CURVATURE AND INJECTIVITY RADIUS ESTIMATES FOR EINSTEIN 4-MANIFOLDS

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Introduction.

We will review work with Gang Tian from 2005 on possibly collapsed Einstein 4-manifolds M^4, with say $|\mathrm{Ric}_{M^4}| \leq 3$.

Key is an ϵ-regularity theorem yielding a uniform curvature bound on $B_{\frac{1}{2}}(p)$, if the L^2-norm of the curvature on $B_1(p)$ is at most ϵ, where $\epsilon > 0$ is independent of the collapsing.

Dimension 4 enters the proof only via the positivity of the Gauss-Bonnet form.

However, in [Ge,Jiang 2017] it is noted that the ϵ-regularity theorem doesn't extend to higher dimensions.

ϵ-regularity in the possibly collapsed case.

Let R denote the Riemann curvature tensor.

Theorem ([Ch,Ti 2005]) There exists $c, \epsilon > 0$ such that if M^4 is Einstein, $|\mathrm{Ric}_{M^4}| \leq 3$, $B_r(p) \subset M^4$, $r \leq 1$, and

$$\int_{B_r(p)} |R|^2 \leq \epsilon\,, \tag{1}$$

then

$$\sup_{B_{r/2}(p)} |R| \leq c r^{-2}\,. \tag{2}$$

Note that the integral in (1) is *not* normalized by volume.

That case is due to Anderson, Tian, Nakajima $\sim$ 1990.

Remarks.

Remark. $B_{r/2}(p)$ may be collapsed with bounded curvature but *not* homeomorphic to $B_1(0^n) \subset \mathbb{R}^n$.

If not, by [Ch,Fukaya,Gromov 1992], there is a neighborhood containing a ball of a definite radius which looks like a tube around an infranil manifold.

Remark. In [Ge,Jiang 2017] the ϵ-regularity theorem is extended to 4-dimensional gradient shrinking Ricci solitons. Similar results are due independently to S. Huang.

General dimensions and bounded Ricci curvature.

From now on, M^n will denote a complete riemannian manifold.

For $n \neq 4$, when we assume $|\mathrm{Ric}_{M^n}| \leq n-1$, we say so.

Our main results for $n = 4$ have direct generalizations to the case in which the Einstein condition is dropped.

In the conclusions, sectional curvature bounds are replaced by $C^{1,\alpha}$ bounds on the metric in harmonic coordinates, for all $\alpha < 1$, as well as $L^{2,p}$ curvature bounds for all $p < \infty$.

Collapse implies L_2 concentration of curvature.

Theorem. ([Ch,Ti 2005]) There exists $v > 0, \beta, c$, such that if M^4 is complete, Einstein, with $|\mathrm{Ric}_{M^4}| \leq 3$,

$$\int_{M^4} |R|^2 \leq C\,,$$

and for all p,

$$\frac{\mathrm{Vol}(B_s(p))}{s^4} \leq v\,,$$

then there exist $p_1, \ldots, p_N$, such that

$$N \leq \beta C\,, \tag{3}$$

(4)

$$\int_{M^4 \setminus (\bigcup_i B_s(p_i))} |R|^2 \leq c \left(\sum_i \frac{\mathrm{Vol}(B_s(p_i))}{s^4} + \lim_{r \to \infty} \frac{\mathrm{Vol}(B_r(p))}{r^4} \right)\,.$$

Indication of proof.

The balls $B_s(p_i)$ are those on which the curvature concentrates.

For the proof, apply Chern-Gauss-Bonnet to the region

$$M^4 \setminus (\bigcup_i B_s(p_i))\,.$$

More precisely, one must first straighten out the boundary via the Equivariant Good Chopping Theorem; see below.

The case $\mathrm{Ric}_{M^4} = \pm 3g$.

Note that if

$$\mathrm{Ric}_{M^4} = \pm 3g\,,$$

and

$$\int_{M^4} |R|^2 \leq C\,,$$

then

$$\mathrm{Vol}(M^4) \leq \frac{C}{6}\,.$$

In this case, the last term on the r.h.s. of (4) vanishes.

Margulis lemma type result in dimension 4.

Theorem. There exists $w(C) > 0$ such that if

$\mathrm{Ric}_{M^4} = \pm 3g$ and

$$\int_{M^4} |R|^2 \leq C\,. \tag{5}$$

Then for some $p \in M^4$,

$$\frac{\mathrm{Vol}(B_1(p))}{\mathrm{Vol}(M^4)} \geq w(C)\,. \tag{6}$$

Proof. If $\mathrm{Ric}_{M^4} = 3g$, a stronger result follows from Meyers' theorem and the Bishop-Gromov inequality.

