Automorphisms of Fano contact manifolds

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Wolf spaces

These are the Riemannian symmetric spaces with a quaternionic structure. One for each simple compact Lie group G parametrizes subalgebras associated to a highest root in $\mathfrak{g}$.

Given a compact simple Lie algebra $\mathfrak{g}$, fix such a "minimal" subalgebra $\mathfrak{su}(2) = \mathfrak{sp}(1)$. The Wolf space is

$$M = \frac{G}{H},$$

where $H = N(\mathfrak{sp}(1)) = \{g \in G : \mathrm{Ad}(g)(\mathfrak{sp}(1)) = \mathfrak{sp}(1)\}$.

If G is centreless, the linear holonomy group is

$$H = KSp(1) \subseteq Sp(n)Sp(1).$$

Each tangent space $T_o M \cong \mathbb{H}^n \otimes \mathbb{H}$ has dimension $4n$, and admits almost complex structures I, J, K, no preferred choice.

Families

Two 4-dimensional examples

$$S^4 = \frac{SO(5)}{SO(4)} = \frac{Sp(2)}{SU(2)\times SU(2)}$$

$$\mathbb{P}^2 = \mathbb{CP}^2 = \frac{SU(3)/\mathbb{Z}_3}{U(2)} = \frac{SU(3)}{U(1)\times SU(2)}.$$

branch out into three classical families of $4n$-manifolds.

$$\mathbb{HP}^n = \frac{Sp(n+1)}{Sp(n)\times Sp(1)}$$

$$\mathbb{G}\mathrm{r}_2(\mathbb{C}^{n+2}) = \frac{SU(n+2)}{S(U(n)\times U(2))}$$

$$\mathbb{G}\mathrm{r}_4(\mathbb{R}^{n+4}) = \frac{SO(n+4)}{SO(n)\times SO(4)}.$$

The exceptional Lie groups G_2, F_4, E_6, E_7, E_8 give Wolf spaces of quaternionic dimension $n = 2,\ 7,\ 10,\ 16,\ 28$.

Only $\mathbb{G}\mathrm{r}_2(\mathbb{C}^{n+2})$ ($= \mathbb{G}\mathrm{r}_4(\mathbb{R}^6)$ if $n=2$) has $b_2 > 0$. The others admit no almost complex structure even stably [GMS].

Reduced holonomy

A *quaternion-Kähler* (QK) manifold is a Riemannian manifold M of dimension $4n \geqslant 8$ with holonomy

$$H \subseteq Sp(n)Sp(1)$$

and not Ricci-flat. There is equality unless M is locally isometric to a symmetric space other than $\mathbb{HP}^n$.

A QK manifold is Einstein, so there is a dichotomy: $s < 0$, $0 < s$. Positive QK manifolds are "nearly hyperkähler" but only the 4-form

$$\Omega = \sum_{i=1}^{3} \omega_i \wedge \omega_i$$

is well defined. $Sp(n)Sp(1)$ is the only member of Berger's list for which the existence of compact examples with $\nabla R \not\equiv 0$ is open, though there are new complete examples with $s < 0$ constructed via Hessian metrics [Cortés et al].

We'll asssume that M is compact QK with $s > 0$ from now on.

Twistor space

The holonomy reduction equips each tangent space T_mM with a 2-sphere

$$Z_m = \{aI + bJ + cK : a^2+b^2+c^2 = 1\}$$

of almost complex structures. The associated bundle Z over M is a complex manifold in which each fibre is a rational curve.

If M has positive scalar curvature, Z is Kähler-Einstein; its 2-form ω is the curvature of an ample line bundle L such that $L|Z_m \cong \mathcal{O}(2)$.

