Gravitational instantons and del Pezzo surfaces

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This lecture is based on arXiv:2111.09287, which is written together with H. Hein, J. Viaclovsky, R. Zhang.

It can be viewed as a continuation of the talk by Ruobing Zhang at the annual meeting in 2021, or the talk by the speaker at the annual meeting in 2018. It is also related to several lectures given at other collaboration meetings by HSVZ.

A Calabi-Yau metric is a Riemannian metric with holonomy contained in $SU(n)$.

This condition is equivalent to the data of a triple (X, ω, Ω), where

- X is a complex manifold;
- ω is a Kähler metric;
- Ω is a holomorphic volume form,

that satisfies the equation

$$\omega^n = C\Omega \wedge \bar{\Omega}.$$

Yau's proof of the Calabi conjecture gives an algebro-geometric description of compact Calabi-Yau manifolds.

In real dimension 4, a Calabi-Yau metric is the same as a hyperkähler metric since $SU(2) = Sp(1)$.

Writing $\omega_1 = \omega$, $\Omega = \omega_2 + \sqrt{-1}\omega_3$, then the Calabi-Yau condition is equivalent to the existence of a triple of closed 2-forms $(\omega_1, \omega_2, \omega_3)$ satisfying

$$\frac{1}{2}\omega_i \wedge \omega_j = \delta_{ij} \cdot \mathrm{dVol}_g.$$

There is an $SO(3)$ symmetry on $(\omega_1, \omega_2, \omega_3)$. So we obtain an S^2 family of compatible complex structures.

The only closed 4-manifolds supporting hyperkähler metrics are T^4 and the K3 manifold.

Hyperkähler metrics on T^4 are flat.

Hyperkähler metrics on the K3 are never flat.

The moduli space $\mathcal{M}$ of all hyperkähler metrics on the K3 are described by the Torelli theorem in terms of an open set in a locally symmetric space $\mathcal{D} = \Gamma \setminus O(3, 19)/(O(3) \times O(19))$.

In the non-compact world the situation is richer.

A gravitational instanton (for our purpose) is by definition a complete non-flat hyperkähler 4-manifold (X, g) with $\int_X |Rm(g)|^2 < \infty$.

Gravitational instantons can exhibit diverse geometric structures at infinity. There are also a variety of constructions in both mathematics and physics.

They provide models for singularity formation of 4 dimensional hyperkähler metrics (and of higher dimensional metrics with special holonomy).

Question: Classify all gravitational instantons.

It can be divided into 2 subquestions

(1). Classify the possible asymptotics at infinity

(2). Classify gravitational instantons with a given asymptotic model at infinity.

(1) is the central issue. It create models at infinity hence allows us to make use of PDE analysis on models. For example, the conical and cylindrical asymptotics have been well-studied in the literature.

For (2) one approach is to make connections with complex/algebraic geometry and make use of the well-studied theory of algebraic surfaces.

Regarding Question (1) we have

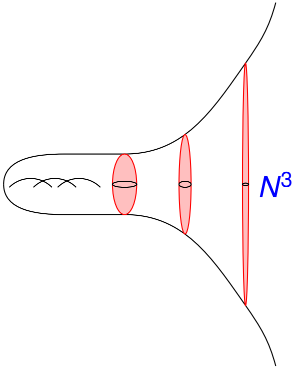
Theorem (S.-Zhang 2021)

A gravitational instanton must be asymptotic to one of the 6 families of model ends: ALE, ALF, ALG/ALG, ALH/ALH*.*

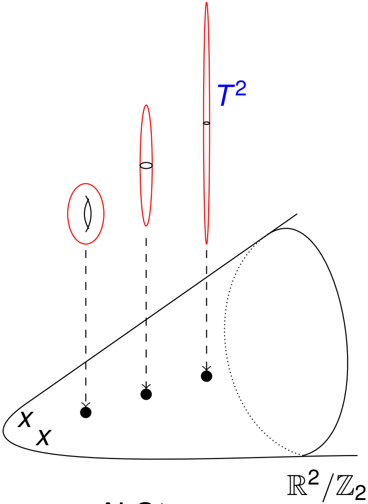
Remark:

1. For AL$\sharp^{(*)}$ model the dimension of the asymptotic cone at infinity is $4-(\sharp - E)$.
2. Assuming the technical assumption $|Rm| = O(r^{-2-\epsilon})$, only ALE, ALF, ALG, ALH appear (Chen-Chen 2015, building on earlier work of Minerbe). The models are locally flat product $T^k \times \mathbb{R}^{4-k}$.

