Solitons in Bryant's G_2-Laplacian flow.

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Rowan Juneman & Johannes Nordström (Bath)

3-forms in 7 variables and G_2-structures

One way to define G_2 is as $G_2 = Aut(\mathbb{O})$ where $\mathbb{O}$ is the octonions.

Define a cross-product and a G_2-**invariant** 3-form φ_0 on $\mathbb{R}^7 = \mathrm{Im}(\mathbb{O})$ using octonionic multiplication and the Euclidean inner product

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$$\varphi_0(u, v, w) := \langle u \times v, w \rangle = \langle uv, w \rangle.$$

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For an oriented smooth 7-manifold M and $p \in M$

$$\mathcal{P}_p(M) := \{\varphi \in \Lambda^3 T_p^* M \,|\, \iota^* \varphi_0 = \varphi \text{ for } \iota : T_p M \to \mathbb{R}^7\}$$

where ι is any orientation-preserving isomorphism.

A 3-form φ on M is *positive* if φ is a section of $\mathcal{P}(M)$, i.e. $\varphi_p \in \mathcal{P}_p(M)\ \forall p$, where $\mathcal{P}(M)$ denotes the bundle over M with fibre $\mathcal{P}_p(M)$.

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Each positive 3-form on M defines a reduction of the frame bundle $\mathcal{F}M$ to a principal subbundle of $\mathcal{F}M$ with fibre G_2, i.e. a G_2-structure on M that induces the given orientation on M.

Positive 3-forms on M $\leftrightsquigarrow$ (oriented) G_2-structures on M.

1st-order PDE system for G_2 holonomy metrics

Holonomy/parallel tensors correspondence: $Hol_g(M) \subseteq G_2 \subset \mathrm{SO}(7)$ implies

M^7 admits a g-parallel positive 3-form φ. *Converse?*

Theorem: Let (M, φ, g_φ) be a G_2-structure; the following are equivalent

1. $\mathrm{Hol}(g_\varphi) \subseteq G_2$ and φ is the induced 3-form
2. $d\varphi = d^*\varphi = 0$, where d^* is defined using Hodge star $*$ w.r.t. g_φ.

2 is *nonlinear* in φ: g_φ depends nonlinearly on φ and d^* depends on g_φ.

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By writing equation for 3-form φ (not metric g directly) and allowing $\mathrm{Hol}(g_\varphi) \subseteq G_2$ we obtain *a PDE system:* (not integro-differential equations) a 1st-order system of 49=(35+21-7) equations on the 35 coeffs of φ! It is an overdetermined diffeomorphism-invariant system.

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Elliptic approach: (singular) perturbation method due to **Joyce**.

All known constructions of G_2-holonomy metrics on compact manifolds use this approach!

Idea: Construct a *closed* G_2-structure φ with $d^*\varphi$ *sufficiently small*. Condition that closed 3-form $\tilde{\varphi} = \varphi + d\eta$ solves **2** yields a nonlinear elliptic PDE for 2-form η which is solvable by an iteration method (η is small).

Difficulty then becomes to construct initial closed G_2-structure φ with sufficiently small torsion: all constructions exploit various *degenerate limits.*

Bryant's Laplacian flow

There is a natural flow on *closed* G_2-structures. Solve

$$\frac{d\varphi_t}{dt} = \Delta_{\varphi_t}\varphi_t \tag{LF}$$

with initial condition φ_0 satisfying $d\varphi_0 = 0$. (Then $d\varphi_t = 0$ for all t.)

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- Stationary points of (LF) are exactly torsion-free G_2-structures.
- (LF) is the (upward) gradient flow for Hitchin's volume functional $\mathrm{vol}(\varphi)$
$$\mathrm{vol}(\varphi) := \frac{1}{7}\int_M \varphi \wedge *\varphi$$
when restricted to cohomology class of φ_0. Critical points of $\mathrm{vol}(\varphi)$ in $[\varphi]$ are **maxima** (strict modulo diffeos).

