Homotopy Associative Submanifolds in G_2-Manifolds

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Outline

Associative Submanifolds

Let (M, φ) be a closed seven-manifold with holonomy G_2
Then $\varphi \in \Omega^3(M)$ is a calibration
An associative submanifold $A \subset M$ is a three-dimensional submanifold that is calibrated by φ (Harvey and Lawson 1982)

Recall that φ defines a metric on M, has $\mathrm{comass}(\varphi) = 1$, and $d\varphi = 0$
Then A is calibrated by φ if and only if $\varphi|_A = d\,\mathrm{vol}_A$
In particular, A is a volume minimising submanifold within its homology class in $H_3(M)$

Generic associative submanifolds are rigid (McLean 1998), in particular

- they are isolated in the space of three-dimensional submanifolds
- they vary smoothly with φ for small deformations of φ

Associative Submanifolds—Counting

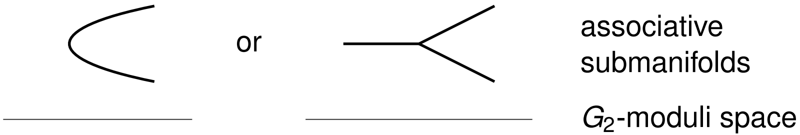
Associative submanifolds share certain properties
with complex curves in Calabi-Yau three-manifolds

Question

Can one count associative submanifolds and get subtle invariants of (M, φ) as for example in Gromov-Witten theory? Related to counting problem for G_2-instantons (Donaldson-Segal 2011, Haydys-Walpuski 2015)

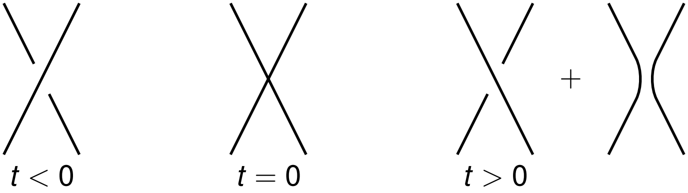
Problem

Naive counting is not invariant under modifications of φ
As φ varies, one may have obstructed or singular associatives like



Associative Submanifolds—Example of a Bifurcation

Let $(\varphi_t)_{t\in(-\varepsilon,\varepsilon)}$ be a family of torsionfree G_2-structures
Assume that A_t is an immersed associative for φ_t
with one selfintersection for $t = 0$ and no other singularities



Then at $t = 0$ another family of associative submanifolds is created or destroyed
that looks like the connected sum of the two branches (Joyce, Nordström, Bera)
It has the local geometry of a Lawlor neck
(a certain special Lagrangian in $\mathbb{C}^3$, Lawlor 1989)

This picture looks like the skein relation from knot theory

Associative Submanifolds—Summary

- Associative submanifolds are calibrated submanifolds in G_2-manifolds
  They are generically rigid, but otherwise hard to control
- Associative submanifolds are expected to behave analogous
  to complex curves in Calabi-Yau threefolds
- Counting associative submanifolds is tricky because of bifurcations
  There may be more problems
- Conjectural counting schemes have been proposed by
  Joyce (2018, $b_1(A) = 0$) and Doan-Walpuski (2019, $b_1(A) > 1$)
- Some pictures of associatives look like pictures from knot theory
  But knot theory and complex curves are not entirely unrelated
  (Ekholm-Shende)

Homotopy Associatives

We (Andriy Haydys and myself) attempt to use topological methods to

- get an overview of all possible associative submanifolds
- get around all the analytic problems—or at least postpone them

There is no naive *h*-principle, so we will loose information

We distinguish associative submanifolds by cobordism classes
Finer than homology, but coarser than homotopy

We also consider normal *G*-structures, for $G \subset SO(4) \subset G_2$, for example

- $G = SO(3)$ describes Joyce's flagged associatives
- $G = \mathrm{Spin}(3)$ describes the deformation operator as a spin Dirac operator
- $G = Sp(1)$ describes associatives with trivialised tangent bundle

