Asymptotically conical G_2-solitons

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These slides available at
`http://people.bath.ac.uk/jlpn20/g2sol.pdf`

Introduction

G_2 **solitons:** self-similar solutions to Bryant's G_2 (closed) Laplacian flow.

Project: Find asymptotically conical (AC) G_2 solitons with cohomogeneity one: $SU(3)$-invariant ones on $\Lambda^2_+\mathbb{C}P^2$; $Sp(2)$-invariant ones on $\Lambda^2_+ S^4$.

Theorem

1-parameter family of steady solitons on $\Lambda^2_+\mathbb{C}P^2$ asymptotic with rate -1 to torsion-free cone (deformations of the Bryant-Salamon AC G_2-manifold).

Theorem

Explicit AC shrinker with rate -2 on $\Lambda^2_+ S^4$ and $\Lambda^2_+\mathbb{C}P^2$.

Possible models for formation of conical singularities in Laplacian flow.

- Shrinkers are rare!
- AC steady solitons a new feature (compared to Ricci/Kähler-Ricci flow).

Theorem

1-parameter family of complete expanders on $\Lambda^2_+ S^4$ and on $\Lambda^2_+\mathbb{C}P^2$.

Models for how Laplacian flow can smooth out certain conical singularities.

Bryant's Laplacian flow

Solve

$$\frac{d\varphi_t}{dt} = \Delta_{\varphi_t}\varphi_t$$

with initial condition φ_0 satisfying $d\varphi_0 = 0$. (Then $d\varphi_t = 0$ for all t.)

- Stationary points are exactly torsion-free G_2-structures.
- Gradient flow for $\mathrm{vol}(\varphi)$ restricted to cohomology class of φ_0.
- Induced metric evolves by

$$\frac{dg_t}{dt} = -2\mathrm{Ric}(g_t) + \text{ terms quadratic in torsion of } \varphi_t$$

Theorem (Bryant-Xu, Lotay-Wei)

Short-time existence and uniqueness.
Torsion-free G_2-structures are stable.

What is long term behaviour?? Expect singularities to form in finite time.
By analogy with other flows, expect solitons as models.

G_2 solitons

G_2-structure φ, vector field X, $\lambda \in \mathbb{R}$ satisfying

$$\begin{cases} d\varphi &= 0, \\ \Delta_\varphi \varphi = & \lambda\varphi + \mathcal{L}_X \varphi. \end{cases}$$

$\Leftrightarrow$ self-similar solution of Laplacian flow

$$\varphi_t = k(t)^3 f^* \varphi, \qquad \frac{df}{dt} = k(t)^{-2} X, \qquad k(t) = \frac{3 + 2\lambda t}{3}$$

$\lambda > 0$: expanders (immortal solutions)
$\lambda = 0$: steady solitons (eternal solutions)
$\lambda < 0$: shrinkers (ancient solutions)

- Non-steady soliton $\Rightarrow \varphi$ exact
- Solitons on a compact manifold are stationary or expanders
- Scaling behaviour: (φ, X) is a λ-soliton $\Leftrightarrow (k^3\varphi, k^{-2}X)$ is a $k^{-2}\lambda$-soliton.

Invariant G_2-structures

Let $M = \mathbb{C}P^2$ or S^4.
$\Lambda^2_+ M$ has a cohomogeneity one action by $G = SU(3)$ or $Sp(2)$.
Complement in $\Lambda^2_+ M$ to zero section is $\mathbb{R}_+ \times \Sigma$, for $\Sigma = SU(3)/T^2$ or $\mathbb{C}P^3$.

There are $\omega_1, \omega_2, \omega_3 \in \Omega^2(\Sigma)$ and $\alpha \in \Omega^3(\Sigma)$ such that any closed G-invariant G_2-structure on $\mathbb{R}_+ \times \Sigma$ with $\|\frac{\partial}{\partial t}\| = 1$ can be written as

$$\varphi = (f_1^2\omega_1 + f_2^2\omega_2 + f_3^2\omega_3) \wedge dt + f_1 f_2 f_3 \alpha, \qquad f_i : \mathbb{R}_+ \to \mathbb{R}_+$$

with

$$\frac{d(f_1 f_2 f_3)}{dt} = \tfrac{1}{2}(f_1^2 + f_2^2 + f_3^2).$$

For $Sp(2)$-invariance in addition require $f_2 = f_3$.

Structure equations the same in both cases $\Rightarrow$
$\Lambda^2_+ S^4$ case can be treated as a special case of $\Lambda^2_+ \mathbb{C}P^2$ case where $f_2 = f_3$.