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- On a compact manifold $\mathrm{vol}(\varphi_t)$ is *increasing* along (LF)
$\Rightarrow$ there are no *compact* shrinking solitons in (LF) (unlike in Ricci flow).

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- Bryant–Xu : Short-time existence & uniqueness of solutions to (LF)
- Lotay–Wei: Torsion-free G_2-structures are stable under (LF).

  Lotay–Wei also establish analogues of results in Ricci flow, e.g. Hamilton's compactness result, Shi estimates, finite-time extension result.

Geometric flows 101

- *Ideal case*: Establish long-time existence & convergence of flow as $t \to \infty$ to a static solution.

e.g. *Hamilton's 3-diml spherical space form theorem:* any initial metric g_0 on a compact 3-manifold with $Ric(g_0) > 0$ converges under normalized Ricci flow to an Einstein metric $Ric(g) = g$ (which must be a spherical space form in 3 dims).

Later influential higher diml results under other *curvature pinching conditions* on initial metric: Huisken, Hamilton, Böhm-Wilking, Brendle-Schoen.

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• In general long-time existence or convergence to a static solution may fail. Both issues arise in 3-dim Ricci flow for *general* initial metrics.

Long-time existence fails when we encounter **finite-time singularities** of the flow, e.g. in Ricci flow when $|Riem(g_t)| \to \infty$ as $t \nearrow T$.

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In 3-diml Ricci flow *some* singularities were expected/needed:
Thurston's Geometrization Conjecture provided important insights into what (finite and infinite time) singularities *should* arise in 3-dim Ricci flow.

e.g. Neck-pinch singularities implementing *connect sum decomposition* of initial 3-manifold; modelled on shrinking cylinder $\mathbb{S}^2 \times \mathbb{R}$ which is a simple example of a **shrinking soliton**.

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In general need to develop a theory of **finite-time singularity models**:

if only *'expected'* finite-time singularities arise can hope to develop a **flow with surgeries** and study its long-time behaviour, eg *Ricci flow with surgery*.

We don't have any analogue of the Geometrization Conjecture to guide us in the G_2 setting but we can study **solitons in the Laplacian flow**.

Solitons in the Laplacian flow

In many geometric flows solitons provide key models for singularity formation (and sometimes singularity resolution).

A **Laplacian soliton** is a G_2-structure φ, vector field X, $\lambda \in \mathbb{R}$ satisfying

$$\begin{cases} d\varphi = 0 \\ \Delta_\varphi \varphi = \lambda\varphi + \mathcal{L}_X \varphi \end{cases} \tag{LSE}$$

$\Leftrightarrow$ self-similar solution of Laplacian flow

$$\varphi_t = k(t)^3 f^*\varphi, \qquad \frac{df}{dt} = k(t)^{-2}X, \qquad k(t) = \tfrac{1}{3}(3+2\lambda t)$$

$\lambda < 0$: **shrinkers** – ancient solutions, i.e. exist backwards to $t = -\infty$

$\lambda = 0$: **steady solitons** – eternal solutions, i.e. exist for all time $t \in \mathbb{R}$

$\lambda > 0$: **expanders** – immortal solutions, i.e. exist up to $t = +\infty$

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- Non-steady soliton $\Rightarrow \varphi$ exact
- Solitons on a *compact* manifold are stationary or expanders
- Scaling behaviour: (φ, X) a λ-soliton $\Longleftrightarrow (k^3\varphi, k^{-2}X)$ a $k^{-2}\lambda$-soliton.
- Bryant (unpublished) has studied the local generality of Laplacian solitons using methods from overdetermined systems of PDE.

Laplacian soliton construction methods

No general analytic machinery to construct *global* solutions to (LSE). Have same problem in Ricci flow (except in the Kähler setting.) Must study Laplacian solitons with *additional geometric structure*.

- *Homogeneous solitons:* Quite a bit of work by Lauret, Fino & coworkers
  Work has a Lie-theoretic flavour – nilpotent and solvable Lie groups
  Relevance to finite-time singularity formation on say compact 1-connected manifolds M with $p_1(M) \neq 0$ is unclear.