All this might work analogously for other kinds of calibrated submanifolds

Homotopy Associatives—Pontryagin-Thom in a nutshell

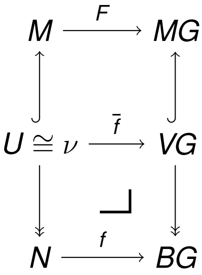
Translate submanifold theory to homotopy theory (Thom 1954, Pontryagin 1955)
Let G be a Lie group with a fixed representation of real dimension k

Let $N \subset M$ be a G-submanifold with normal bundle ν
The G-structure on ν is classified by $f\colon N \to BG$

A tubular neighbourhood $U \cong \nu$ maps properly
to the universal vector bundle $VG = EG \times_G \mathbb{R}^k \to BG$

The Thom space MG is the one-point compactification of VG
Send $M \setminus U$ to ∞

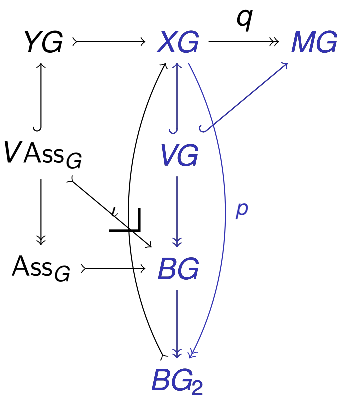
Conversely if $F \pitchfork BG$ recover $N = F^{-1}(BG)$ and $\bar{f} = dF|_\nu$



Theorem (Pontryagin-Thom)

$$\Omega^k_G(M) \;\cong\; [M_+, MG]$$

Homotopy Associatives—The space XG



Let $G \subset SO(4)$ be a subgroup and $\mathrm{Ass}_G = G_2/G$
Take $BG = EG_2/G \cong EG_2 \times_{G_2} \mathrm{Ass}_G$

Let $V\mathrm{Ass}_G = G_2 \times_G \mathbb{R}^4 \to \mathrm{Ass}_G$ and
take $VG = EG_2 \times_G \mathbb{R}^4 \cong EG_2 \times_{G_2} V\mathrm{Ass}_G$

Let YG denote the Thom space of $V\mathrm{Ass}_G \to \mathrm{Ass}_G$
and define $XG = EG_2 \times_{G_2} YG$

The points at infinity give a map $\iota \colon BG_2 \to XG$
Collapsing them to one point gives $q \colon XG \to MG$

Recover $MG \hookleftarrow VG \twoheadrightarrow BG$ from the
Pontryagin-Thom construction

Regard $XG \hookleftarrow VG \twoheadrightarrow BG$
as a bundle version over BG_2

Homotopy Associatives—Definition

Definition

A homotopy G-associative in a G-bordism class $[\alpha] \in [M_+, MG]$ and over the G_2-structure $\tau \colon M \to BG_2$ is a map $F \colon M \to XG$ such that $p \circ F = \tau$ and $q \circ F \sim \alpha$

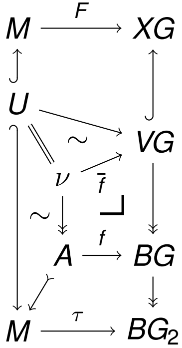
$$
\begin{array}{ccc}
MG & \overset{q}{\twoheadleftarrow} & XG \\
{\scriptstyle\alpha}\uparrow & \overset{F}{\nearrow} & \downarrow{\scriptstyle p} \\
M & \xrightarrow{\tau} & BG_2
\end{array}
$$

Write $h\mathcal{A}sso_G(M,\tau) = \Gamma(\tau^* XG \to M)$ and $hA_G(M,\tau) = h\mathcal{A}sso_G(M,\tau)/\sim$

If $F \pitchfork BG$, let $A = F^{-1}(BG)$, so $A \in [\alpha] \in \Omega^4_G(M)$ for $\alpha = q \circ F$
Let $\nu \to A$ be the normal bundle and $a \in A$, then $dF_a|_\nu \colon \nu_a \to VG_{F(a)} \cong V\mathrm{Ass}_G$ identifies ν_a with a coassociative subspace of T_aM with a G-structure