The twistor space Z is locally a $\mathbb{C}^*$ quotient of a hyperkähler manifold U, with fibre $T^*\mathbb{P}^1$ over M, mimicking

$$\begin{array}{ccc} U & \subset & \mathbb{H}^{n+1}/\mathbb{Z}_2 \\ \downarrow & & \\ \mathbb{P}^{2n+1} & & \\ \downarrow & & \\ \mathbb{HP}^n. & & \end{array}$$

Holomorphic contact structure

The symplectic form $\omega_2 + i\omega_3$ on U induces a short exact sequence

$$0 \to D \to TZ \xrightarrow{\theta} L \to 0,$$

for which $\theta \in H^0(Z, T^*Z \otimes L)$ satisfies

$$0 \neq \theta \wedge (d\theta)^n \in H^0(Z, \kappa \otimes L^{n+1}).$$

This defines a complex contact structure, and $\overline{\kappa} \cong L^{n+1}$.

Locally, $D \cong L^{1/2} \otimes \pi^* E$, where E is the instanton bundle with fibre $\mathbb{H}^n$ and $\Omega_i^j \in \bigcap \Lambda^{1,1}$. Thus $D \cong D^* \otimes L$, and if $z \in Z$ is a fixed point of a torus action then

$$L_z \otimes T_z^* Z \cong \mathbb{C} + D_z$$

defines a "compass" of weights $(0, \nu_1, \nu - \nu_1, \ldots, \nu_n, \nu - \nu_n)$.

Homogeneous contact manifolds

When M^{4n} is a Wolf space,

$$Z = \frac{G}{KU(1)} \quad \downarrow \quad \frac{G}{KSp(1)} = M.$$

The twistor space Z can be described as

- a minimal adjoint orbit in $\mathfrak{g}$;
- a closed nilpotent orbit in $\mathbb{P}(\mathfrak{g}_c^*)$ with $L \cong \iota^* \mathcal{O}(1)$.

All homogeneous complex contact manifolds arise in this way.

Another TDS $\mathfrak{su}(2) \subset \mathfrak{g}$ will give rise to an open nilpotent coadjoint orbit U and an incomplete QK space $U/\mathbb{H}^*$. E.g.,

$$\mathbb{P}(\{A \in SL(3, \mathbb{C}) : A^3 = 0,\ A^2 \neq 0\})$$

is a $\mathbb{Z}_3$ quotient of $Z(G_2/SO(4) \setminus \mathbb{P}^2)$ [BK, "shared orbits"].

Fano twistor spaces

The classical twistor spaces of complex dimension $2n+1$ are

$$\begin{aligned} Z\,\mathbb{HP}^n &= \mathbb{P}^{2n+1} \\ Z\,\mathbb{G}\mathrm{r}_2(\mathbb{C}^{n+2}) &= \mathbb{P}(T^*\mathbb{P}^{n+1}) \\ Z\,\mathbb{G}\mathrm{r}_4(\mathbb{R}^{n+4}) &= \mathbb{G}\mathrm{r}(\mathbb{P}^1, \mathbb{Q}^{n+2}). \end{aligned}$$

- $\mathbb{P}^{2n+1}$ has index $2n+2$ because $L = \mathcal{O}(2)$. Recall that a Fano manifold of index $2n+2$ must be $\mathbb{P}^n$ or a quadric $\mathbb{Q}^n \subset \mathbb{P}^{n+1}$ [Kobayashi-Ochiai].
- $\mathbb{P}(T^*\mathbb{P}^{n+1})$ has $b_2 = 2$. Any twistor space Z with $b_2 > 1$ admits a contraction $Z \to \mathbb{P}^{n+1}$ with fibres tangent to D [Wiśniewski].
- $\mathbb{G}\mathrm{r}(\mathbb{P}^1, \mathbb{Q}^{n+2})$ parametrizes lines on the quadric. It has $b_2 = 1$ and $\mathrm{Pic} = \mathbb{Z} \cdot L$ for $n > 2$.

Fano contact manifolds

Let Z^{2n+1} be a contact manifold with L ample.

Theorem [LeBrun]. A Fano contact manifold Z is a QK twistor space if (and only if) Z admits a Kähler-Einstein metric.

Z is homogeneous iff it is the twistor space of a Wolf space.