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- Cohomogeneity-one solitons: a Lie group acts with codimension-one generic orbit; reduces (LSE) to nonlinear system of ODEs.
  Various important Ricci solitons are cohomogeneity one: Cigar soliton; shrinking round cylinders; Bryant's steady soliton and expanders; Cao's Kähler expander and steady soliton; FIK Kähler shrinkers

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- *Bundle constructions*, e.g. a circle/torus bundle over base manifold with special geometry. Often leads to spaces foliated by special hypersurfaces and governed by the same ODEs as some cohomogeneity one examples.

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- S^1-collapsed solitons?: Foscolo-H-Nordström (accepted **Duke Math J**) constructed many new complete noncompact torsion-free G_2-structures on circle bundles over AC Calabi-Yau 3-folds. ? $\exists$ construction of S^1-collapsed solitons that is related to solitons of Lagrangian MCF in CY 3-folds?

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  Minimal symmetry assumption, but not clear that highly-collapsed solitons could appear as finite-time singularity models. But (as in torsion-free case) might also suggest existence of new non-collapsed solitons.

Our motivation and high-level overview of results

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Telegraphic overview of results:

- We find complete shrinkers, steady solitons and expanders
- Shrinkers are the rarest and most rigid
- Expanders are the most abundant
- Steady solitons are intermediate between shrinkers and expanders; our steady solitons behave very differently to steady Ricci solitons.

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- Almost all our complete examples are asymptotically conical (AC); **but** there are complete steady solitons with exponential volume growth that appear at the boundary of the space of AC steady solitons. Related to scalar curvature of closed G_2-structures being non-positive.

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- There are important differences between the $Sp(2)$ and $SU(3)$-invariant cases.

Theorems and conjectures on Laplacian shrinkers

Theorem Shrink1

There exists an explicit complete AC shrinker with rate -1 on $\Lambda^2_- \mathbb{S}^4$ and on $\Lambda^2_- \mathbb{C}P^2$.

- Shrinkers are rare! Possible models for *formation of conical singularities*.

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Theorem Shrink2

Let G be $SU(3)$ or $Sp(2)$. For every closed G-invariant G_2-cone C there exists a unique G-invariant shrinker AC end (i.e. need not extend to a complete AC shrinker) asymptotic to C.

- The space of G-invariant G_2-cones is 1-dimensional for $G = Sp(2)$ and 2-dimensional for $G = SU(3)$. So previous theorem yields continuous families of AC shrinker *ends*.

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Shrinker Conjectures

(i) The explicit $Sp(2)$-invariant AC shrinker on $\Lambda^2_-\mathbb{S}^4$ is the unique complete AC $Sp(2)$-invariant shrinker.
(ii) The explicit $Sp(2)$-invariant AC shrinker on $\Lambda^2_-\mathbb{S}^4$ is the unique complete AC shrinker asymptotic to an $Sp(2)$-invariant closed G_2-cone, i.e. we assume symmetry only of the cone not of the AC shrinker.
(iii) There are only finitely many complete $SU(3)$-invariant AC shrinkers.

So MOST AC shrinker ends from **Thm Shrink2** SHOULDN'T extend to complete AC shrinkers.

Theorems and conjectures on Laplacian expanders

Theorem Expand1

(i) There exists a 1-parameter family of complete $Sp(2)$-invariant AC expanders with rate -1 on $\Lambda^2_-\mathbb{S}^4$ and a 1-parameter family of $SU(3)\times\mathbb{Z}_2$-invariant AC expanders on $\Lambda^2_-\mathbb{C}P^2$.

(ii) *Every* $Sp(2)$-invariant closed G_2-cone 'on one side' of the torsion-free cone arises as the asymptotic cone of a unique (up to scale) complete AC $Sp(2)$-invariant expander.

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Theorem Expand2

Let G be $SU(3)$ or $Sp(2)$ and k be the dimension of the space of G-invariant closed G_2-cones.

$\exists$ a k-diml family of G-invariant AC expander ends asymptotic to any closed G-invariant G_2-cone; the difference between two such AC expanders is of order $\exp(-\frac{\lambda}{6}t^2)\times$ polynomial, where $\lambda > 0$ is the dilation constant of the expander.

