Codimension seven revisited

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Riemannian manifolds with special holonomy are ideal spaces on which to compactify M-theory, since the covariantly-constant spinors typical of such spaces give rise to supersymmetry in the effective, lower-dimensional theory. However, it has long been recognized that other important features of the effective theory (including nonabelian gauge symmetry and massless chiral matter in four dimensions) cannot be realized by compactification on manifolds, and so it has been proposed to compactly M-theory on singular spaces as well. The effective theory in such a case is not derived exclusively from supergravity, but must contain other massless fields such as nonabelian gauge fields or massless matter fields corresponding to physics localized at singularities. One of the challenges is identify the new massless fields representing the new physics, based on the geometry of the singularities.

The case of gauge fields is very well studied and is understood to be derived from ADE singularities in (real) codimension four. The other cases (codimensions six and seven for compactifications to four dimension as well as codimension eight for compactifications to three dimension) are less well understood. Many examples are known, but in examples it is often assumed that the singularity is asymptotically a metric cone, which seems to have been justified on the basis of simplicity rather than on physical grounds.

We will propose a mathematical framework for studying singular spaces which contain a manifold with a metric of special holonomy as a dense open set, and we hope that this framework will capture all the relevant physical phenomena. We also hope that this is a reasonable framework mathematically for example, one might hope that limits of metrics which are at finite distance from the bulk in a Weil-Petersson type metric would always fall into this class (as is already known for K3 surfaces).

K3 surfaces and gauge fields

The first place we see non-flat metrics with a covariantly constant spinor is the case of K3 surfaces. The space of all such metrics has natural limit points (in the Weil–Petersson or Gromov–Hausdorff senses) in which certain configurations of S^2's shrink to zero area, and it is known by work of Kobayashi and Todorov (generalizing Yau's theorem) that the limit of nearby Ricci-flat metrics is an asymptotically orbifold metric on the limiting metric space. When compactifying M-theory on a nonsingular K3 surface close to such a limit, one finds massive 7-dimensional gauge fields in the spectrum whose masses are proportional to the areas of the shrinking S^2's. (These fields arise in the effective theory from wrapping M2-branes on the shrinking spheres.)

It is then natural to compactify a coupled theory (11-dimensional supergravity coupled to 7-dimension gauge theory, with masses of the gauge bosons among the parameters) on a possibly singular K3 surface in such a way that the areas of the shrinking spheres correspond to the masses of the gauge bosons. Massless gauge bosons will appear in the orbifold limit in which spheres have shrunk to zero area.

Note that this (real) codimension four behavior is exactly what we obtain in non-collapsed Gromov–Hausdorff limits (work of Cheeger, Tian, and Naber).

We really don't need compact K3 surfaces for this construction (except the final step where we actually compactify) – the entire thing can be done with ALE spaces, using Kronheimer's construction to vary the areas of the spheres. And in fact, we can repeat the construction for spaces containing a family of ALE singularities. There are various ways to set this up, using Higgs bundles, spectral covers, and the like. (See the other talks/discussions.)

The new gauge field which is introduced (and which is massive for certain parameter ranges) will interact with the metric being used in the supergravity theory and there will be some resulting "equations of motion" (which include Ricci-flatness of the metric, and covariant constancy of an appropriate spinor field, as well as equations on the gauge field itself) which should be imposed.

One possible complication, as discussed earlier in this meeting in the context of the Joyce-Karigiannis work, is the possibility that although the singularities can be resolved locally, there may be global obstructions to doing so. Such global obstructions are likely to be interpreted in terms of certain global properties of a "families" construction (such as an unexpected zero in a one-form).

Calabi–Yau threefolds and matter fields

The next case in which covariantly constant spinors occur is the case of Calabi–Yau threefolds. In this case, as in the case of K3 surfaces, we rely heavily on Yau's solution to the Calabi conjecture to guarantee the existence of appropriate metrics. But while in the K3 case much was governed by the periods of 2-forms (and the hyperKähler nature of the metrics) and this was enough, for example, to see that we had asymptotically orbifold metrics, in the Calabi–Yau threefold case we know much less about the metrics.