Closed G_2 cones

Helpful to analyse invariant G_2-structures on $\mathbb{R}_+ \times \Sigma$ in terms of scale and homothety class of invariant metrics on Σ:

$$\text{scale} \quad g := \sqrt[3]{f_1 f_2 f_3} = \sqrt[6]{\mathrm{vol}(\Sigma)}$$
$$\text{homethety class} \quad \frac{f_1}{g}, \frac{f_2}{g}, \frac{f_3}{g}$$

φ closed and homothety class constant implies g linear and φ conical:

$$d\varphi = 0 \;\Rightarrow\; \frac{dg}{dt} = \frac{1}{6}\left(\frac{f_1^2}{g^2} + \frac{f_2^2}{g^2} + \frac{f_3^2}{g^2}\right) \;\Rightarrow\; f_i = c_i t$$

with

$$6c_1c_2c_3 = c_1^2 + c_2^2 + c_3^2. \qquad (*)$$

Note: any positive triple (c_1, c_2, c_3) can be uniquely rescaled to satisfy $(*)$
$\rightsquigarrow$ 2-parameter family of closed conical G_2-structures on $\mathbb{R}_+ \times SU(3)/T^2$.

In other words, given homothety class on Σ, there is a unique choice of "cone angle" to make a closed cone.

Evolution

On the face of it, the soliton condition for

$$\varphi = (f_1^2\omega_1 + f_2^2\omega_2 + f_3^2\omega_3) \wedge dt + f_1 f_2 f_3 \alpha, \qquad X = u\frac{\partial}{\partial t}$$

is 2nd-order ODE system for (f_1, f_2, f_3, u) (with some constraints).
Can rewrite as a 1st-order system in 5 variables.

Tendency: if $\frac{f_1}{g}, \frac{f_2}{g}, \frac{f_3}{g}$ bounded as $t \to \infty$ then asymptotic to closed cone.

Rough strategy for finding AC solitons on $\Lambda^2_+ M = M \sqcup \mathbb{R}_+ \times \Sigma$.

1. Solutions on $(0, \epsilon) \times \Sigma$ that extend smoothly across M at $t = 0$?
2. Solutions for large t asymptotic to prescribed closed cone (c_1, c_2, c_3)?
3. Do they fit together?

Picture for **1.** is clearest.

Initial value problem

Understand solutions near zero section of $\Lambda^2_+ M$ à la Eschenburg-Wang.

$$\varphi = (f_1^2\omega_1 + f_2^2\omega_2 + f_3^2\omega_3) \wedge dt + f_1 f_2 f_3 \alpha$$

on $\mathbb{R}_+ \times \Sigma$ extends to smooth G_2-structure on $\Lambda^2_+ M$ iff f_1 is odd with $f_1'(0) = 1$, and f_2 and f_3 are even with $m := f_2(0) = f_3(0) \neq 0$.

Solve resulting singular initial value problem by power series.

Proposition

For each $\lambda \in \mathbb{R}$, there is

- *a 2-parameter family $\varphi_{m,c}$ of solutions defined for small t that extend smoothly to a λ-soliton on (neighbourhood of zero section in) $\Lambda^2_+ \mathbb{C}P^2$;*
- *1-parameter subfamily $\varphi_m = \varphi_{m,0}$ also defines λ-solitons on $\Lambda^2_+ S^4$.*

Two scale-invariant parameters: λm^2 and c.

So up to scale there are 2-parameter families of local expanders and shrinkers on $\Lambda^2_+ \mathbb{C}P^2$, and 1-parameter family of local steady solitons.

Expanders

Theorem

For $\lambda > 0$, each φ_m extends to a complete solution with $f_2 = f_3$, and

$$\frac{f_i}{t} \to c_i$$

for (c_1, c_2, c_2) a closed cone with $c_1 \leq c_2$.

So this gives 1-parameter families of expanders on both $\Lambda^2_+ S^4$ and $\Lambda^2_+ \mathbb{C}P^2$.

Expectations

- These solitons are all AC, with rate -2
- 1-1 correspondence with closed cones such that $c_1 < c_2$:
  any closed cone with $c_2 = c_3$ on "one side" of the torsion-free cone $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is the AC end of a unique expander

Conjecture

For $\lambda > 0$, an open subfamily of $\varphi_{m,c}$ (but not all) extend to complete solutions, defining 2-parameter family of AC solitons on $\Lambda^2_+ \mathbb{C}P^2$.

Stability/rigidity of AC ends

Given $\lambda > 0$ and any closed cone (c_1, c_2, c_3), we expect:

- There is a 2-parameter family of solutions defined for large t asymptotic to the given cone.
- Difference between two solutions is of order $\exp(-\frac{\lambda}{6}t^2) * \text{polynomial}$.
- If $c_2 = c_3$, then a 1-parameter subfamily has $f_2 = f_3$.

Flow lines of this 4-parameter family of solutions fill open subset of 5-dimensional phase space, so AC expander ends are **stable**.