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Expander Conjectures

(i) There is a 2-parameter family of complete $SU(3)$-invariant AC expanders on $\Lambda^2_-\mathbb{C}P^2$.
(ii) The set of asymptotic cones of complete $SU(3)$-invariant AC expanders is a proper open subset of the 2-diml space of all $SU(3)$-invariant closed G_2-cones.

We have a precise conjecture for what this proper open subset should be.

$Sp(2)$-invariant singularity formation

In other geometric flows, often there exists an AC shrinker and an AC expander that share a common (asymptotic) cone; combining them yields a ‘weak solution’ to the flow that is singular only at the time instant $t = 0$ where the common cone appears.

Feldman-Ilmanen-Knopf (FIK) called this ‘flowing through the singularity’.

e.g. in Kähler-Ricci flow there is such an AC shrinker (due to FIK) / AC expander (due to Cao) pair for blowing-down a (-1) curve in a Kähler surface

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Feldman-Ilmanen-Knopf (FIK) called this 'flowing through the singularity'.

No-Flow-Through Theorem

There is no complete $Sp(2)$-invariant AC expander whose cone coincides with the cone of the explicit AC $Sp(2)$-invariant shrinker on $\Lambda^2_-\mathbb{S}^4$.

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Obvious questions this raises:

1. Does the explicit AC $Sp(2)$-invariant shrinker arise as the singularity model for a finite-time singularity of Laplacian flow for some closed G_2-structure on a compact 7-manifold?
2. If **yes** to **Q1** how should we continue Laplacian flow past this singularity?

   Should we consider a singular version of Laplacian flow where the conical singularity persists but can vary within the set of closed G_2-cones?

   How does the bulk geometry drive the evolution of the conical singularity?

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- We expect *different* behaviour for the explicit $SU(3)$-invariant AC shrinker on $\Lambda^2_- \mathbb{C}P^2$!

$SU(3)$-invariant steady Laplacian solitons

Theorem Steady

There exists (up to scale) a 1-parameter family of complete $SU(3)$-invariant steady solitons on $\Lambda^2_-\mathbb{C}P^2$ parameterised by $s \in [-1, 1]$.

(i) For $s = 0$ it is the standard $SU(3)$-invariant torsion-free AC G_2 structure on $\Lambda^2_-\mathbb{C}P^2$.

(ii) For $\pm s \in (0, 1)$ it is a nontrivial steady soliton asymptotic with rate -1 to the unique $SU(3)$-invariant torsion-free cone.

(iii) For $s = \pm 1$ it is an *explicit* nontrivial steady soliton with *exponential volume growth*.

Asymptotically this steady soliton has constant negative scalar curvature and approaches a flat T^2-bundle over the sinh-cone over $(\mathbb{C}P^2, g_{\mathrm{FS}})$ where g_{FS} denotes the Fubini-Study metric.

(The sinh-cone of $\mathbb{C}P^2$ is a noncompact Einstein 5-manifold with negative Einstein constant.)

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Remarks:

- Any complete $Sp(2)$-invariant steady soliton must be a trivial steady soliton.
- **AC** steady solitons a new feature compared to Ricci/Kähler-Ricci flow.
- the asymptotic behaviour of the explicit steady soliton is impossible for steady Ricci solitons because they have scalar curvature $S \geq 0$.
- Understanding complete $SU(3)$-invariant steady solitons is also useful for proving results about complete $SU(3)$-invariant expanders, e.g. via blow-down arguments.