Homotopy Associatives versus Associative Submanifolds

Let M be a G_2-manifold with G_2-structure $\tau\colon M \to BG_2$
so $BG_{\tau(p)} = \{\, \text{coassociative } G\text{-subspaces of } T_pM \,\} \cong \mathrm{Ass}_G$
Let $A \subset M$ be an associative G-submanifold
For $a \in A$, get $f(a) \in BG_{\tau(a)}$ and $\bar{f}\colon \nu_a \overset{\cong}{\to} VG_{f(a)} \subset VG_{\tau(a)}$
Let $U \cong \nu$ be a tubular neighbourhood of A
Using fibre transport, map $U \to VG$ over τ
Finally, map $p \in M \setminus U$ to $\infty_{\tau(p)} \in XG_{\tau(p)} \cong YG$
This turns an associative G-submanifold A
into a homotopy G-associative $F \in h\mathcal{A}sso_G(M, \tau)$



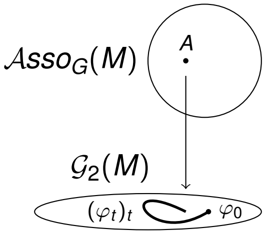
Homotopy Associatives—the Homotopy Fibre

More generally, let $\mathcal{A}sso_G(M) \to \mathcal{G}_2(M)$ describe all associative G-submanifolds for all G_2-structures on M and construct a map

$$\mathrm{hofib}\big(\mathcal{A}sso_G(M) \to (\mathcal{G}_2(M), \varphi_0)\big) \quad \longrightarrow \quad h\mathcal{A}sso_G(M, \tau_0)$$

Recall that elements of this homotopy fibre are pairs $\big((\varphi_t)_{t\in[0,1]}, A\big)$ of

- A path $(\varphi_t)_t$ of torsion-free G_2-structures
- A G-associative A in (M, φ_1)



Using fibre transport, we can turn A into a homotopy G-associative in (M, φ_0)

We might even hope for a naive *h*-principle: Each class in $hA_G(M, \tau_0)$ would then be realised by an element of $\mathrm{hofib}\big(\mathcal{A}sso_G(M) \to (\mathcal{G}_2(M), \varphi_0)\big)$

Homotopy Associatives—Stabilisation

Idea. Add extra dimensions to make life easier
But never leave the G_2-world! In particular, do not make $G \subset SO(4)$ larger

- Consider compact threedimensional submanifolds of $M \times \mathbb{R}^k$ with normal bundle $\nu = \nu' \oplus \underline{\mathbb{R}}^k$ having a G-structure on ν'

$$\Omega_G^{4+k}(M \times \mathbb{R}^k) \cong [S^k M_+, S^k MG]$$

- Replace YG by $S^k YG$ and XG by $X^k G = EG_2 \times_{G_2} S^k YG$
Define $h\mathcal{A}sso_G^k(M, \tau) = \Gamma(\tau^* X^k G)$ and $h\mathcal{A}sso_G^s = \operatorname{colim}_{k \to \infty} h\mathcal{A}sso_G^k$
- Turn immersed G-associatives into stable homotopy G-associatives
- Get group structures on $\Omega_G^{4+k}(M \times \mathbb{R}^k)$ and $hA_G^k(M, \tau) = h\mathcal{A}sso_G^k(M, \tau)/{\sim}$
Note: $k \geq 1$ suffices and gives abelian groups

We will see that $G = SO(3)$ and $k \geq 1$ fits with Joyce's proposal

Homotopy Associatives—Summary

- The Pontryagin-Thom construction gives an isomorphism

$$[M_+, MG] \xrightarrow{\cong} \Omega^k_G(M)$$

- We replace MG by a new space XG that fibres over BG_2
  It contains a copy of BG_2 at infinity, and collapsing it gives back MG
- Homotopy G-associatives for a G_2-structure $\tau\colon M \to BG_2$ are sections of $\tau^* XG \to M$
- A modified Pontryagin-Thom construction turns (immersed) associative G-submanifolds into (stable) homotopy G-associatives
- The space of homotopy G-associatives captures the full homotopy fibre of associatives over arbitrary G_2-structures on M