If $\mathrm{Ric}_{M^4} = -3g$, and $\mathrm{diam}(M^4) \leq d(C)$, then the same argument applies.

The case $\mathrm{diam}(M^4) \geq d(C)$.

By the Bishop-Gromov inequality and $N \leq \beta C$, if $\mathrm{diam}(M^4) \geq d(C)$, there exists $\rho(C) > 0$ with

$$\mathrm{Vol}(M^4 \setminus \bigcup_i B_s(p_i)) \geq \rho(C) \cdot \mathrm{Vol}(M^4)\,. \tag{7}$$

Thus, the l.h.s. of (4) is $\geq 6\rho(C) \cdot \mathrm{Vol}(M^4)$.

By (3), the r.h.s. of (4) is $\leq c\beta C \cdot v$.

Thus,

$$6\rho(C) \cdot \mathrm{Vol}(M^4) \leq c\beta C \cdot v\,.$$

Cross multiplying gives (6).

Exponential decay of volume.

Theorem. There exist $\beta, \gamma > 0$, c, such that if M^4 is complete Einstein satisfying, $\mathrm{Ric}_{M^n} = -3$ and

$$\int_{M^4} |R|^2 \leq C\,,$$

then there exist $p_1, \ldots, p_N$, with

$$N \leq \beta C\,,$$

such that for $r \geq 5$,

$$\mathrm{Vol}(M^4 \setminus \bigcup_i B_r(p_i)) \leq c \cdot C \cdot e^{-\gamma r}\,.$$

Gravitational instantons for $n = 4$.

Before describing the proof of the ϵ-regularity theorem we mention the following recent related results.

There is a classification of complete Ricci flat hyper-Kähler 4-manifolds with faster than quadratic curvature decay given by Gao Chen and Xiuxiong Chen.

For this, see [Chen,Chen 2015 I, II], [Chen,Chen 2016 III].

Remark. [Ch,Ti 2005] should eventually lead to further general classification results for Einstein 4-manifolds.

Positivity and Chern-Gauss-Bonnet, $n = 4$.

Let ω_n denote the volume form for some local orientation.

If $n = 4$ the Chern-Gauss-Bonnet form, P_χ, satisfies

$$P_\chi = \frac{1}{8\pi^2} \cdot |R|^2 \cdot \omega_n \,. \tag{8}$$

In particular, for M^4 compact,

$$\frac{1}{8\pi^2} \int_{M^4} |R|^2 = \chi(M^4)\,. \tag{9}$$

Remark. As mentioned above, the remaining ingredients in the proof are valid in any dimension.

The curvature radius.

Definition. The *curvature radius*, $r_{|R|}(p) > 0$, is the supremal s such that

$$\sup_{B_s(p)} |R| \leq s^{-2}\,. \tag{10}$$

In particular,

$$|R(p)| \leq (r_{|R|}(p))^{-2}\,. \tag{11}$$

It follows easily that either $R \equiv 0$ and $r_{|R|} \equiv \infty$ or the function $r_{|R|}$ is 1-Lipschitz and hence, it *varies moderately on its own scale.*

Explanations concerning the curvature radius.

On the ball $B_{r_{|R|}(p)}(p)$ with rescaled metric $(r_{|R|}(p))^{-2} \cdot g$, the curvature is bounded in absolute value by 1.

Note that the rescaled ball has radius 1.

The fact that $r_{|R|}$ varies moderately on its own scale enables one to generalize global constructions from the theory of bounded curvature to ones on the scale of $r_{|R|}$.

In these constructions, structures on overlapping regions have to be perturbed slightly in order to match.

This applies to F-structures and N-structures.

Collapse with locally bounded curvature.

Definition. U is *v-collapsed with locally bounded curvature* if for all p in U,

$$\mathrm{Vol}(B_{r_{|R|}(p)}(p)) \leq v \cdot (r_{|R|}(p))^n \,. \tag{12}$$

Existence of N-structures.

By [Ch,Fu,Gv 1992], if $v \leq t(n)$, there exists $V \supset U$, which carries an *N-structure of positive rank.*

Thus, V is the disjoint union of nilmanifolds, $\mathcal{O}$, of positive dimension > 0 called *orbits*.