The KE metric is not really used in any of the classification results. Unless $Z = \mathbb{P}^{2n+1}$,

$$H^0(Z, TZ) \xrightarrow{\cong} H^0(Z, L)$$

is the space of infinitesimal contact automorphisms, known to be non-zero if $n \leqslant 4$:

n	1	2	3	4	5	$\cdots$
$h^0(L) \geqslant$	4	6	**5**	**8**	?	

Narrowing the field

One approach is to analyse the meromorphic map

$$\mu\colon\ Z \longrightarrow \mathbb{P}(H^0(Z,L)^*),$$

try to prove that $B=\varnothing$ and construct a ladder of polarized varieties (Z^i,L) for $i<n$. This works for $n=2$. Moreover, a Fano contact manifold Z^{2n+1} is homogeneous if μ is generically finite [Beauville].

Theorem [LeBrun-S]. There are finitely many Fano contact manifolds in each dimension. If $b_2(Z)>1$ then $Z\cong\mathbb{P}(T^*\mathbb{P}^{n+1})$.

From now on, we may therefore assume that $b_2(Z)=1$.

Theorem [Buczyński-Wiśniewski-Weber]. A Fano contact manifold is necessarily homogeneous if $\mathrm{Aut}\,Z$ is reductive of rank at least $n-2$, in particular if $n=3$ or 4.

Isometry rank

Let M^{4n} denote a complete QK manifold with $s > 0$.

Theorem [Besse, Poon-S, BWW]. M^{4n} is a Wolf space if $n \leqslant 4$, i.e. one of eleven: $\mathbb{HP}^n$ or $\mathbb{G}\mathrm{r}_2(\mathbb{C}^{n+2})$ for $n = 1, 2, 3, 4$,
$\mathbb{G}\mathrm{r}_4(\mathbb{R}^{n+4})$ for $n = 3, 4$,
or $G_2/SO(4)$.

The table shows the lower bound on the isometry rank currently needed to infer that M^{4n} is symmetric:

$n \to$	3	4	5	6	7	8	9	10	11	12	13	14	15
BWW	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	10	11	12	13
Fang	5	5	6	6	7	7	8	**8**	**9**	**9**	**10**	**10**	**11**
Bielawski	4	5	6	7	8	9	10	11	12	13	14	15	16

Part two

From now on, we suppose that Z^{2n+1} is a *Fano contact manifold* with $\mathrm{Pic}\,Z = \mathbb{Z}\cdot L$, and that $\mathrm{Aut}\,G$ is reductive with a maximal torus $T = (\mathbb{C}^*)^r$.

This places one in the realm of *T-varieties* with relatively high *complexity* $2n+1-r$, but simplifications arise from the contact structure $TZ/D \cong L$ and the isomorphism $H^0(Z, L) \cong \mathfrak{g}$.

In any case, it is often useful to extend the toric case to study *downgrading*, i.e. the action of a lower-dimensional torus.

Buczyński, Wiśniewski and Weber use Białynicki-Birula's decomposition [BB 1973, Carrell] and localization formulae to analyse the action of complex tori on Z. A special role is played by fixed points that define an extreme weight or vertex.

Subsequent slides highlight a selection of their results.

Torus actions and polytopes

We suppose that Z^{2n+1} is a *Fano contact manifold*, $\operatorname{Pic} Z = \mathbb{Z} \cdot L$, and that $\operatorname{Aut} G$ is reductive with maximal $T = (\mathbb{C}^*)^r$. The latter will have fixed components $Y_i \subset Z$, *points* if we're lucky.

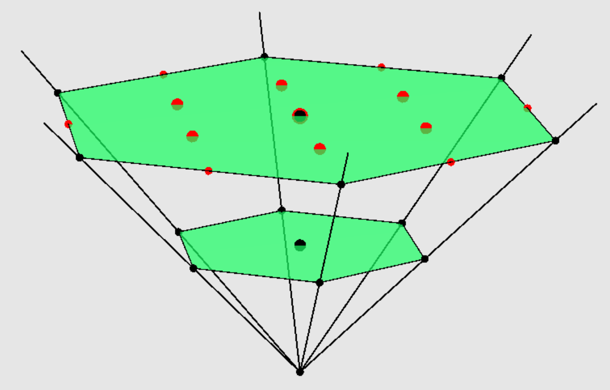
This set-up gives rise to two polytopes in $\mathfrak{t}^*$:

- $\Gamma(L)$, the convex hull of the weights of T on $H^0(Z, L)$. If L^m is very ample then $\Gamma(L^m)$ is Kostant's polytope, the projection of an adjoint orbit to $\mathbb{R}^r$.
- $\Delta(L)$, the convex hull of the weights λ of T on L_y for $y \in Y_i$. It is the image of the abelian moment mapping [Guillemin-Sternberg, Atiyah, Brion,…].

$\Delta(L^m)$ merely scales with m. One always has $\Gamma(L) \subseteq \Delta(L)$, equality if (for example) $B = \varnothing$. For a twistor space, λ is a "rotation index" on the fibre $S^2 \ni y$. The shape of $\Delta(L)$ at a vertex is determined by the compass of weights at y.

Weight cone

Consider the example of $Z\mathbb{P}^2 = SU(3)/T^2 = \mathbb{F}$:
$\Gamma(L^2)$ is generated by the 27 weights of $H^0(\mathbb{F}, L^2)$;
$\Delta(L^2)$ is generated the action of T^2 at the 6 fixed points.



A similar picture describes a hypothetical action by $SU(3)$ on a singular Fano 5-fold $Z^5 \subset Q_2 \cap C_3 \subset \mathbb{P}^7$.

Extremal fixed points

We suppose that Z^{2n+1} is a *Fano contact manifold*, $\operatorname{Pic} Z = \mathbb{Z} \cdot L$, and that $\operatorname{Aut} G$ is reductive with maximal $T = (\mathbb{C}^*)^r$.

Corollary 3.8 [BWW]. If every extremal component $Y \subset Z^H$ has dimension at most 3 then $\Gamma(L) = \Delta(L)$.

Proof. The key point is that the restriction

$$H^0(Z, L) \to H^0(Y, L\,|\,Y)$$

is surjective:

- If Y is a point it is the source of a $\mathbb{C}^*$ action and $\exists\, X \in |L|$ such that $X \cap Y = \varnothing$.
- If not, Y is Fano, $h^0(L\,|\,Y) \geqslant 2$, and any holomorphic section s of $L|Y$ extends to $\widehat{s} \in H^0(Z, L)$ with weight λ matching the action on $L\,|\,y$.

These conclusions are based on the BB-decomposition or similar techniques. □

As a consequence:

Lemma 4.7 [BWW]. If $r \geqslant n-2$ then $\Gamma(L) = \Delta(L)$ and all extremal fixed point components are isolated points.

Proof. Construct a chain of i-dimensional faces of $\Delta(L)$ that determine subtori

$$T = T^r \supsetneq T^{r-1} \supsetneq \cdots \supsetneq T^1$$

and extremal fixed-point components

$$Y^0 \subsetneq Y^1 \subsetneq \cdots \subsetneq Y^{r-1}$$

(with $Y_i \subset Z^{T_{r-i}}$) of increasing dimensions. The last is isotropic, so

$$\dim Y^{r-1} \leqslant \tfrac{1}{2} \dim D = n \;\Rightarrow\; \dim Y^0 \leqslant 3.$$

Moreover, $h^0(Y^0, L|_{Y^0})$ is the multiplicity of a root in $\mathfrak{g}$, i.e. 1, which forces Y to be a point. □

Proposition 4.8 [BWW]. If $r \geqslant n-2$ then G is simple.

Proof. We know that $\Gamma(L) = \Delta(L)$. The former arises from the roots of $\mathfrak{g}$. The latter spans the same dimension as $\Delta(L^m)$, i.e. r (if L^m is very ample), so G is semisimple.