Closed invariant G_2-structures on $\Lambda^2_- M^4 \setminus M$

For $M = \mathbb{C}P^2$ or $\mathbb{S}^4$:

- $\Lambda^2_- M$ has a cohomogeneity one action by $G = SU(3)$ or $Sp(2)$
- $\Lambda^2_- M \setminus M$ is diffeomorphic to $\mathbb{R}_+ \times \Sigma$, for $\Sigma = SU(3)/T^2$ or $\mathbb{C}P^3$

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Any closed G-invariant G_2-structure φ on $\mathbb{R}_+ \times \Sigma$ can be written as

$$\varphi = (f_1^2\omega_1 + f_2^2\omega_2 + f_3^2\omega_3) \wedge dt + f_1 f_2 f_3\, \alpha = \omega_f \wedge dt + \mathrm{Re}\, \Omega_f$$

where $\omega_1, \omega_2, \omega_3 \in \Omega^2(\Sigma)$ and $\alpha \in \Omega^3(\Sigma)$ are G-invariant forms on Σ and $f_i : \mathbb{R}_+ \to \mathbb{R}_+$ satisfy

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For $Sp(2)$-invariance in addition we require $f_2 = f_3$. Structure equations for ω_i, α the same in both cases $\Rightarrow$ $\Lambda^2_- \mathbb{S}^4$ case can be treated as a special case of $\Lambda^2_- \mathbb{C}P^2$ case where $f_2 = f_3$.

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Discrete symmetries for $G = SU(3)$: When $f_j = f_k$ for $j \neq k$ metric on the corresponding principal orbit has an extra free isometric $\mathbb{Z}_2$-action, which does NOT preserve the $SU(3)$ structure (ω_f, Ω_f). When $f_1 = f_2 = f_3$ the orbit has an additional free isometric action of the symmetric group S_3, and the subgroup $A_3 < S_3$ preserves (ω_f, Ω_f).

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$$(\#) \Rightarrow \tfrac{d}{dt}(f_1 f_2 f_3)^{1/3} \geq \tfrac{1}{2} \quad \text{i.e. volume grows at least like } t^7.$$

The soliton equations and their local solutions

The **torsion** τ of a closed G_2-structure φ is the 2-form (of type 14) s.t.

$$d(*\varphi) = \tau \wedge \varphi.$$

In our G-invariant setting the torsion of φ_f is $\tau = \sum \tau_i \omega_i$ where ω_i are the G-invariant 2-forms on Σ and

$$\tau_i = (f_i^2)' + \frac{f_i^2}{f_1 f_2 f_3}\left(2f_i^2 - \sum f_i^2\right). \tag{1}$$

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The soliton condition for $(\varphi_f, X = u\frac{\partial}{\partial t}, \lambda)$ is the (mixed-order) ODE system

$$2(f_1 f_2 f_3)' = f_1^2 + f_2^2 + f_3^2, \tag{2a}$$

$$(\tau_i - uf_i^2)' = \lambda f_i^2, \quad \text{for } i = 1, 2, 3, \tag{2b}$$

$$\tau_1 + \tau_2 + \tau_3 = u(f_1^2 + f_2^2 + f_3^2) + 2\lambda f_1 f_2 f_3. \tag{2c}$$

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We can rewrite (2) as a real analytic *1st-order system in the 6 variables* (f_i, τ_i) and the type 14 condition on τ imposes the algebraic condition $\sum \tau_i f_j^2 f_k^2 = 0$.

Then u is determined algebraically from (f, τ) by equation (2c).

$\Rightarrow \exists$ a 4-diml family of $SU(3)$-invariant solitons & 2-diml family of Sp_2-invariant solitons.

In the *steady case* $\lambda = 0$ the action of scaling reduces these parameter counts by 1.

Smooth extension over the zero section of $\Lambda^2_- M^4$

Understand solutions defined near zero section of $\Lambda^2_- M$ that extend smoothly over it.

Proposition

- *For each $\lambda \in \mathbb{R}$, there is a 2-parameter family $\varphi_{b,c}$ of solutions defined for small t that extend smoothly to a λ-soliton on (nhd of zero section in) $\Lambda^2_- \mathbb{C}P^2$;*
- *the 1-parameter subfamily $\varphi_b = \varphi_{b,0}$ also defines λ-solitons on $\Lambda^2_- \mathbb{S}^4$.*

Two scale-invariant parameters: λb^2 and c. So up to scale:

- *2-parameter families of smoothly-closing expanders/shrinkers* on $\Lambda^2_- \mathbb{C}P^2$
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Idea of proof: The closed G-invariant G_2-structure

$$\varphi = (f_1^2\omega_1 + f_2^2\omega_2 + f_3^2\omega_3) \wedge dt + f_1 f_2 f_3 \alpha$$

on $\mathbb{R}_+ \times \Sigma$ extends to smooth G_2-structure on $\Lambda^2_- M$ if and only if

f_1 is odd with $f_1'(0) = 1$, and f_2 and f_3 are even with $b := f_2(0) = f_3(0) \neq 0$.