Consequently, V has vanishing Euler characteristic,

$$\chi(V) = 0\,.$$

Good chopping.

Let $T_r(K)$ denote the r-tubular neighborhood of $K \subset M^n$.

Put $A_{r_1,r_2}(K) := T_{r_2}(K) \setminus \overline{T_{r_1}(K)}$.

Theorem. ([Ch,Gv 1990], [Ch,Ti 2005]) There exists a submanifold with boundary, Z^n, such that

(13) $$T_{\frac{1}{3}r}(K) \subset Z^n \subset T_{\frac{2}{3}r}(K)\,,$$

(14) $$\mathrm{Vol}(\partial Z) \leq r^{-1} \cdot \mathrm{Vol}(A_{\frac{1}{3}r,\frac{2}{3}r}(K)\,,$$

(15) $$|II_{\partial Z}(z)| \leq c(n) \cdot (r^{\cdot} r_{|R|})^{-1})\,.$$

Equivariant good chopping.

Let $K \subset M^n$ be closed.

If $T_r(K)$ is $\epsilon(n)$ collapsed with locally bounded curvature, Z^n can be chosen to be a union of orbits of the associated N-structure and so,

$$\chi(Z^n) = 0\,. \tag{16}$$

Below, we use the notation,

$$A_{a,b}(K) := T_b(K) \setminus T_a(K)\,.$$

$II_{\partial Z^n}$ denotes the second fundamental form of ∂Z^n.

For the Chern-Gauss-Bonnet boundary term.

Theorem. There exists $c = c(n) < \infty$, and Z^n with

$$T_{\frac{1}{3}r}(K) \subset Z^n \subset T_{\frac{2}{4}r}(K)\,,$$

$$|II_{\partial Z^n}| \leq c \cdot (r^{-1} + (r_{|R|})^{-1})\,,$$

and for all $k_1, k_2 > 0$,

(17)
$$\int_{\partial Z^n} |II_{\partial Z}|^{k_1} \cdot |R|^{k_2} \leq \frac{c}{r} \cdot \int_{A_{\frac{1}{3}r,\frac{2}{3}r}(K)} \left(r^{-(k_1+2k_2)} + (r_{|R|})^{-(k_1+2k_2)}\right)\,.$$

The proof uses a covering argument and $\operatorname{Lip} r_{|R|} \leq 1$.

Chern-Gauss-Bonnet in the locally collapsed case.

(18)

$$\Big|\int_{\partial Z^n} TP_\chi\Big| \le c(n) r^{-1}\cdot \int_{A_{\frac{1}{3}r,\frac{2}{3}r}(K)} \left(r^{-(n-1)} + (r_{|R|})^{-(n-1)}\right)\,.$$

Since $\chi(Z^n)=0$, we get

(19)

$$\Big|\int_{\partial Z^n} P_\chi\Big| \le c(n) r^{-1}\cdot \int_{A_{\frac{1}{3}r,\frac{2}{r}(K)}} \left(r^{-(n-1)} + (r_{|R|})^{-(n-1)}\right)\,.$$

$|R|_{L^{n/2}}$ sufficiently small with respect to volume.

Let $\underline{p} \in M^n_{-1}$, simply connected curvature $\equiv -1$.

Theorem (Anderson) There exists $\tau(n)$ such that if M^n is Einstein, $|\mathrm{Ric}_{M^n}| \leq n-1$, and

$$\frac{\mathrm{Vol}(B_r(\underline{p}))}{\mathrm{Vol}(B_r(p))} \cdot \int_{B_r(p)} |R|^{\frac{n}{2}} \leq \tau(n)\,, \tag{20}$$

then

$$\sup_{B_{r/2}(p)} |R| \leq c \cdot r^{-2} \cdot \left(\frac{\mathrm{Vol}(B_r(\underline{p}))}{\mathrm{Vol}(B_r(p))} \cdot \int_{B_r(p)} |R|^{\frac{n}{2}} \right)^{\frac{2}{n}} . \tag{21}$$

Indication of proof. The proof uses Moser iteration.

The local scale $\rho(p)$.

If (20) holds for $r = 1$ put $\rho(p) := 1$.

Otherwise, define $\rho(p)$ to be the largest solution of

$$\frac{\mathrm{Vol}(B_{\rho(p)}(\underline{p}))}{\mathrm{Vol}(B_{\rho(p)}(p))} \cdot \int_{B_{\rho(p)}(p)} |R|^{\frac{n}{2}} := \tau(n)\,. \tag{22}$$

By (21), we have

$$\frac{1}{2}\rho(p) \leq r_{|R|}(p)\,. \tag{23}$$

If $\rho(p) = 1$, then

$$\sup_{B_{1/2}(p)} |R| \leq 4\,. \tag{24}$$

The local scale and the maximal function.