Basic Results

Before we look at concrete constructions,
we need some basic facts about bordism sets and groups

- Can every class in $H_3(M) \cong H^4(M)$ be realised as a G-bordism class?—Yes
- Is the representation in $\Omega^4_G(M)$ unique?—Only for $G = SO(4)$
- Does stabilisation introduce new bordism classes?—No
- What are the preimages of $\Omega^4_G(M) \twoheadrightarrow \Omega^{4+k}_G(M \times \mathbb{R}^k)$?
  Do they have a geometric meaning?

We also want to know how bordism classes refine to homotopy G-associatives

- Can every class in $\Omega^{4+k}_G(M)$ be realised in $hA^k_G(M, \tau)$?—Yes
- Is this representation unique?—Only for $G = SO(4)$ and $k \geq 1$
- What are the preimages of $hA^k_G(M, \tau) \twoheadrightarrow \Omega^{4+k}_G(M \times \mathbb{R}^k)$?
  Do they have a geometric meaning?

Basic Results—Bordism Sets and Groups

Let $\dim M \leq 7$
Thom (1954) has shown that $\Omega^4_{SO(4)}(M) \cong H^4(M)$
By Freudenthal's suspension theorem also $\Omega^{4+k}_{SO(4)}(M \times \mathbb{R}^k) \cong H^4(M)$

Let M be a spin 7-manifold. For other groups $G \subset SO(4)$, we get

$$\Omega^4_G(M) \longrightarrow \Omega^{4+k}_G(M \times \mathbb{R}^k) \longrightarrow \Omega^{4+k}_{SO(4)}(M \times \mathbb{R}^k) \cong H^4(M)$$

To understand the preimages of the second map,
obstruction theory tells us compute

$$\pi_{k+\ell}\big(MSO(4+k), S^k MG\big) \qquad \text{for } \ell \leq 8 \ .$$

For simply connected groups G like $\mathrm{Spin}(3)$, $Sp(1)$ or $\{e\}$,
there are too many obstruction groups for a simple answer

For $G = SO(3)$, Pontryagin-Thom gives

$$\Omega^4_{SO(3)}(M) \cong [M_+, S^1 MSO(3)] \cong \{\, N \subset M \text{ with normal bundle } \nu \cong \nu' \oplus \underline{\mathbb{R}} \,\} \;/\; \sim$$

Adapting Joyce's terminology, we call this "flagged oriented cobordism"

Because $SO(3) \to SO(4+k)$ is 2-connected, we compute

$$\pi_{k+\ell}\big(MSO(4+k), S^k MSO(3)\big) \cong \begin{cases} 0 & \text{for } \ell \leq 7, \\ \mathbb{Z} & \text{for } \ell = 8 \text{ and } k = 0, \text{ and} \\ \mathbb{Z}/2 & \text{for } \ell = 8 \text{ and } k \geq 1. \end{cases}$$

Hence, $\mathbb{Z}$ acts on $\Omega^4_{SO(3)}(M)$ with quotient $\Omega^4_{SO(4)}(M)$, and for $k \geq 1$,

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \Omega^{4+k}_{SO(3)}(M \times \mathbb{R}^k) \longrightarrow \Omega^4_{SO(4)}(M) \longrightarrow 0$$

Theorem

Let $[\alpha] \in [M_+, MG]$ *describe a G-bordism class in M and let* $\tau\colon M \to BG_2$ *describe a (topological)* G_2*-structure on M*

- *Then there exist (stable) homotopy G-associatives* $F\colon M \to XG$ *in* $[\alpha]$ *over* τ
- *The group* $\operatorname{coker}\big(R_*\colon \pi_{8+k}(S^k MG) \to \pi_{8+k}(S^k MG, S^k YG)\big)$ *acts on* $hA^k_G(M,\tau)$*, and* $\Omega^{4+k}_G(M \times \mathbb{R}^k) \cong hA^k_G(M,\tau)/\operatorname{coker}(R_*)$