Resulting singular initial value problem for the higher order terms has formal power series solutions that are convergent. (It is a **regular singular point** of 1st-order ODE system).

Proof sketch of Theorem Steady

1. **Decoupling**
   - For $\lambda = 0$, the ODEs can be separated into evolution of *scale* g and evolution of 4 scale-normalised variables.
   - Unique fixed point for scale-normalised flow is torsion-free cone; it is a *stable* fixed point.
2. **Smoothly-closing solutions**
   - Near special orbit $\mathbb{C}P^2$, $\exists$ a 1-parameter family of solutions φ_c up to scale.
   - Unique one with $f_2 = f_3$: static soliton from Bryant–Salamon AC G_2-mfd has $c = 0$.

1 & 2: Stability of fixed point and continuous dependence of smoothly-closing solutions on c $\Rightarrow$ persistence of AC asymptotics for c sufficiently small.

3. **The explicit solution**

Numerical simulations suggested critical value c_{crit} of c at which AC asymptotics terminated. Inspection of power series solutions for c_{crit} led to initial guess for explicit solution.

4. **Trapping by the explicit solution**

Evolution of a quantity G suggested by the explicit solution (G is constant on it) guarantees that for any $c < c_{crit}$ the smoothly-closing solution φ_c is complete and has AC asymptotics.

Shrinkers: consequences of AC end rigidity (Thm Shrink2)

Heuristic for $\lambda < 0$: Invariant shrinkers on $\mathbb{R}_+ \times SU(3)/T^2$ are flow lines in 5-dim phase space.
In 4-dimensional space of flow lines

- Proposition $\Rightarrow$ a 2-dimensional submanifold of solutions extend across zero section $\mathbb{C}P^2 \subset \Lambda^2_- \mathbb{C}P^2$
- Thm Shrink2 $\Rightarrow$ a 2-dimensional submanifold of solutions has AC behaviour

Expect transverse intersections $\rightsquigarrow$ *finitely* many AC shrinkers on $\Lambda^2_- \mathbb{C}P^2$.

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Similarly, restricting attention to solutions with $f_2 = f_3$:

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In fact, can spot one explicit solution!

Theorem Shrink1: For $\lambda = -1$

$$f_1 = t, \quad f_2^2 = f_3^2 = \tfrac{9}{4} + \tfrac{1}{4}t^2, \quad u = \frac{t}{3} + \frac{4t}{9+t^2}$$

is an AC shrinker with rate -1 asymptotic to the cone $(1, \frac{1}{2}, \frac{1}{2})$.

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Conjecture: *This is the unique Sp_2-invariant AC shrinker on $\Lambda^2_- \mathbb{S}^4$.*

Thanks for your attention!

Bonus slides!

Expectations about $SU(3)$-invariant expanders.

Conjecture

(i) There is a 2-parameter family of complete $SU(3)$-invariant expanders on $\Lambda^2_-\mathbb{C}P^2$ asymptotic to a closed non-torsion free $SU(3)$-invariant G_2-cone.

(ii) The set of possible asymptotic cones of complete $SU(3)$-invariant expanders on $\Lambda^2_-\mathbb{C}P^2$ (with singular orbit type $\mathbb{C}P^2_1$, i.e. $f_1 \to 0$ as $t \to 0$) is a proper connected open subset of the 2-dimensional space of all closed $SU(3)$-invariant G_2-cones; it is bounded by those cones with $f_1 = f_2$ or $f_1 = f_3$, i.e. cones that possess either of the two possible extra $\mathbb{Z}_2$-isometries that differ from the extra asymptotic $\mathbb{Z}_2$-isometry present close to the singular orbit $\mathbb{C}P^2_1$.

