals

$$dR^2 + \tfrac{1}{2}R^2\left[d\theta^2 + \tfrac{1}{4}(3-\cos 2\theta)d\phi^2\right].$$

This is isometric to a cone over a surface of revolution, illustrated next.

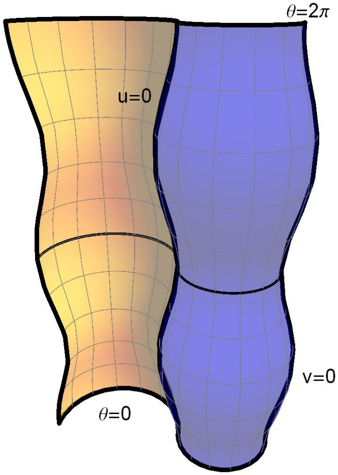
3.4 Slices (continued)

Let $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \in \mathrm{SO}(3)$.

Then $\mathscr{L}^3$ is the cone over the blue surface, and $P\cdot\mathscr{L}^3$ the cone over the yellow surface. Together these patches close up topologically to define a torus $\mathscr{T}$ and $\mu^{-1}(\mathscr{T})$ is a cone over $S^1 \times S^2$.

Relative to the metric g, vectors in the respective the singular $\mathbb{R}^3$ axes meet at an angle of

$$\tfrac{1}{2}\pi \leqslant \pi\sqrt{\tfrac{3}{8} - \tfrac{1}{8}\cos\theta} \leqslant \tfrac{1}{\sqrt{2}}\pi.$$



3.5 The symplectic form

Recall that $\mu\colon \mathscr{C}^7 = \mathbb{R}^+ \times \mathbb{CP}^3 \longrightarrow \mathbb{R}^6$, and that $R = u + v = |\mathbf{u}| + |\mathbf{v}|$. An almost Kähler structure on $\mathbb{R}^6$ is defined by scaling g so that the symplectic form

$$\sigma = X \lrcorner\, \varphi$$

has constant norm. Here X is the Killing field generating $\mathrm{U}(1)_L$.

Theorem. The vectors $\mathbf{p} = \mathbf{u} + \mathbf{v}$ and $\mathbf{q} = R(\mathbf{u} - \mathbf{v})$ furnish Darboux coordinates:

$$\sigma = -\tfrac{1}{2} \sum_{i=1}^{3} dp_i \wedge dq_i.$$

Note that σ extends to $\mathbb{R}^3 \cup \mathbb{R}^3$ and is non-degenerate on $\mathbb{R}^6 \setminus \mathbf{0}$. The projections $(\mathbf{u}, \mathbf{v}) \mapsto R^{1/2}\mathbf{u}$ and $(\mathbf{u}, \mathbf{v}) \mapsto R^{1/2}\mathbf{v}$ also have Lagrangian fibres.

3.6 The SU(3) structure

This is determined by g and the $\mathrm{SL}(3,\mathbb{C})$ structure encoded in a complex volume form Ψ. From the theory of stable forms, Ψ is determined by the closed 3-form

$$\mathrm{Re}\,\Psi = X \lrcorner\, (*\varphi),$$

which will involve the function $N = h(X,X) = 6uv - 2\mathbf{u}\cdot\mathbf{v}$.

Proposition.

$$\begin{aligned} 8uv\,\mathrm{Re}\,\Psi \;=\; & \tfrac{1}{6}v(N+4v^2)\{d\mathbf{u},d\mathbf{u},d\mathbf{u}\} \\ & -v(4u^2+3uv+\mathbf{u}\cdot\mathbf{v})\{d\mathbf{v},d\mathbf{u},d\mathbf{u}\} \\ & +\big((u+2v)\mathbf{v}\cdot d\mathbf{v}+v\mathbf{u}\cdot d\mathbf{v}\big)\wedge\{\mathbf{u},d\mathbf{u},d\mathbf{u}\} \\ & +(v\mathbf{u}\cdot d\mathbf{v}-u\mathbf{v}\cdot d\mathbf{v})\wedge\{\mathbf{v},d\mathbf{u},d\mathbf{u}\} \\ & +\text{ terms interchanging } \mathbf{u} \text{ and } \mathbf{v} \end{aligned}$$

This is the closest we can get to an explicit description of the (non-integrable) almost complex structure $\mathbb{J}$ on $\mathbb{R}^6$, as there are no easy expressions for $(1,0)$ forms.

3.7 Pseudo holomorphic surfaces

Proposition. The linear subvariety

$$\mathscr{L}^4 = \{(0, u_2, u_3; 0, v_2, v_3),\ uv \neq 0\}$$

is $\mathbb{J}$-holomorphic for the induced $\mathrm{SU}(3)$ structure on $\mathbb{R}^6$.

Applying $\mathrm{SO}(3)$, there will be a family of such subvarieties (parametrized by $\mathbb{RP}^2$) that exhaust $\mathbb{R}^6$. Any two intersect in a $\mathbb{J}$-holomorphic curve, isomorphic to $\mathscr{L}^2$.

Unlike the case of standard $\mathbb{C}^3 = \mathbb{R}^3 \oplus \mathrm{J}\mathbb{R}^3$, we cannot extend this $\mathbb{RP}^2$ to $\mathbb{G}\mathrm{r}_2(\mathbb{C}^3)$.

The action of $\mathrm{SO}(3)$ on $\mathbb{C}^3$ has been used to construct invariant Kähler-Einstein metrics on $\mathbb{CP}^2$ minus the conic curve $u = v$ with cone angle lying in $(\frac{1}{2}\pi, 2\pi]$ [C. Li, Dancer-Strachan] and associated Calabi-Yau cones.

3.8 Conclusion

We have analysed a quotient of nearly Kähler $\mathbb{CP}^3$ and its G_2 cone by $\mathrm{U}(1)$. It is convenient to work on $\mathbb{C}^4$ and (via the Gibbons-Hawking ansatz) identify the quotient with $\mathbb{C}^4/T^2 \cong \mathbb{R}^6$. In the holomorphic setting, all the formulae are simpler and related to the Kähler quotient

$$\mathbb{CP}^3/\!/\mathrm{U}(1) \;\cong\; \mathbb{CP}^1 \times \mathbb{CP}^1.$$

For G_2, we can easily describe the symplectic form and also the curvature 2-form $F = d\Theta$ of μ, but pinning $\mathbb{J}$ down is more difficult. Some modification is necessary when starting with the complete G_2 metric on $\Lambda^- T^* S^4$.

There remains the motivating conjecture that $\mathbb{R}^+ \times \mathbb{WCP}^3_{p,p,q,q}$ carries a metric with holonomy G_2 [Acharya-Witten]. The constructions can be generalized to a circle acting with different weights on $\mathbb{C}^4$, or actions on other G_2 manifolds, though this study will involve real invariant theory outside the familiar hyperkähler setting.