For $\lambda < 0$, for each closed cone (c_1, c_2, c_3) there is a unique solution defined for large t asymptotic to the given cone; shrinker ends are **rigid**.

(For $\lambda = 0$, only possible asymptotic cone is Bryant-Salamon cone $c_1 = c_2 = c_3 = \frac{1}{2}$.)

Shrinkers: consequences of AC end rigidity

Heuristic for $\lambda < 0$:

Invariant shrinkers on $\mathbb{R}_+ \times SU(3)/T^2$ are flow lines in 5-dim phase space. In 4-dimensional space of flow lines

- 2-dimensional submanifold extends across zero section $\mathbb{C}P^2 \subset \Lambda^2_+\mathbb{C}P^2$
- 2-dimensional submanifold has AC behaviour

Expect transverse intersections $\rightsquigarrow$ *finitely* many AC shrinkers on $\Lambda^2_+\mathbb{C}P^2$.

Similarly, restricting attention to solutions with $f_2 = f_3$:

2-dimensional space of flow lines; 1-dim submanifold extends over special orbit; 1-dim submanifold has AC behaviour.

Expect transverse intersections $\rightsquigarrow$ *finitely* many AC shrinkers on $\Lambda^2_+ S^4$.

In fact, can spot one explicit solution! For $\lambda = -1$

$$f_1 = t, \quad f_2^2 = f_3^2 = \tfrac{9}{4} + \tfrac{1}{4}t^2, \quad u = \frac{t}{3} + \frac{4t}{9 + t^2}.$$

AC with rate -2 to cone $(1, \tfrac{1}{2}, \tfrac{1}{2})$.

Steady solitons

Significant qualitative differences from $\lambda \neq 0$:

Near special orbit, only a 1-parameter family of solutions up to scale.
Unique one with $f_2 = f_3$: static soliton from Bryant-Salamon AC G_2-mfd.

Theorem
No non-stationary steady solitons on $\Lambda^2_+ S^4$.

Decoupling

- For $\lambda = 0$, the flow can be separated into evolution of *scale* g and evolution of 4 scale-normalised variables.
- Unique fixed point for the scale-normalised flow is the torsion-free cone; It is a *stable* fixed point.

Theorem
There exists a 1-parameter family (up to scale) of AC steady solitons on $\Lambda^2_+\mathbb{C}P^2$ all asymptotic to the torsion-free cone over $SU(3)/T^2$; the family includes steady solitons with arbitrarily small torsion.

Comparison with other flows: the steady case

• All known *steady* solitons in Ricci flow have *sub-Euclidean* volume growth:
◦ the Bryant soliton; Appleton's resolutions of (some of) its quotients.
◦ Bryant soliton known to appear in a finite-time singularity of RF.
◦ known Kähler examples have at most half-dimensional volume growth (Cao, Conlon–Deruelle). *Not* seen in finite-time singular behaviour of KRF.

• Our steady AC G_2 solitons most closely resemble Joyce-Lee-Tsui's (JLT) *translating solitons* in Lagrangian mean curvature flow (LMCF).
◦ Joyce conjectures JLT translating solitons *can* appear in finite-time singularities of LMCF if Floer homology is obstructed.
◦ Speculate that our steady G_2 solitons can also arise as finite-time singularities of Laplacian flow on a compact 7-manifold.

(Our 2-parameter family of AC G_2 *expanders* on $\Lambda^2_+\mathbb{C}P^2$ resembles JLT's family of exact Maslov-zero LMCF expanders asymptotic to pairs of transverse Lagrangian 3-planes).

Comparison with other flows: shrinkers

Ricci flow: One obvious significant difference: absence of *compact* shrinkers in G_2 flow; associated with positive curvature in RF, whereas scalar curvature is non-positive for closed G_2-structures.

General theory for *noncompact complete shrinkers* in RF is well-developed:
◦ their properties are a hybrid of those of positively curved Einstein manifolds and spaces with non-negative Ricci, e.g. at most Euclidean volume growth.
◦ AC (gradient) shrinkers are extremely rigid–manifestation of parabolic backwards uniqueness phenomenon, also seen in MCF.
◦ AC end behaviour of our (highly symmetric) G_2 shrinkers some indication such strong rigidity also holds for AC G_2 (gradient?) shrinkers.

LMCF: self-shrinkers exist and do occur but *not* in the Maslov-zero (graded) setting. **Q:** Is there any natural condition to impose in the G_2 setting that would rule out our AC shrinkers on $\Lambda^2_+ S^4$ and $\Lambda^2_+ \mathbb{C}P^2$?

KRF: Feldman-Ilmanen-Knopf (FIK) constructed symmetric ALE Kähler shrinkers; simplest FIK shrinker does appear as a finite-time blowup of KRF on 1-point blowup of $\mathbb{C}P^2$ and is associated with blowing down the point.