Homotopy Associative Submanifolds in $G_2$-Manifolds

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Outline

- Associative submanifolds
  Intro, properties, examples, questions
- Homotopy associative submanifolds
  Definition and first properties
- Basic Results
  Bordism sets and groups, and classes of homotopy associatives
- Singularities
  Some constructions and computations
Let \((M, \varphi)\) be a closed seven-manifold with holonomy \(G_2\). Then \(\varphi \in \Omega^3(M)\) is a calibration.

An **associative submanifold** \(A \subset M\) is a three-dimensional submanifold that is calibrated by \(\varphi\) (Harvey and Lawson 1982).

Recall that \(\varphi\) defines a metric on \(M\), has \(\text{comass}(\varphi) = 1\), and \(d\varphi = 0\). Then \(A\) is **calibrated** by \(\varphi\) if and only if \(\varphi|_A = d\text{vol}_A\).

In particular, \(A\) is a volume minimising submanifold within its homology class in \(H_3(M)\).

Generic associative submanifolds are **rigid** (McLean 1998), in particular:

- they are isolated in the space of three-dimensional submanifolds
- they vary smoothly with \(\varphi\) for small deformations of \(\varphi\).
Associative submanifolds share certain properties with complex curves in Calabi-Yau three-manifolds

**Question**
Can one count associative submanifolds and get subtle invariants of $(M, \varphi)$ as for example in Gromov-Witten theory? Related to counting problem for $G_2$-instantons (Donaldson-Segal 2011, Haydys-Walpuski 2015)

**Problem**
Naive counting is not invariant under modifications of $\varphi$. As $\varphi$ varies, one may have obstructed or singular associatives like

\[
\begin{align*}
\text{or} & & \langle \quad \rangle \\
& & \text{associative submanifolds}
\end{align*}
\]

$G_2$-moduli space
Let \((\varphi_t)_{t \in (-\varepsilon, \varepsilon)}\) be a family of torsionfree \(G_2\)-structures. Assume that \(A_t\) is an immersed associative for \(\varphi_t\) with one selfintersection for \(t = 0\) and no other singularities.

\[
\begin{align*}
\text{\(t < 0\)} & \quad \begin{array}{c}
\text{\(t = 0\)}
\end{array} & \quad \begin{array}{c}
\text{\(t > 0\)}
\end{array}
\end{align*}
\]

Then at \(t = 0\) another family of associative submanifolds is created or destroyed that looks like the connected sum of the two branches (Joyce, Nordström, Bera). It has the local geometry of a Lawlor neck (a certain special Lagrangian in \(\mathbb{C}^3\), Lawlor 1989).

This picture looks like the skein relation from knot theory.
Associative submanifolds—Summary

- Associative submanifolds are calibrated submanifolds in $G_2$-manifolds. They are generically rigid, but otherwise hard to control.
- Associative submanifolds are expected to behave analogous to complex curves in Calabi-Yau threefolds.
- Counting associative submanifolds is tricky because of bifurcations. There may be more problems.
- Conjectural counting schemes have been proposed by Joyce (2018, $b_1(A) = 0$) and Doan-Walpuski (2019, $b_1(A) > 1$).
- Some pictures of associatives look like pictures from knot theory. But knot theory and complex curves are not entirely unrelated (Ekholm-Shende).
We (Andriy Haydys and myself) attempt to use topological methods to

- get an overview of all possible associative submanifolds
- get around all the analytic problems—or at least postpone them

There is no naive $h$-principle, so we will loose information

We distinguish associative submanifolds by cobordism classes
Finer than homology, but coarser than homotopy

We also consider normal $G$-structures, for $G \subset SO(4) \subset G_2$, for example

- $G = SO(3)$ describes Joyce’s flagged associatives
- $G = \text{Spin}(3)$ describes the deformation operator as a spin Dirac operator
- $G = \text{Sp}(1)$ describes associatives with trivialised tangent bundle

All this might work analogously for other kinds of calibrated submanifolds
Translate submanifold theory to homotopy theory (Thom 1954, Pontryagin 1955)