Remark

- We will see that $\pi_{8+k}(S^k MG, S^k YG) \cong \mathbb{Z}$
  So $\operatorname{coker}(R_*)$ is a cyclic group
- There is no naive *h*-principle in this setting
  For an associative $A \in [\alpha]$ in (M,φ), we must have $\varphi[A] > 0$
  Choose $[\alpha]$ representing $\beta^2 \in H^4(M)$ for $\beta \in H^2(M)$, then $\varphi[A] < 0$

$$
\begin{array}{ccc}
\pi_{k+\ell}(S^k MG) & \xrightarrow{R_*} & \pi_{k+\ell}(S^k MG, S^k YG) \\
\downarrow H & & \cong \downarrow H \\
H_{k+\ell}(S^k MG) & \longrightarrow & H_{k+\ell}(S^k MG, S^k YG) \\
\cong \downarrow \Theta & & \cong \downarrow \Theta \\
H_{\ell-4}(BG) & \xrightarrow{r_*} & H_{\ell-4}(BG, \mathrm{Ass}_G) \\
 & & \cong \uparrow H \\
 & & \pi_{\ell-4}(BG, \mathrm{Ass}_G) \\
 & & \cong \downarrow \pi_* \\
 & & \pi_{\ell-4}(BG_2)
\end{array}
$$

Consider potential obstructions against F
Because BG_2 is 3-connected,
we have a chain of isomorphisms
(H: Hurewicz, Θ: Thom)
For $\ell \leq 7$ this column is 0
Hence, no obstructions against F
For $\ell = 8$ this column is $\mathbb{Z}$
Possibly different choices for F

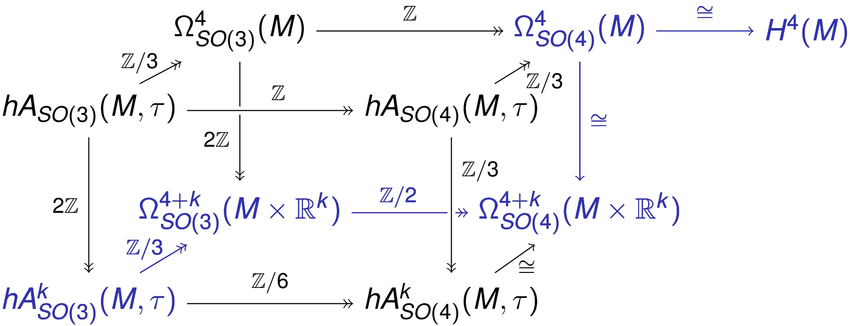
The image of R_* does not affect F
The map $r_* \colon H_4(BG) \to \mathbb{Z}$ for $\ell = 8$
is given by evaluating $p_1 + e \in H^4(BG)$

Basic Results—Relations

We look for generators $[N]$ of $\pi_{8+k}(S^k MG)$ that provide relations for $\mathrm{coker}(R_*)$

G	k	$\pi_{8+k}(S^k MG)$	$[N]$	$(p_1 + e)(\nu)[N]$	$\mathrm{coker}(R_*)$
$SO(4)$	0	$\mathbb{Z}$	$[\mathbb{C}P^2]$	$-3+0$	$\mathbb{Z}/3$
	1	$\mathbb{Z}^2$	$[\mathbb{C}P^2], [S^4, \mathrm{id}]$	$-3+0, 0+2$	0
$SO(3)$	0	$\mathbb{Z} \oplus \pi_4(S^3)$	$[\mathbb{C}P^2], [S^4, \eta]$	$-3+0, 0+0$	$\mathbb{Z}/3$
	1	$\mathbb{Z} \oplus ?$	$[\mathbb{C}P^2], ???$	$-3+0, 0+0$	$\mathbb{Z}/3$
$Sp(1)$	0	$\pi_7(S^3)$	???	$0+0$	$\mathbb{Z}$
	1	$\mathbb{Z} \oplus ?$	$[K3]$	$48-24$	$\mathbb{Z}/24$
$\mathrm{Spin}(3)$	0	$\pi_7(S^3)$	???	$0+0$	$\mathbb{Z}$
	1	$\mathbb{Z} \oplus ?$	$[K3]$	$48+24$	$\mathbb{Z}/72$
$\{e\}$	k	$\pi_{8+k}(S^{4+k})$	???	$0+0$	$\mathbb{Z}$