ALG* and ALH* are interesting in that they have inhomogeneous geometry at infinity.



ALH* space



ALG* space

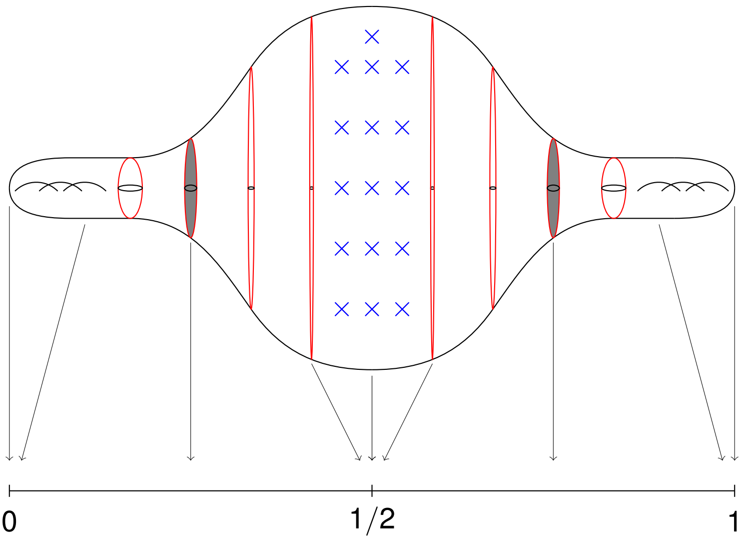
There are many recent papers on Question (2). The interested audience can easily find those by typing “gravitational instantons" in the arXiv search. Please be cautious: for example, some papers use a more general definition. In this talk we will focus on the classification of ALH^* gravitational instantons by HSVZ 2021.

One motivation came from the work HSVZ 2018, where it is demonstrated that these metrics arise from collapsing of hyperkähler metrics on K3 to an interval.

Another motivation is that the classification theorem is a special case of a result that holds for a class of Calabi-Yau metrics in all dimensions. The latter has their significance in the study of “small" complex structure degenerations of Calabi-Yau manifolds (S.-Zhang 2019).

I plan to explain the ideas involved in our proof with sufficient details, with the hope that it will have applications in other related contexts.

An example of K3 collapsing to the interval (from HSVZ 2018)



We first define the Calabi model end in general dimensions.

Given a compact Calabi-Yau manifold (D, ω_D, Ω_D) in complex dimension $n-1$. Suppose $[\omega_D] = c_1(L)$ for a holomorphic line bundle L. Let h be a hermitian metric on L with curvature form ω_D.

Denote by

$$\mathcal{C} = \{0 < |\xi|_h < 1\}$$

the open subset in the total space of L. It admits a natural holomorphic volume form $\Omega_{\mathcal{C}} = \Omega_D \wedge \xi^{-1} d\xi$.

Calabi ansatz provides an S^1 invariant Calabi-Yau metric on $\mathcal{C}$, explicitly given by

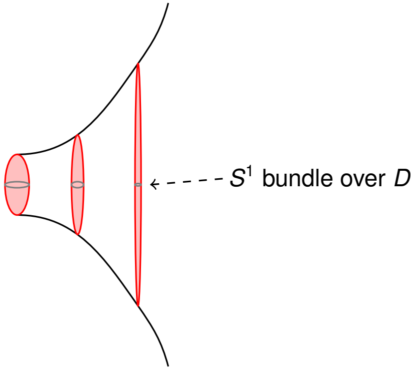
$$\omega_{\mathcal{C}} = \sqrt{-1}\partial\bar{\partial}(-\log|\xi|_h^2)^{\frac{n+1}{n}}$$

It is complete as $|\xi|_h \to 0$. We call such $(\mathcal{C}, \omega_{\mathcal{C}}, \Omega_{\mathcal{C}})$ a *Calabi model end*.