Let $G$ be a Lie group with a fixed representation of real dimension $k$

Let $N \subset M$ be a $G$-submanifold with normal bundle $\nu$

The $G$-structure on $\nu$ is classified by $f : N \to BG$

A tubular neighbourhood $U \cong \nu$ maps properly to the universal vector bundle $VG = EG \times G \mathbb{R}^k \to BG$

The Thom space $MG$ is the one-point compactification of $VG$

Send $M \setminus U$ to $\infty$

Conversely if $F \pitchfork BG$ recover $N = F^{-1}(BG)$ and $\overline{f} = dF|_\nu$

Theorem (Pontryagin-Thom)

$$\Omega^k_G(M) \cong [M_+, MG]$$
Let $G \subset SO(4)$ be a subgroup and $Ass_G = G_2/G$
Take $BG = EG_2/G \cong EG_2 \times G_2 \ Ass_G$
Let $VAss_G = G_2 \times G \mathbb{R}^4 \to Ass_G$ and
take $VG = EG_2 \times G \mathbb{R}^4 \cong EG_2 \times G_2 \ VAss_G$
Let $YG$ denote the Thom space of $VAss_G \to Ass_G$
and define $XG = EG_2 \times G_2 \ YG$
The points at infinity give a map $\iota : BG_2 \to XG$
Collapsing them to one point gives $q : XG \to MG$
Recover $MG \leftarrow VG \to BG$ from the
Pontryagin-Thom construction
Regard $XG \leftarrow VG \to BG$
as a bundle version over $BG_2$
A homotopy $G$-associative in a $G$-bordism class $[\alpha] \in [M_+, MG]$ and over the $G_2$-structure $\tau: M \to BG_2$ is a map $F: M \to XG$ such that $p \circ F = \tau$ and $q \circ F \sim \alpha$

\[
\begin{array}{c}
MG \xleftarrow{\alpha} XG \\
M \xrightarrow{\tau} BG_2
\end{array}
\]

Write $h\text{Asso}_G(M, \tau) = \Gamma(\tau^*XG \to M)$ and $hA_G(M, \tau) = h\text{Asso}_G(M, \tau)/\sim$

If $F \pitchfork BG$, let $A = F^{-1}(BG)$, so $A \in [\alpha] \in \Omega^4_G(M)$ for $\alpha = q \circ F$

Let $\nu \to A$ be the normal bundle and $a \in A$, then $dF_a|_{\nu}: \nu_a \to VG_{F(a)} \cong V\text{Ass}_G$ identifies $\nu_a$ with a coassociative subspace of $T_aM$ with a $G$-structure
Let $M$ be a $G_2$-manifold with $G_2$-structure $\tau: M \to BG_2$
so $BG_\tau(p) = \{ \text{coassociative } G\text{-subspaces of } T_pM \} \cong \text{Ass}_G$
Let $A \subset M$ be an associative $G$-submanifold
For $a \in A$, get $f(a) \in BG_\tau(a)$ and $\bar{f}: \nu_a \xrightarrow{\sim} VG_{f(a)} \subset VG_\tau(a)$
Let $U \cong \nu$ be a tubular neighbourhood of $A$
Using fibre transport, map $U \to VG$ over $\tau$
Finally, map $p \in M \setminus U$ to $\infty_\tau(p) \in XG_\tau(p) \cong YG$
This turns an associative $G$-submanifold $A$
into a homotopy $G$-associative $F \in h\text{Asso}_G(M, \tau)$
More generally, let \( \text{Asso}_G(M) \rightarrow G_2(M) \) describe all associative \( G \)-submanifolds for all \( G_2 \)-structures on \( M \) and construct a map

\[
\text{hofib}(\text{Asso}_G(M) \rightarrow (G_2(M), \varphi_0)) \rightarrow h\text{Asso}_G(M, \tau_0)
\]

Recall that elements of this homotopy fibre are pairs \( ((\varphi_t)_{t \in [0,1]}, A) \) of

- A path \( (\varphi_t)_t \) of torsion-free \( G_2 \)-structures
- A \( G \)-associative \( A \) in \( (M, \varphi_1) \)

Using fibre transport, we can turn \( A \) into a homotopy \( G \)-associative in \( (M, \varphi_0) \)