- Bordism sets and groups ($k \geq 1$) and classes of homotopy associatives



- In each G-cobordism class $[\alpha]$ there are homotopy G-associatives
  Hence there is no naive h-principle

For F in an open dense subset $h\mathcal{A}sso_G^{k,\mathrm{reg}}(M,\tau)$ of $h\mathcal{A}sso_G^k(M,\tau)$
we have $F \pitchfork BG$, so we get smooth submanifolds $A = F^{-1}(BG) \subset M$
The connected components of $h\mathcal{A}sso_G^{k,\mathrm{reg}}(M,\tau)$ are separated by subsets of "mildly singular" homotopy G-associatives

The same is expected to happen in the analytic description
But the codimensions of corresponding subsets may differ

We need to describe certain types of singularities to complete our overview both of associative G-submanifolds, and of homotopy G-associatives

Singularities—Strategy

Given a family A_t of G-associatives
that are submanifolds for $t \neq 0$

Construct a cobordism W between A_{-t} and A_t
for some $t > 0$

Turn it into a G-cobordism
If necessary, change the normal G-structure

Obstruction against a lift F $\rightsquigarrow$
Difference between A_{-t} and A_t in $hA_G(M, \tau)$

$$\begin{array}{ccc} M \times \{-t, t\} & \longrightarrow & XG \\ \downarrow & \overset{F}{\nearrow} & \downarrow \\ & & MG \\ \downarrow & \nearrow & \downarrow \\ M \times [-t, t] & \longrightarrow & MSO(4) \end{array}$$

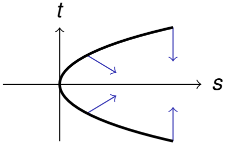
Maybe we can “see” a passage through a singularity by a change
in the G-cobordism class or the class of the homotopy G-associative
Try to compare this with known results or conjectures

Singularities—Changes of Orientation

Let $(\varphi_s)_s$ be a family of torsion free G_2-structures on M
Let $(A_t)_t$ be a smooth family of flagged associative submanifolds in (M, φ_{t^2})

Under certain assumptions, the A_t are unobstructed for $t \neq 0$
and have opposite Joyce orientation (flag) for $t > 0$ and for $t < 0$

View each A_t as a homotopy $SO(3)$-associative for (M, φ_0)
Consider the bordism $\bigcup_t A_t \times \{(t^2, t)\} \subset M \times \mathbb{R}^2$
Regarding $s = t^2$ as bordism parameter,
we have $0 = [A_t] + [A_{-t}] \in \Omega^5_{SO(3)}(M \times \mathbb{R})$



To acchieve compatibility with Joyce, we may try to

- identify $-[A] \in \Omega^5_{SO(3)}(M \times \mathbb{R})$ with $[A]$, equipped with the opposite flag
- using a flag structure, identify $\Omega^5_{SO(3)}(M \times \mathbb{R}) / \sim$ with $\Omega^4_{SO(4)}(M) \cong H_3(M)$

This could even lead to a “twisted” h-principle

Let $(A_t)_t$ be a family of G-associatives that are submanifolds for $t \neq 0$
Assume that A_0 has only one isolated singularity at x_0
All generic singularities of homotopy associatives are of this type