It has volume growth $\mathrm{Vol}(B(p,r)) \sim r^{\frac{2n}{n+1}}$.

As $r \to \infty$, S^1 fiber has length $\sim r^{-\frac{n-1}{n+1}}$, the base D has uniform size $\sim r^{\frac{1}{n+1}}$.

Asymptotic cone is a ray.



Remark

When $n = 2$:

- ω_D *is flat, so it has finite* $\int |Rm|^2$*, and is called an* ALH^* *model end.*
- *it can be alternatively described in terms of the Gibbons-Hawking ansatz.*

It is convenient to introduce a coordinate $z = (-\log|\xi|_h^2)^{\frac{1}{2}}$. We have $r \sim z^{\frac{n+1}{n}}$.

Harmonic functions on $\mathcal{C}$ have two different classes of growth rates at infinity:

- (zero mode) S^1 invariant harmonic functions have rates $e^{\lambda z} z^t$.
  Roughly speaking, these reduce to solutions to elliptic equations on the quotient space $\mathcal{C}/S^1$, which is (almost) cylindrical.

- (non-zero mode) Harmonic functions of weight $k \neq 0$ (with respect to the S^1 action) have rates $e^{\frac{k}{2}z^2} z^t$.
  Example: any $s \in H^0(D, L^k)$ gives a holomorphic function f_s on $\mathcal{C}$ by

$$f_s(x, \xi) = s(x)/\xi^{\otimes k}.$$

Remark: There is ongoing work by Mazzeo-X. Zhu developing sharp Fredholm theory on these spaces.

Definition

We say a complete Calabi-Yau manifold (X, ω, Ω) is *asymptotically Calabi* if the end of X can be smoothly identified with a Calabi model end $\mathcal{C}$ in such a way that the difference between (ω, Ω) and $(\omega_{\mathcal{C}}, \Omega_{\mathcal{C}})$ decays at the rate $e^{-\delta z}$ for some $\delta > 0$.

Remark

1. *When $n = 2$, ALH* gravitational instantons are asympotically Calabi.*

2. *The decay rate was designed in HSVZ 2018 and SZ 2019 so that certain Liouville theorem for harmonic functions holds. For our classification result a sufficiently large polynomial decay rate z^{-t} also works. A posteriori the decay rate of the Kähler form is improved to $e^{-\delta z}$, and the decay rate of the holomorphic volume forms can be even improved to $e^{-(\frac{1}{2}-\epsilon)z^2}$.*

Examples can be produced via the Tian-Yau construction from algebro-geometric input. We explain here a slight generalization.

Let M be a projective manifold with a meromorphic volume form Ω having exactly a simple pole along a smooth divisor D with ample normal bundle (In particular, M is weakly Fano).

The residue of Ω gives a holomorphic volume form Ω_D on D. Yau's theorem implies the existence of hermitian metric h on $K_M^{-1}|_D$ whose curvature form is a Calabi-Yau metric ω_D on D. Then extend h to a global smooth hermitian metric on K_M^{-1} which is positive in a neighborhood of D.

Denote $X = M \setminus D$. Then Ω is a holomorphic volume form on X.

We can define a class of model metrics near the infinity on X: for $\tau > 0$, $A \in \mathbb{R}$,

$$\omega_{\tau,A} = \tau \cdot \sqrt{-1}\partial\bar{\partial}(-\log|S|^2_{he^{-A}})^{\frac{n+1}{n}}.$$

The function $\tilde{z} = (-\log|S|^2_{he^{-A}})^{\frac{1}{2}}$ is uniformly comparable to z.

Denote by $H^{2,+}_c(X)$ the subset of $Im(H^2_c(X) \to H^2(X))$ consisting of those β such that $\int_Y \beta^p > 0$ for any non-trivial compact analytic set Y of dimension $p > 0$.