We might even hope for a naive \( h \)-principle: Each class in \( hA_G(M, \tau_0) \) would then be realised by an element of \( \text{hofib}(\text{Asso}_G(M) \rightarrow (G_2(M), \varphi_0)) \)
Idea. Add extra dimensions to make life easier
But never leave the $G_2$-world! In particular, do not make $G \subset SO(4)$ larger

- Consider compact threedimensional submanifolds of $M \times \mathbb{R}^k$
  with normal bundle $\nu = \nu' \oplus \mathbb{R}^k$ having a $G$-structure on $\nu'$

$$\Omega^{4+k}_G(M \times \mathbb{R}^k) \cong [S^k M_+, S^k MG]$$

- Replace $YG$ by $S^k YG$ and $XG$ by $X^k G = EG_2 \times_{G_2} S^k YG$
  Define $hAso^k_G(M, \tau) = \Gamma(\tau^* X^k G)$ and $hAso^s_G = \colim_{k \to \infty} hAso^k_G$

- Turn immersed $G$-associatives into stable homotopy $G$-associatives

- Get group structures on $\Omega^{4+k}_G(M \times \mathbb{R}^k)$ and $hA^k_G(M, \tau) = hAso^k_G(M, \tau)/\sim$
  Note: $k \geq 1$ suffices and gives abelian groups

We will see that $G = SO(3)$ and $k \geq 1$ fits with Joyce’s proposal
The Pontryagin-Thom construction gives an isomorphism

\[ [M_+, MG] \xrightarrow{\cong} \Omega^k_G(M) \]

We replace \( MG \) by a new space \( X_G \) that fibres over \( BG_2 \)
It contains a copy of \( BG_2 \) at infinity, and collapsing it gives back \( MG \)

Homotopy \( G \)-associatives for a \( G_2 \)-structure \( \tau : M \to BG_2 \)
are sections of \( \tau^*X_G \to M \)

A modified Pontryagin-Thom construction turns (immersed)
associative \( G \)-submanifolds into (stable) homotopy \( G \)-associatives

The space of homotopy \( G \)-associatives captures the full homotopy fibre
of associatives over arbitrary \( G_2 \)-structures on \( M \)
Basic Results

Before we look at concrete constructions, we need some basic facts about bordism sets and groups

- Can every class in $H_3(M) \cong H^4(M)$ be realised as a $G$-bordism class?—Yes
- Is the representation in $\Omega^4_G(M)$ unique?—Only for $G = \text{SO}(4)$
- Does stabilisation introduce new bordism classes?—No
- What are the preimages of $\Omega^4_G(M) \rightarrow \Omega^{4+k}_G(M \times \mathbb{R}^k)$? Do they have a geometric meaning?

We also want to know how bordism classes refine to homotopy $G$-associatives

- Can every class in $\Omega^{4+k}_G(M)$ be realised in $hA^k_G(M, \tau)$?—Yes
- Is this representation unique?—Only for $G = \text{SO}(4)$ and $k \geq 1$
- What are the preimages of $hA^k_G(M, \tau) \rightarrow \Omega^{4+k}_G(M \times \mathbb{R}^k)$? Do they have a geometric meaning?
Let $\dim M \leq 7$
Thom (1954) has shown that $\Omega^4_{SO(4)}(M) \cong H^4(M)$
By Freudenthal's suspension theorem also $\Omega^4_{SO(4)}(M \times \mathbb{R}^k) \cong H^4(M)$

Let $M$ be a spin 7-manifold. For other groups $G \subset SO(4)$, we get
$$
\Omega^4_G(M) \rightarrow \Omega^4_{G}(M \times \mathbb{R}^k) \rightarrow \Omega^4_{SO(4)}(M \times \mathbb{R}^k) \cong H^4(M)
$$

To understand the preimages of the second map, obstruction theory tells us compute
$$
\pi_{k+\ell}(MSO(4+k), S^k MG)
$$
for $\ell \leq 8$.