Corollary

There are two distinct complete AC $SU(3)$-invariant expanders on $\Lambda^2_-\mathbb{C}P^2$ asymptotic to the cone of the explicit $SU(3)$-invariant shrinker on $\Lambda^2_-\mathbb{C}P^2$ constructed in Theorem Shrink1. They differ by their singular orbit types: $\mathbb{C}P^2_2$ and $\mathbb{C}P^2_3$ (versus $\mathbb{C}P^2_1$).

$\Rightarrow$ In the $SU(3)$-invariant case we CAN 'flow through the singularity': there is a 'weak solution' to Laplacian flow obtained by using the explicit $SU(3)$-invariant AC shrinker for $t < 0$, the cone for $t = 0$ and either of the two compatible AC expanders for $t > 0$.

We clearly don't have a unique weak solution in this case.

Constructing invariant non-steady AC ends

The problem of constructing an *invariant non-steady soliton AC end asymptotic to a given closed invariant cone* can be written as another singular initial value problem (SIVP) for the first-order ODE system: this time the SIVP is **irregular**.

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Theorem 1 *For any* $\lambda \neq 0$, $\exists$ a ! *formal power series* solution $\mathcal{P}$ in t^{-1} determined by the cone; $\exists$ a solution of the ODE system that is smooth in a nhd of $t = +\infty$ and whose Taylor series is $\mathcal{P}$.

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$\Rightarrow$ invariant AC shrinker ends are rigid.

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Theorem 3 Given $\lambda > 0$ and any closed cone (c_1, c_2, c_3)

- $\exists$ a 2-parameter family of AC soliton ends asymptotic to the given cone.
- Difference between two solutions is of order $\exp(-\frac{\lambda}{6}t^2) * \text{polynomial}$.
- If $c_2 = c_3$, then a 1-parameter subfamily has $f_2 = f_3$.

Flow lines of this 4=(2+2)-parameter family of solutions fill an open subset of 5-dimensional phase space. $\Rightarrow$ **invariant AC expander ends are stable**.

Closed invariant G_2 cones

Helpful to analyse invariant G_2-structures on $\mathbb{R}_+ \times \Sigma$ in terms of scale and homothety class of invariant metrics on Σ:

$$\text{scale} \quad g := \sqrt[3]{f_1 f_2 f_3} = \sqrt[6]{\mathrm{vol}(\Sigma)}$$

$$\text{homethety class} \quad \frac{f_1}{g}, \frac{f_2}{g}, \frac{f_3}{g}$$

φ closed and homothety class constant implies g linear and φ conical:

$$d\varphi = 0 \ \Rightarrow \ \frac{dg}{dt} = \frac{1}{6}\left(\frac{f_1^2}{g^2} + \frac{f_2^2}{g^2} + \frac{f_3^2}{g^2}\right) \ \Rightarrow \ f_i = c_i t$$

with

$$6c_1 c_2 c_3 = c_1^2 + c_2^2 + c_3^2. \qquad (*)$$

Closed invariant G_2 cones

Helpful to analyse invariant G_2-structures on $\mathbb{R}_+ \times \Sigma$ in terms of scale and homothety class of invariant metrics on Σ:

$$\text{scale} \quad g := \sqrt[3]{f_1 f_2 f_3} = \sqrt[6]{\mathrm{vol}(\Sigma)}$$

$$\text{homethety class} \quad \frac{f_1}{g}, \frac{f_2}{g}, \frac{f_3}{g}$$

φ closed and homothety class constant implies g linear and φ conical:

$$d\varphi = 0 \;\Rightarrow\; \frac{dg}{dt} = \frac{1}{6}\left(\frac{f_1^2}{g^2} + \frac{f_2^2}{g^2} + \frac{f_3^2}{g^2}\right) \;\Rightarrow\; f_i = c_i t$$

with

$$6c_1 c_2 c_3 = c_1^2 + c_2^2 + c_3^2. \qquad (*)$$

Note: *any* positive triple (c_1, c_2, c_3) can be uniquely rescaled to satisfy $(*)$
$\rightsquigarrow$ 2-parameter family of closed conical G_2-structures on $\mathbb{R}_+ \times SU(3)/T^2$.