$\rho(p)$ is not a priori 1-Lipschitz, so it can't be used directly.

We address this issue via the maximal function.

Definition. For (X, μ) a metric measure space put

$$⨍_A |f| \, d\mu := \frac{1}{\mu(A)} \int_A |f| \, d\mu \, .$$

Define the maximal function for balls of radius $\leq r$,

$$M_f(x, r) := \sup_{s \leq r} ⨍_{B_s(x)} |f| \, d\mu \, .$$

Slightly nonstandard maximal function estimate.

Lemma. If for $x \in W$, $s \leq 4r$,

$$\mu(B_{2s}(x)) \leq 2^\kappa \mu(B_s(x))\,,$$

then for all $\alpha < 1$,

$$(25)\quad \left(⨍_W M_f(x,r)^\alpha \, d\,\mu \right)^{\frac{1}{\alpha}} \leq c(\kappa,\alpha) \cdot \frac{1}{\mu(W)} \cdot \int_{T_{6r}(W)} |f| \, d\mu \,.$$

Indication of proof. Similar to that of more standard maximal function estimates.

Bounding the local scale from below.

Since

$$\mathrm{Vol}(B_s(\underline{p})) \leq c(n)s^n \qquad (s \leq 1)\,,$$

we get

$$\rho(p)^{-1} \leq c \cdot \max(M_{|R|^{\frac{n}{2}}}(p,s)^{\frac{1}{n}}, s^{-1})\,. \tag{26}$$

Thus, with Bishop's inequality,

$$(r_{|R|}(p))^{-(n-1)} \leq c \cdot \left(s^{-(n-1)} + (M_{|R|^{\frac{n}{2}}}(p,s))^{\frac{n-1}{n}}\right) \tag{27}$$

Chern-Gauss-Bonnet boundary estimate.

By (17) and (27) for $s \le r \le 1$,

(28)
$$\Big|\int_{\partial Z^n} TP_\chi\Big| \le c(n) r^{-1} \cdot \int_{A_{\frac{1}{3}r,\frac{2}{3}r}(K)} \left(s^{-(n-1)} + (M_{|R|^{\frac{n}{2}}})^{\frac{n-1}{n}}\right) .$$

Choosing $s = \frac{r}{512}$, we get from (25), (28),

$$(\mathrm{Vol}(A_{0,r}(K))^{-1} \cdot \Big|\int_{\partial Z^n} TP_\chi\Big|$$

(29)
$$\le c(n) \left(r^{-(n-1)} + r^{-1} \left(\frac{1}{\mathrm{Vol}(A_{0,1}(K)} \int_{A_{\frac{1}{4}r,\frac{3}{4}r}(K)} |R|^{\frac{n}{2}} \right)^{\frac{n-1}{n}} \right) .$$

ϵ-regularity in the collapsed case, $n = 4$.

Recall, $P_\chi = \frac{1}{8\pi^2} \cdot |R|^2$.

Also, t-collapse with locally bounded curvature, $t = t(4)$, implies the existence of an N-structure.

So, if $T_1(E)$ is t-collapsed with locally bounded from equivariant good chopping, we have by ((16)), $\chi(Z^n) = 0$, and by (29), we get for c independent of M^4,

(30)

$$\frac{\mathrm{Vol}(E)}{\mathrm{Vol}(A_{0,1}(E))} \cdot \fint_E |R|^2 \leq c \cdot \left(1 + \left(\frac{1}{\mathrm{Vol}(A_{0,1}(E))} \int_{A_{\frac{1}{4},\frac{3}{4}}(E)} |R|^2 \right)^{\frac{3}{4}} \right)$$

$x_{i+1} \leq a_i + b_i \cdot x_{i+1}^{\alpha}, \; i = 2, \cdots .$

Similarly, short argument (using $\mathrm{Lip}\, r_{|R|} \leq 1$), shows

$$\int_{A_{2^{-(i-1)},1-2^{-(i-1)}}(E)} |R|^2$$

(31)

$$\leq c 2^{4i} \cdot \mathrm{Vol}(A_{0,1}(E)) \cdot \left(1 + \left(\frac{1}{\mathrm{Vol}(A_{0,1}(E))} \int_{A_{2^{-i},1-2^{-i}}(E)} |R|^2\right)^{\frac{3}{4}}\right)$$

The exponentially growing factor 2^{4i} arises because we are applying equivariant good chopping in the narrow region

$$A_{0,2^{-i}}(A_{2^{-(i-1)},1-2^{-(i-1)}}(E))\,.$$

Lemma on sequences.