For simply connected groups $G$ like $\text{Spin}(3)$, $\text{Sp}(1)$ or $\{e\}$, there are too many obstruction groups for a simple answer
For $G = SO(3)$, Pontryagin-Thom gives

$$
\Omega^4_{SO(3)}(M) \cong [M_+, S^1 MSO(3)] \cong \{ N \subset M \text{ with normal bundle } \nu \cong \nu' \oplus \mathbb{R} \} / \sim
$$

Adapting Joyce’s terminology, we call this “flagged oriented cobordism”

Because $SO(3) \to SO(4 + k)$ is 2-connected, we compute

$$
\pi_{k+\ell}(MSO(4 + k), S^k MSO(3)) \cong \begin{cases} 
0 & \text{for } \ell \leq 7, \\
\mathbb{Z} & \text{for } \ell = 8 \text{ and } k = 0, \text{ and} \\
\mathbb{Z}/2 & \text{for } \ell = 8 \text{ and } k \geq 1.
\end{cases}
$$

Hence, $\mathbb{Z}$ acts on $\Omega^4_{SO(3)}(M)$ with quotient $\Omega^4_{SO(4)}(M)$, and for $k \geq 1$,

$$
0 \to \mathbb{Z}/2 \to \Omega^4_{SO(3)}(M \times \mathbb{R}^k) \to \Omega^4_{SO(4)}(M) \to 0
$$
Basic Results—Finding Homotopy Associatives

**Theorem**

Let \([\alpha] \in [M_+, MG]\) describe a \(G\)-bordism class in \(M\)
and let \(\tau: M \to BG_2\) describe a (topological) \(G_2\)-structure on \(M\).

- Then there exist (stable) homotopy \(G\)-associatives \(F: M \to XG\) in \([\alpha]\) over \(\tau\).
- The group \(\text{coker}(R_*: \pi_{8+k}(S^k MG) \to \pi_{8+k}(S^k MG, S^k YG))\)
  acts on \(hA^k_G(M, \tau)\), and \(\Omega^{4+k}_G(M \times \mathbb{R}^k) \cong hA^k_G(M, \tau)/\text{coker}(R_*)\).

**Remark**

- We will see that \(\pi_{8+k}(S^k MG, S^k YG) \cong \mathbb{Z}\)
  So \(\text{coker}(R_*)\) is a cyclic group.
- There is no naive \(h\)-principle in this setting.
  For an associative \(A \in [\alpha]\) in \((M, \varphi)\), we must have \(\varphi[A] > 0\)
  Choose \([\alpha]\) representing \(\beta^2 \in H^4(M)\) for \(\beta \in H^2(M)\), then \(\varphi[A] < 0\).
Consider potential obstructions against $F$.

Because $BG_2$ is 3-connected, we have a chain of isomorphisms

$$(H: \text{Hurewicz}, \Theta: \text{Thom})$$

For $\ell \leq 7$ this column is 0.

Hence, no obstructions against $F$.

For $\ell = 8$ this column is $\mathbb{Z}$.

Possibly different choices for $F$.

The image of $R_*$ does not affect $F$.

The map $r_*: H_4(BG) \to \mathbb{Z}$ for $\ell = 8$ is given by evaluating $p_1 + e \in H^4(BG)$.
We look for generators $[N]$ of $\pi_{8+k}(S^k MG)$ that provide relations for $\text{coker}(R_*)$

