Pseudo-Convexity for the Special Lagrangian Potential Equation

Blaine Lawson

ABSTRACT

This is an equation which Reese Harvey and I found years ago, when we were first working on calibrations. For a function $u : \Omega \to \mathbf{R}$ on a domain $\Omega \subset \mathbf{R}^n$ it is given by

$$\mathrm{tr}\{\arctan(D^2 u)\} \;=\; \theta$$

for a constant $\theta \in (-n\frac{\pi}{2}, n\frac{\pi}{2})$. For C^2 solutions the graph of Du in $\Omega \times \mathbf{R}^n$ is a special Lagrangian submanfold. Much has been understood about the Dirichlet problem for this equation, but the existence result relies on explicitly computing the associated boundary conditions (or, otherwise said, computing the pseudo-convexity for the associated potential theory). This will be done in the talk, and the answer is interesting. The result carries over to many related equations – for example, those obtained by taking $\sum_k \arctan \lambda_k^g = \theta$ where $g : \mathrm{Sym}^2(\mathbf{R}^n) \to \mathbf{R}$ is a Gårding-Dirichlet polynomial which is hyperbolic with respect to the identity. A particular example of this is the deformed Hermitian-Yang-Mills equation which appears in mirror symmetry. Another example is $\sum_j \arctan \kappa_j = \theta$ where $\kappa_1, ..., \kappa_n$ are the principal curvatures of the graph of u in $\Omega \times \mathbf{R}$.

We also discuss the inhomogeneous Dirichlet Problem

$$\mathrm{tr}\{\arctan(D_x^2 u)\} = \psi(x)$$

where $\psi : \overline{\Omega} \to (-n\frac{\pi}{2}, n\frac{\pi}{2})$. This equation has the feature that the pull-back of ψ to the Lagrangian submanifold $L \equiv \mathrm{graph}(Du)$ is the phase function θ of the tangent spaces of L. On L it satisfies the equation $\nabla\psi = -JH$ where H is the mean curvature vector field of L.