We do know, thanks to work of Hayakawa and Wang, which singularities (in the sense of algebraic geometry) are at finite distance from the "bulk" of the moduli space – they are the canonical threefold singularities. This includes the same (real) codimension four ALE singularities, as well as some new singularities in (real) codimension six. (Note that the new singularities can either be isolated, or can reside within the old singular locus where they appear as a worsening of the singularity type.)

We do *not* know much about the metric behavior of these singularities. The simplest "conifold" singularities admit a cone-type metric written down long ago by Candelas and de la Ossa in the physics literature (and by Stenzel in the math literature), and this may have been one factor which has influnced people to assume that these singularities are metrically cone-like. But this has been known to be false for some time, for some singularities of the form $x^2 + y^2 + z^2 + t^k = 0$, with non-existence of cone metrics for $k \geq 3$ due to Gauntlett, Martelli, Sparks, and Yau and existence of non-cone metrics for $k \geq 4$ due to Hein and Nadel. (The case $k = 2$ is the conifold mentioned above, and the case $k = 3$ is still partially open.)

For many of these singularities, the physics content is again captured by adding (five-dimensional) fields – scalar fields this time – corresponding to shrinking S^2's in a "crepant" resolution of singularities, with wrapped M2-branes again being responsible for massive fields which become massless in the singular limit.

This explanation fails to be complete in two ways: first, when a divisor shrinks to a point, the physics – a conformal field theory – is much more complicated.

Second, not every canonical threefold singularity admits a crepant resolution – sometimes one must deal with residual "$\mathbb{Q}$-factorial terminal singularities." (This includes the $k = 3$ case in the examples mentioned above.) Those have also been studied in physics (by Grassi and Weigand) and a description can be given.

In any event, whether we are associating simple five-dimensional "matter" fields to shrinking spheres, or modeling more complicated phenomena, we can do so in families, based on a local analysis of the singularity type, As in the case of gauge fields there may be global obstructions to a resolution of singularities in this case, analogous to the zeros in a one-form encountered by Joyce and Karigiannis.

Holonomy G2

The last case which we will have time for in this lecture is the case of holonomy G2. Here we are on even shakier ground, in terms of understanding the structure of the metric. I wish to emphasize that althugh a number of examples have been found in which the codiemsnion seven phenomena correspond to a metrically conical point, there seems no good reason to assume this is always true (and experience with metrics in the Calabi–Yau threefold case suggests that it is probably not true).

It would be great to have a classification of such singularities, and of the resulting effective physics which could hopefully be modeled by four-dimensional physical fields.

Even without an explicit classification or full understanding of the metric behavior, there is still a way that we can get some information about the structure, and give a mathematical description which may be sufficiently detailed for both further study, and for the construction of global examples – which are sorely lacking.

The basic structure I propose is a seven-dimensional space which has singular sets of codimensions four, six, and seven (some of which may be contained in others). Away from the singular locus, a metric with covariantly constant spinor field is expected. One possible model is to remove small neighborhoods of connected components of the singular locus: so one could remove a set fibered over a 3-manifold with fibers being small 4-balls mod finite group, a set fibered over a 1-manifold with fibers being small 6-dimensional neighborhoods of canonical 3-fold singularities, and small 7-dimensional neighborhoods of codimension 7 singularities. Some information about the associated physics is then captured in the boundary of the removed set, and in particular by integrals of appropriate differential forms over that boundary. This was the approached used by Witten to measure chirality for the new physical fields created by singularities of codimension 7, and one could hope that many relevant physical features (charges, for example) could be measured in this way.

conclusions

Our understanding of codimension 7 singularities is quite unsatisfactory. However, it seems we can build a general framework which has a chance of capturing all the phenomena we need. We hope that this leads to further progress in constructing local and global examples, and understanding the physics thereof.