<table>
<thead>
<tr>
<th>$G$</th>
<th>$k$</th>
<th>$\pi_{8+k}(S^k MG)$</th>
<th>$[N]$</th>
<th>$(p_1 + e)(\nu)[N]$</th>
<th>$\text{coker}(R_*)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$SO(4)$</td>
<td>0</td>
<td>$\mathbb{Z}$</td>
<td>$[\mathbb{C}P^2]$</td>
<td>$-3 + 0$</td>
<td>$\mathbb{Z}/3$</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>$\mathbb{Z}^2$</td>
<td>$[\mathbb{C}P^2], [S^4, \text{id}]$</td>
<td>$-3 + 0, 0 + 2$</td>
<td>0</td>
</tr>
<tr>
<td>$SO(3)$</td>
<td>0</td>
<td>$\mathbb{Z} \oplus \pi_4(S^3)$</td>
<td>$[\mathbb{C}P^2], [S^4, \eta]$</td>
<td>$-3 + 0, 0 + 0$</td>
<td>$\mathbb{Z}/3$</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>$\mathbb{Z} \oplus ?$</td>
<td>$[\mathbb{C}P^2], ???$</td>
<td>$-3 + 0, 0 + 0$</td>
<td>$\mathbb{Z}/3$</td>
</tr>
<tr>
<td>$Sp(1)$</td>
<td>0</td>
<td>$\pi_7(S^3)$</td>
<td>???</td>
<td>$0 + 0$</td>
<td>$\mathbb{Z}$</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>$\mathbb{Z} \oplus ?$</td>
<td>$[K3]$</td>
<td>$48 - 24$</td>
<td>$\mathbb{Z}/24$</td>
</tr>
<tr>
<td>$\text{Spin}(3)$</td>
<td>0</td>
<td>$\pi_7(S^3)$</td>
<td>???</td>
<td>$0 + 0$</td>
<td>$\mathbb{Z}$</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>$\mathbb{Z} \oplus ?$</td>
<td>$[K3]$</td>
<td>$48 + 24$</td>
<td>$\mathbb{Z}/72$</td>
</tr>
<tr>
<td>${e}$</td>
<td>$k$</td>
<td>$\pi_{8+k}(S^{4+k})$</td>
<td>???</td>
<td>$0 + 0$</td>
<td>$\mathbb{Z}$</td>
</tr>
</tbody>
</table>
Bordism sets and groups \((k \geq 1)\) and classes of homotopy associatives

\[
\begin{align*}
\Omega^4_{SO(3)}(M) \xrightarrow{\mathbb{Z}/3} \Omega^4_{SO(3)}(M) \xrightarrow{\mathbb{Z}} \Omega^4_{SO(4)}(M) \xrightarrow{\mathbb{Z}/3} H^4(M) \\
hA^k_{SO(3)}(M, \tau) \xrightarrow{2\mathbb{Z}} hA^k_{SO(3)}(M, \tau) \xrightarrow{\mathbb{Z}/2} hA^k_{SO(4)}(M, \tau) \xrightarrow{\mathbb{Z}/3} H^4(M) \\
\Omega^{4+k}_{SO(3)}(M \times \mathbb{R}^k) \xrightarrow{\mathbb{Z}/3} \Omega^{4+k}_{SO(3)}(M \times \mathbb{R}^k) \xrightarrow{\mathbb{Z}/6} hA^k_{SO(4)}(M, \tau) \xrightarrow{\mathbb{Z}/3} H^4(M)
\end{align*}
\]

In each \(G\)-cobordism class \([\alpha]\) there are homotopy \(G\)-associatives
Hence there is no naive \(h\)-principle
For $F$ in an open dense subset $h\text{Asso}^k_{G,\text{reg}}(M,\tau)$ of $h\text{Asso}^k_{G}(M,\tau)$ we have $F \pitchfork BG$, so we get smooth submanifolds $A = F^{-1}(BG) \subset M$. The connected components of $h\text{Asso}^k_{G,\text{reg}}(M,\tau)$ are separated by subsets of “mildly singular” homotopy $G$-associatives.

The same is expected to happen in the analytic description. But the codimensions of corresponding subsets may differ.

We need to describe certain types of singularities to complete our overview both of associative $G$-submanifolds, and of homotopy $G$-associatives.
Singularities—Strategy

Given a family $A_t$ of $G$-associatives that are submanifolds for $t \neq 0$.

Construct a cobordism $W$ between $A_{-t}$ and $A_t$ for some $t > 0$.

Turn it into a $G$-cobordism.

If necessary, change the normal $G$-structure.

Obstruction against a lift $F$ \( \rightsquigarrow \) Difference between $A_{-t}$ and $A_t$ in $hA_G(M, \tau)$.

Maybe we can “see” a passage through a singularity by a change in the $G$-cobordism class or the class of the homotopy $G$-associative.