In other words: **given a homothety class on Σ, there is a unique choice of "cone angle" that makes it a closed cone.**

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Rough strategy for finding AC solitons on $\Lambda^2_- M = M \sqcup \mathbb{R}_+ \times \Sigma$.

1. Solutions on $(0, \epsilon) \times \Sigma$ that extend smoothly across M at $t = 0$?
2. Solutions for large t asymptotic to prescribed closed cone (c_1, c_2, c_3)?
3. Do solutions from **1** and **2** fit together?

Comparison with other flows: the steady case

- All known *steady* solitons in Ricci flow have *sub-Euclidean* volume growth:

◦ the Bryant soliton; Appleton's resolutions of (some of) its quotients.

◦ Bryant soliton known to appear in a finite-time singularity of RF.

◦ known Kähler examples have at most half-dimensional volume growth (Cao, Conlon–Deruelle). *Not* seen in finite-time singular behaviour of KRF.

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• Our steady AC G_2 solitons most closely resemble Joyce-Lee-Tsui's (JLT) *translating solitons* in Lagrangian mean curvature flow (LMCF).
◦ Joyce conjectures JLT translating solitons *can* appear in finite-time singularities of LMCF if Floer homology is obstructed.
◦ Speculate that our steady G_2 solitons can also arise as finite-time singularities of Laplacian flow on a compact 7-manifold.

(Our 2-parameter family of AC G_2 *expanders* on $\Lambda^2_-\mathbb{C}P^2$ resembles JLT's family of exact Maslov-zero LMCF expanders asymptotic to pairs of transverse Lagrangian 3-planes).

Comparison with other flows: shrinkers

Ricci flow: One obvious significant difference: absence of *compact* shrinkers in G_2 flow; associated with positive curvature in RF, whereas scalar curvature is non-positive for closed G_2-structures.

General theory for *noncompact complete shrinkers* in RF is well-developed:

∘ their properties are a hybrid of those of positively curved Einstein manifolds and spaces with non-negative Ricci, e.g. at most Euclidean volume growth.

∘ AC (gradient) shrinkers are extremely rigid–manifestation of parabolic backwards uniqueness phenomenon, also seen in MCF.

∘ AC end behaviour of our (highly symmetric) G_2 shrinkers some indication such strong rigidity also holds for AC G_2 (gradient?) shrinkers.

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LMCF: self-shrinkers exist and do occur but *not* in the Maslov-zero (graded) setting. **Q:** Is there any natural condition to impose in the G_2 setting that would rule out our AC shrinkers on $\Lambda^2_+\mathbb{S}^4$ and $\Lambda^2_+\mathbb{C}P^2$?

KRF: Feldman-Ilmanen-Knopf (FIK) constructed symmetric ALE Kähler shrinkers; simplest FIK shrinker does appear as a finite-time blowup of KRF on 1-point blowup of $\mathbb{C}P^2$ and is associated with blowing down the point.

Expanders

Theorem (C)

For $\lambda > 0$, each φ_m extends to a complete solution with $f_2 = f_3$, and

$$\frac{f_i}{t} \to c_i$$

for (c_1, c_2, c_2) a closed cone with $c_1 \leq c_2$. These expanders are all smoothly asymptotic with rate -1 to a unique invariant closed G_2-cone.

This gives 1-parameter families of expanders on both $\Lambda^2_- \mathbb{S}^4$ and $\Lambda^2_- \mathbb{C}P^2$.

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Very Strong Expectation

- 1-1 correspondence with closed cones such that $c_1 < c_2$:
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Conjecture

For $\lambda > 0$, an **open** *subfamily of $\varphi_{m,c}$ (but not all) extend to complete solutions, defining a 2-parameter family of AC solitons on $\Lambda^2_- \mathbb{C}P^2$.*