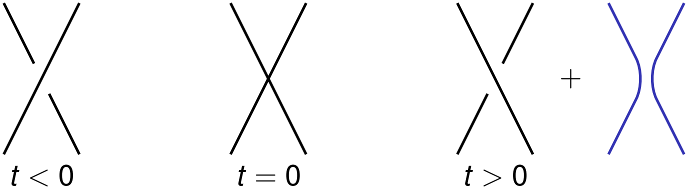
Let B be a cobordism between A_{-t} and A_t for some $t > 0$
Assume that there exists a ball $U \cong B^7$ around x_0 such that

$$B \setminus (U \times [0,1]) \quad = \quad (A_{-t} \setminus U) \times [0,1]$$

If we can choose a normal frame along $W = B \cap (U \times [0,1])$,
we obtain an element in $[\partial W, G_2]$
Use this to determine the difference between A_{-t} and A_t in $hA_G(M,\tau)$

Singularities—Selfintersections

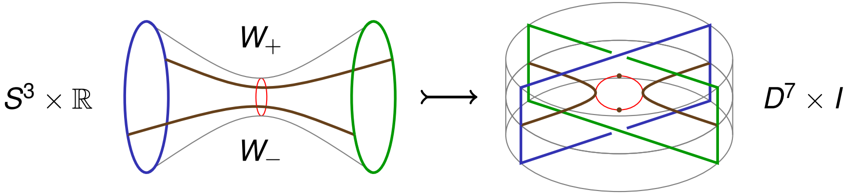
Let $(\varphi_t)_{t\in(-\varepsilon,\varepsilon)}$ be a family of torsionfree G_2-structures
Assume that A_t is an immersed associative for φ_t
with one selfintersection for $t = 0$ and no other singularities



We may assume that $\bigcup_t A_t \times \{t\}$ has a transversal selfintersection in $M \times \mathbb{R}$
Then A_{-t} and A_t are stably isotopic

But what about the extra family with a Lawlor neck?

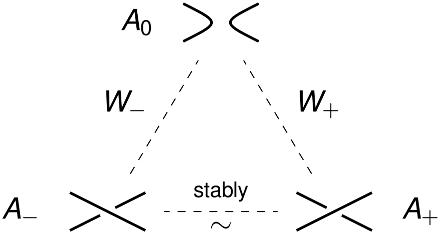
Construct an unstable $SO(4)$-cobordism $W = W_- \cup_{W_0} W_+ \subset M \times I$ between first and second branches on both sides with a Lawlor neck in the middle



W_- contains a critical point of index 1
W_+ contains a critical point of index 3

One can compute $p_1(\nu)[W_\pm, \partial W_\pm] = 0$ and $e(\nu)[W_\pm, \partial W_\pm] = 1$
Hence, $[A_-]$, $[A_0]$, $[A_+]$ are three different lifts of $[A] \in \Omega^4_{SO(4)}(M)$ to $hA_{SO(4)}(M)$
Regarding W as a stable $SO(3)$-cobordism, get different flags on A_-, A_0, A_+

Singularities—Stable versus Unstable



Consider the three known resolutions of a generic selfintersection singularity
They live in $\mathbb{C}^3 \subset \mathbb{R}^7$
So there are preferred flags

We have unstable $SO(4)$-bordisms $W_\pm$
They do not respect the preferred flag

And A_- and A_+ are stably isotopic

- Unstably, $A_\pm$ and A_0 realise all three lifts of their bordism class to $hA_{SO(4)}(M, \tau)$
- Unstably, $A_\pm$ and A_0 have pairwise different $SO(3)$-structures (flags)
- Stably, $[A_+] = [A_-] \in hA^s_{SO(3)}(M, \tau)$
  But the preferred stable flag on A_0 is opposite to the one from $A_\pm$

Singularities—Summary

- We need to understand generic singularities to get an overview over all possible (homotopy) associatives
- Generic topological singularities are isolated and realised by surgery
- Some “geometrically generic” singularities are not “topologically generic”
- The “type” of a (stable) homotopy G-associative can change after passing through a singularity—depending on G and stabilisation

- There is a space of homotopy G-associatives
  It captures the full homotopy fibre of true associative G-submanifolds
- One can study the subset of regular homotopy G-associatives
- Topology can help to understand the counting problem
  But some geometry is still needed to solve it

Thanks a lot for your attention!