Try to compare this with known results or conjectures.
Let \((\varphi_s)_s\) be a family of torsion free \(G_2\)-structures on \(M\)
Let \((A_t)_t\) be a smooth family of flagged associative submanifolds in \((M, \varphi_{t^2})\)
Under certain assumptions, the \(A_t\) are unobstructed for \(t \neq 0\)
and have opposite Joyce orientation (flag) for \(t > 0\) and for \(t < 0\)

View each \(A_t\) as a homotopy \(SO(3)\)-associative for \((M, \varphi_0)\)
Consider the bordism \(\bigcup_t A_t \times \{(t^2, t)\} \subset M \times \mathbb{R}^2\)
Regarding \(s = t^2\) as bordism parameter,
we have \(0 = [A_t] + [A_{-t}] \in \Omega^5_{SO(3)}(M \times \mathbb{R})\)

To achieve compatibility with Joyce, we may try to

- identify \(-[A] \in \Omega^5_{SO(3)}(M \times \mathbb{R})\) with \([A]\), equipped with the opposite flag
- using a flag structure, identify \(\Omega^5_{SO(3)}(M \times \mathbb{R}) / \sim \) with \(\Omega^4_{SO(4)}(M) \cong H_3(M)\)

This could even lead to a “twisted” \(h\)-principle
Let \((A_t)_t\) be a family of \(G\)-associatives that are submanifolds for \(t \neq 0\). Assume that \(A_0\) has only one isolated singularity at \(x_0\). All generic singularities of homotopy associatives are of this type.

Let \(B\) be a cobordism between \(A_{-t}\) and \(A_t\) for some \(t > 0\). Assume that there exists a ball \(U \cong B^7\) around \(x_0\) such that

\[
B \setminus (U \times [0,1]) = (A_{-t} \setminus U) \times [0,1]
\]

If we can choose a normal frame along \(W = B \cap (U \times [0,1])\), we obtain an element in \([\partial W, G_2]\). Use this to determine the difference between \(A_{-t}\) and \(A_t\) in \(hA_G(M, \tau)\).
Let \((\varphi_t)_{t \in (-\varepsilon, \varepsilon)}\) be a family of torsionfree \(G_2\)-structures. Assume that \(A_t\) is an immersed associative for \(\varphi_t\) with one selfintersection for \(t = 0\) and no other singularities.

We may assume that \(\bigcup_t A_t \times \{t\}\) has a transversal selfintersection in \(M \times \mathbb{R}\). Then \(A_{-t}\) and \(A_t\) are stably isotopic.

But what about the extra family with a Lawlor neck?
Construct an unstable $SO(4)$-cobordism $W = W_- \cup_{W_0} W_+ \subset M \times I$ between first and second branches on both sides with a Lawlor neck in the middle.

$W_-$ contains a critical point of index 1
$W_+$ contains a critical point of index 3

One can compute $p_1(\nu)[W_\pm, \partial W_\pm] = 0$ and $e(\nu)[W_\pm, \partial W_\pm] = 1$

Hence, $[A_-], [A_0], [A_+]$ are three different lifts of $[A] \in \Omega^4_{SO(4)}(M)$ to $hA_{SO(4)}(M)$

Regarding $W$ as a stable $SO(3)$-cobordism, get different flags on $A_-, A_0, A_+$
Consider the three known resolutions of a generic selfintersection singularity. They live in $\mathbb{C}^3 \subset \mathbb{R}^7$. So there are preferred flags. We have unstable $SO(4)$-bordisms $W_\pm$. They do not respect the preferred flag. And $A_-$ and $A_+$ are stably isotopic.

- Unstably, $A_\pm$ and $A_0$ realise all three lifts of their bordism class to $hA_{SO(4)}(M, \tau)$.
- Unstably, $A_\pm$ and $A_0$ have pairwise different $SO(3)$-structures (flags).
- Stably, $[A_+] = [A_-] \in hA_{SO(3)}^s(M, \tau)$. But the preferred stable flag on $A_0$ is opposite to the one from $A_\pm$. 
We need to understand generic singularities to get an overview over all possible (homotopy) associatives.

Generic topological singularities are isolated and realised by surgery.

Some “geometrically generic” singularities are not “topologically generic”.

The “type” of a (stable) homotopy $G$-associative can change after passing through a singularity—depending on $G$ and stabilisation.

There is a space of homotopy $G$-associatives. It captures the full homotopy fibre of true associative $G$-submanifolds.

One can study the subset of regular homotopy $G$-associatives.

Topology can help to understand the counting problem but some geometry is still needed to solve it.

Thanks a lot for your attention!