Lemma. Let $0 \leq \alpha < 1$, for $i = 2, 3, \ldots,$, a_i, b_i, x_i be nonnegative real numbers satisfying

$$x_i \leq a_i + b_i \cdot x_{i+1}^{\alpha}\,,$$

$$\lim_{i\to\infty} x_i^{\alpha^i} = 1\,,$$

$$\max(a_i, b_i) \leq c \cdot K^i \qquad (K \geq 1)\,.$$

Then

$$x_2 \leq (2c)^{1+\alpha+\alpha^2+\cdots} \cdot K^{1+\alpha+2\alpha^2+3\alpha^3+\cdots}\,. \tag{32}$$

Proof. It follows by (upward) induction from

$$x_i^{\alpha^i} \leq 2\max(a_i^{\alpha^i}, b_i^{\alpha^i} \cdot x_{i+1}^{\alpha^{i+1}})\,.$$

The key estimate $n = 4$.

Theorem. There exist $\delta > 0$, $c > 0$, such that if (1), (2) hold and $E \subset M^4$ denotes a bounded open subset such that $T_1(E)$ is t-collapsed with

$$\int_{B_1(p)} |R|^2 < \delta \qquad (\text{for all } p \in T_1(E))\,. \tag{33}$$

Then

$$\int_E |R|^2 \leq c \cdot \mathrm{Vol}(A_{0,1}(E))\,. \tag{34}$$

Proof. Combine (31) with the lemma on sequences.

Proof of ϵ-regularity for $n = 4$.

Step 1. (Getting in range) There exists $\epsilon, c_1 > 0$ such that if $|\mathrm{Ric}_{M^4}| \leq 3$ and for $r \leq 1$,

$$\int_{B_r(p)} |R|^2 \leq \epsilon \,,$$

then for $p \in M^4_{-3}$,

$$\frac{\mathrm{Vol}(B_r(\underline{p}))}{\mathrm{Vol}(B_r(p))} \cdot \int_{B_r(p)} |R|^2 \leq c_1 \,. \tag{35}$$

However, to apply Anderson's ϵ-regularity theorem directly, we would need $c_1 \leq \epsilon$.

The decay estimate.

Step 2. It suffices to show that there exist $\gamma < 1$, $\beta > 0$, such that (35) implies

$$\text{(36)} \qquad \frac{\mathrm{Vol}(B_\beta(\underline{p}))}{\mathrm{Vol}(B_\beta(p))} \cdot \int |R|^2 \leq \gamma \cdot \frac{\mathrm{Vol}(B_r(\underline{p}))}{\mathrm{Vol}(B_r(p))} \cdot \int |R|^2$$

We argue by contradiction.

If not, after rescaling, by Anderson's ϵ-regularity theorem, we get a sequence of balls, $\{B_{1/n}(p_i)\}$ such that

$$\text{(37)} \qquad \sup_{A_{1/n.1}(p_i)} |R| \to 0\,.$$

Continued.

Also, using the volume cone implies metric cone theorem and *local bounded covering geometry*, the rescalings by a factor n of the annuli $\{A_{\frac{1}{n},\frac{2}{n}}(p_i)\}$ have finite normal coverings converging to a flat noncollapsed annulus.

Continued.

By a refinement of equivariant good chopping, we get *almost conical* Z^4 with $B_{\frac{1}{n}}(p_i) \subset Z_i^4 \subset B_{\frac{2}{n}}(p_i)$, such that

$$\chi(Z_i^4) = 0\,, \tag{38}$$

$$\int_{\partial Z_i^4} TP_\chi > 0\,, \tag{39}$$

$$\int_{Z_i^4} P_\chi > 0\,. \tag{40}$$

Proof of ϵ-regularity concluded.

By (38)–(40) we contradict Chern-Gauss-Bonnet.

This completes the proof of the ϵ-regularity theorem. $\square$

Note that above, we can and do assume $\not\equiv 0$.

Also, the *refined* equivariant good chopping leading to the positivity in (39) relies on local bounded covering geometry and convergence to the flat case.

Note that terms in both (39) and (40) are strictly positive, though they may be arbitrarily small.