Theorem (Tian-Yau 1990, Hein 2010)

For any $\beta \in H^{2,+}_c(X)$ and $\tau > 0$, there exists a unique $A = A(\tau)$ such that there is a complete Calabi-Yau metric $\omega = \omega_{\beta,\tau}$ with

- $\omega^n = \tau^n(\sqrt{-1})^{n^2}\Omega \wedge \bar{\Omega}$;
- $[\omega] = [\beta] \in H^2_c(X)$;
- $|\omega - \omega_{\tau,A}|_{\omega_{\tau,A}} = O(e^{-\delta\tilde{z}})$ *for some* $\delta > 0$.

In particular, ω is asymptotically Calabi.

Remark

From the construction of these Tian-Yau metrics, we know

the holomorphic volume form Ω *(hence the complex structure) decays at a faster rate* $e^{-(\frac{1}{2}-\epsilon)z^2}$*;*

the Kähler form ω *(hence the Riemannian metric) decays at a rate* $e^{-\delta z}$*.*

The reason is that Ω *is explicitly given in terms of holomorphic coordinates, whereas* ω *comes from solving the non-linear PDE, which in turn is governed by the behavior of harmonic functions.*

The main result is

Theorem (HSVZ 2021)

Any complete Calabi-Yau metric which is asymptotically Calabi is given by the Tian-Yau construction.

Corollary (HSVZ 2021)

Any ALH^* gravitational instanton is given by the Tian-Yau construction on a weak del Pezzo surface.

In the rest of the talk we discuss the proof.

Let (X, ω, Ω) be the metric we want to classify.

We will compactify X to a projective manifold with controlled asymptotics on the Kähler form, and then prove a uniqueness result to the Monge-Ampère equation.

We smoothly identify the end of X with $\mathcal{C}$, then there is a compactification to a smooth manifold $\overline{X}$ by adding the zero section $\mathbf{0} \simeq D$ in L.

The first try is to directly prove the complex structure extends smoothly across D using a Newlander-Nirenberg theorem. This has been carried out in other settings, for example by Haskins-Hein-Nördstrom in the asymptotically cylindrical case. We did not succeed this way.

Second try is to construct (many) holomorphic functions on X. The standard strategy is to take a holomorphic function f from $\mathcal{C}$. Viewed on X, it has the property that $\|\bar{\partial} f\| \ll \|f\|$. Then do a small perturbation $\tilde{f} = f + u$ so that $\bar{\partial}\tilde{f} = 0$.

For this one needs to solve $\bar{\partial} u = -\bar{\partial} f$ with estimates $\|u\| \leq C\|\bar{\partial} f\|$. Often one can solve the easier equation $\Delta u = -\Delta f$ using some generalized Hodge theory, so that $\zeta = \bar{\partial} u$ satisfies $\bar{\partial}^* \zeta = \bar{\partial} \zeta = 0$. Then one needs certain Liouville theorem to conclude that $\zeta = 0$. We did not succeed this way either.

Both failures are related to the fact that the decay rate $O(e^{-\delta z})$ can not beat the fast growth rate $O(e^{\frac{k}{2} z^2})$ of holomorphic functions on the Calabi model $\mathcal{C}$.

Instead, we solve the original $\bar{\partial}$ equation directly, using the Hörmander technique in several complex variables.

Theorem (Hörmander)

Let φ be a smooth function on X with $\sqrt{-1}\partial\bar{\partial}\varphi \geqslant \Upsilon\omega$ for some non-negative continuous function Υ. For any $(0,1)$ form η with $\bar{\partial}\eta = 0$ and $\int \Upsilon^{-1}|\eta|^2_\omega e^{-\varphi}\omega^n < \infty$, there exists a unique function u on X satisfying

- $\bar{\partial}u = \eta$;
- $$\int |u|^2 e^{-\varphi}\omega^n \leqslant \int \Upsilon^{-1}|\eta|^2_\omega e^{-\varphi}\omega^n;$$
- $\int u\bar{w}e^{-\varphi}\omega^n = 0$ *for all holomorphic functions w with* $\int |w|^2 e^{-\varphi}\omega^n < \infty$.