If (for example) $G = G_1 \times G_2$ then $\Delta(L) = \mathrm{Conv}\langle R_1 \cup R_2 \rangle$ contains vertices u_1, u_2 and (unless G_1, G_2 are of type A_1, C), $u_2 - u_1$ must be a weight for the action of H on D_y^*. But then so is $-u_1$ implying that $u_1 - au_2 \in \Delta(L)$ with $a > 0$, contradiction.

A refined argument is needed for the other cases in which $\frac{1}{2}(u_1 + u_2)$ lies in the root lattice. $\square$

From now on, we assume that $r \geqslant n-2$.

Eliminating E and F

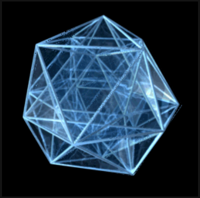
If $Z = ZM$ is a twistor space, then the maximal torus T will fix $m \in M$ and so $T \subset Sp(n)Sp(1)$. Thus $r \leqslant n+1 = \mathrm{rank}(\mathbb{HP}^n)$.

We assume $n-2 \leqslant r$, so that $\Gamma(L) = \Delta(L)$, extreme fixed points are isolated and $G = \mathrm{Aut}\, Z$ is simple.

Lemma. If G is one of F_4, E_6, E_7, E_8, then $\dim Z = 2n+1$ is at least that of the homogeneous space, contrary to assumption:

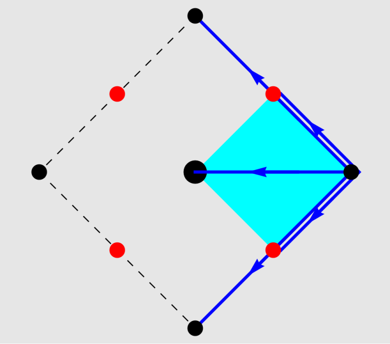
$G \rightarrow$	F_4	E_6	E_7	E_8
$n \geqslant$	7	10	16	28

Proof. The (odd) complex dimension of Z exceeds the number of edges emanating from a vertex of the root polytope $\Gamma(L)$. That suffices for E_r. For F_4 one needs to add interior pointing arrows to 6 short roots and the origin:



The case of Sp(n)=C$_n$

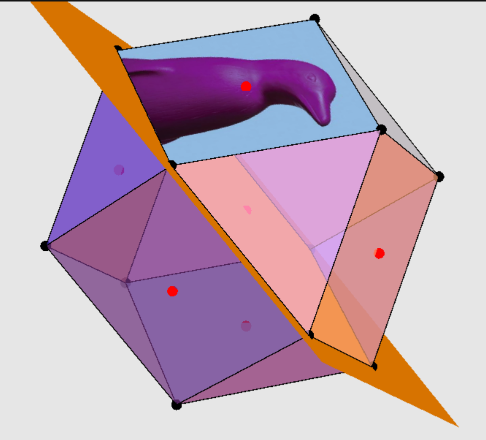
Illustrated for $n = 2$:



Midpoints of edges are also roots for this group, and each is the image of a $\mathbb{P}^2$ (rather than a $\mathbb{P}^1$). But this would imply a weight vector in T_zZ exiting the "contact domain", contradiction.

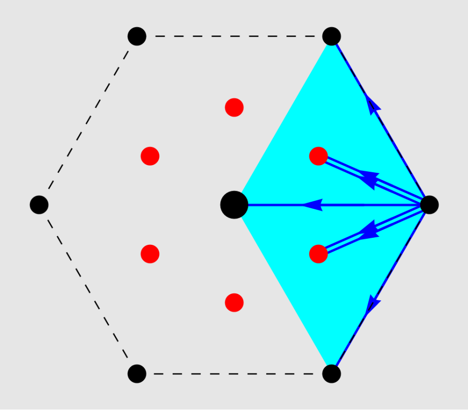
A model for $\mathbf{Z}^7$

T^3 acts on $\mathbb{R}^7 = \mathbb{R}^2 \oplus \mathbb{R}^2 \oplus \mathbb{R}^2 \oplus \mathbb{R}$ and on $Z\,\mathbb{G}\mathrm{r}_4(\mathbb{R}^7)$. In the root polytope for $SO(7) = B_3$, there are 12 fixed points, all extreme:



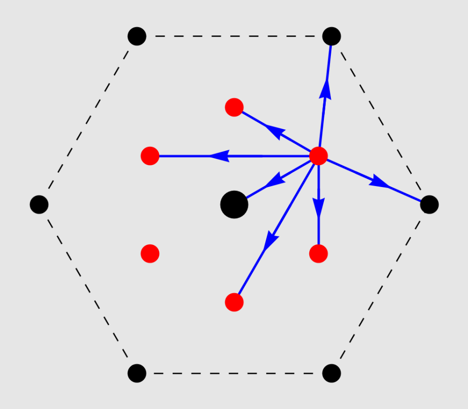
Let T^2 be a subtorus with LA orthogonal to a long diagonal.

Reduction to G_2



The resulting moment map is the diagram for both the action of G_2 on $\mathbb{G}\mathrm{r}(\mathbb{P}^1, \mathbb{Q}^5)$, *and* on a hypothetical non-homogeneous Z^7. Arrows represent weights on $T^*_y Z$ at an extreme fixed point y with target within the diamond-shaped contact domain.

The inner compass



This time the arrows represent weights on T_xZ at an inner fixed point x, and they all have multiplicity 1. One deduces that there is again a unique fixed point for each such weight, exactly as in the homogeneous case, and none at the origin.

Localization

For an arbitrary G_2 action on Z^7, the fixed points and their compass weights are all completely determined.

In general, suppose that (Z, L) is a polarized variety acted upon by $T = (\mathbb{C}^*)^r$, such that:

- T has isolated fixed points $y_1, \ldots, y_k$,
- λ_i is the weight on L_{y_i} as used to define $\Delta(L)$,
- $\nu_{i,j}$ are the weights on $T_{y_i}Z$.

Theorem [Atiyah-Bott]. Given a weight $\lambda = (a_1, \ldots, a_r)$, set $\mathbf{t}^\lambda = \prod_{k=1}^{r} t_k^{a_k}$. Then $\chi^T(Z, L) = \sum_{i=1}^{k} \frac{\mathbf{t}^{\lambda_i}}{\prod_j (1 - \mathbf{t}^{\nu_{i,j}})}$.

When (Z, L) is Fano, this equivariant Euler characteristic yields $H^0(Z, L)$ as a T-module.

$\mathbf{G}_2$ acting on $\mathbf{Z}^7$

For G_2 acting on a Fano contact 7-fold Z, the fixed points and the weights $(\lambda_i, \nu_{i,j})$ are all completely determined from the diagram within the polytope $\Gamma(L) = \Delta(L)$. Localization yields

Conclusion. The character of $H^0(Z, L) \cong \mathfrak{g}_2$ equals

$$\frac{1}{s^3t^2}\Big[1 + t + 2st + 2s^2t + s^3t + 2s^2t^2 + 3s^3t^2 \\ +2s^4t^2 + s^3t^3 + 2s^4t^3 + 2s^5t^3 + s^6t^3 + s^6t^4\Big].$$

Setting $s = t = 1$ gives $h^0(L) = 21$, contradiction!

In fact, the numerical data coincides with that for the action of $G_2 \subset SO(7)$ on $Z\,\mathbb{G}\mathrm{r}(\mathbb{P}^1, \mathbb{Q}^5)$. The case $\mathrm{Aut}\,Z^7 \cong SU(3)$ is similar, though for Z^9 more work is needed to eliminate the possibility of fixed-point components of dimension 1.

Analysis of groups of type A, B, D remains!

What next?

- Interpret the fixed point analysis by means of Morse theory directly on the QK manifold M.
- It is conceivable that $H^0(Z,L)$ is non-zero for higher dimensional Fano contact manifolds using as yet unknown inequalities on characteristic numbers, as in the hyperkähler case.
- But a simple group with $r \geqslant 3$ on Z^{11} requires $h^0(L) \geqslant 15$, which seems out of reach.
- Do all Fano contact manifolds admit a Kähler-Einstein metric?