Remark

- *This is a robust technique which does not require very precise information of φ at infinity.*
- *One can get bounds on the derivatives of u in terms of bounds on the derivatives of η and local elliptic estimates for the Laplacian. In our setting the latter holds with at worst polynomial error in z.*

We need to construct a function φ in our setting. To obtain the best outcome we need to make φ and Υ^{-1} both grow as slowly as possible. It involves some experiment.

A typical holomorphic function f on $\mathcal{C}$ grow at the order $O(e^{\frac{k}{2}z^2})$. So $\eta = \bar{\partial} f$ grow at the order $O(e^{\frac{k}{2}z^2 - \delta_0 z})$ for some $\delta_0 > 0$. To ensure L^2 integrability we may want to take $\varphi = \frac{k}{2}z^2$.

Notice $\sqrt{-1}\partial\bar{\partial}z^2 = \pi^*\omega_D \geq 0$, where $\pi : \mathcal{C} \subset L \to D$ is the natural projection map. To get strict positivity we take $\varphi = \frac{k}{2}z^2 - \delta z$ for $\delta > 0$ small. The corresponding Υ decays at a rate z^{-t} for some t. By composing with a smooth increasing convex function we may assume φ is defined globally with $\sqrt{-1}\partial\bar{\partial}\varphi \geq 0$.

Recall that any holomorphic section s of L^k gives a holomorphic function f_s on $\mathcal{C}$.

Applying the Hörmander construction we get an injective linear map

$$\mathcal{L} : \bigoplus_{k \geq 0} H^0(D, L^k) \to H^0(X, \mathcal{O}_X)$$

such that

$$\mathcal{L}(s) - f_s = O(e^{\frac{k}{2} z^2 - \delta_k z}).$$

Fix $k \gg 1$ such that L^m is very ample for all $m \geq k$. We have a natural holomorphic embedding

$$F : \mathcal{C} \to \mathbf{P} \equiv \mathbb{P}\big(H^0(D, L^k)^* \bigoplus (H^0(D, L^{k+1})^*\big);$$

$$q \mapsto (ev_{q,k}, ev_{q,k+1}),$$

where $ev_{q,k}(s) = f_s(q)$ is the evaluation map.

This embedding is indeed defined on the total space of L. The zero section $\mathbf{0}_L$ is mapped to $\mathbf{D}$ in the linear subspace $\mathbb{P}(H^0(D, L^{k+1})^*)$.

Accordingly we can define the map

$$G : X \to \mathbf{P}$$

$$q \mapsto (\widetilde{ev}_{q,k}, \widetilde{ev}_{q,k+1}),$$

where $\widetilde{ev}_{q,k}(s) = \mathcal{L}(s)(q)$.

One can check that G is a holomorphic embedding on $X \setminus K$ for some compact K. Furthemore it extends to a topological embedding of $\overline{X} \setminus K$ into **P**, which maps $D = \overline{X} \setminus X$ homeomorphically onto $F(\mathbf{0}_L)$.

By the Remmert-Stein extension theorem we know the image Z of G is a complex-analytic set in a neighborhood of **D**. One then show that the normalization Z' is smooth near **D**.

Gluing X and Z' give a compactification of X as a smooth complex manifold. We still denote this by $\overline{X}$.

One can check the following

- Ω has a simple pole along **D**.
- The normal bundle $N_{\mathbf{D}}$ is isomorphic to L. The hermitian metric h on L is naturally viewed as a hermitian metric on $N_{\mathbf{D}}$.
- $\omega - \sqrt{-1}\partial\bar{\partial}(-\log|S_D|_h^2)^{\frac{n+1}{n}} = O(e^{-\delta z})$.
- $\overline{X}$ is Moishezon.

With some extra work one show that $\overline{X}$ is Kähler, hence is projective and is furthermore weak Fano.

By Tian-Yau construction, there is a $\omega' = \omega_{\beta,\tau}$ with $\omega - \omega' = O(e^{-\delta\tilde{z}})$ and $[\omega] = [\omega']$. Then it is relatively standard exercise (following Hein's thesis) to conclude $\omega = \omega'$.