Constructing compact 8-manifolds with holonomy Spin(7)

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J. Diff. Geom. 53 (1999), 89–130; and
'Compact Manifolds with Special Holonomy', OUP, 2000.

These slides available at
`http://people.maths.ox.ac.uk/~joyce/`.

Plan of talk:

1. The holonomy group Spin(7)
2. Spin(7)-manifolds from resolutions of T^8/Γ
3. Spin(7)-manifolds from Calabi–Yau 4-orbifolds
4. Open problems in Spin(7) geometry

Apology

This talk contains no new work since 2000, except open problems.

1. The holonomy group Spin(7)

The holonomy group $\mathrm{Spin}(7)$ in 8 dimensions is one of the exceptional cases $G_2, \mathrm{Spin}(7)$ in Berger's classification. The action of $\mathrm{Spin}(7)$ on $\mathbb{R}^8$ preserves the Euclidean metric $g_0 = \mathrm{d}x_1^2 + \cdots + \mathrm{d}x_8^2$, the orientation, and the 4-form

$$\begin{aligned}\Omega_0 = {} & \mathrm{d}\mathbf{x}_{1234} + \mathrm{d}\mathbf{x}_{1256} + \mathrm{d}\mathbf{x}_{1278} + \mathrm{d}\mathbf{x}_{1357} - \mathrm{d}\mathbf{x}_{1368} - \mathrm{d}\mathbf{x}_{1458} - \mathrm{d}\mathbf{x}_{1467} \\ & - \mathrm{d}\mathbf{x}_{2358} - \mathrm{d}\mathbf{x}_{2367} - \mathrm{d}\mathbf{x}_{2457} + \mathrm{d}\mathbf{x}_{2468} + \mathrm{d}\mathbf{x}_{3456} + \mathrm{d}\mathbf{x}_{3478} + \mathrm{d}\mathbf{x}_{5678},\end{aligned}$$

where $\mathrm{d}\mathbf{x}_{ijkl} = \mathrm{d}x_i \wedge \mathrm{d}x_j \wedge \mathrm{d}x_k \wedge \mathrm{d}x_l$. If (X, g) is a Riemannian 8-manifold with holonomy $\mathrm{Spin}(7)$ then X has a natural 4-form Ω with $\nabla\Omega = 0$ isomorphic to Ω_0 at each $x \in X$, and g is Ricci-flat. We call a pair (Ω, g) a $\mathrm{Spin}(7)$-*structure* on X if at each $x \in X$ there is an isomorphism $T_xX \cong \mathbb{R}^8$ identifying $(\Omega|_x, g|_x)$ with (Ω_0, g_0). We call (Ω, g) *torsion-free* if $\nabla\Omega = 0$ for ∇ the Levi-Civita connection of g, or equivalently if $\mathrm{d}\Omega = 0$ (though this is apparently weaker). Then $\mathrm{Hol}(g) \subseteq \mathrm{Spin}(7)$.

One difference with G_2 is that $\mathrm{Spin}(7)$-forms Ω are not generic. Call a 4-form Ω on X *admissible* if it is pointwise isomorphic to Ω_0, and write $\mathcal{A}X \subset \Lambda^4T^*X$ for the bundle of admissible forms. Then $\mathcal{A}X \to X$ has fibre $\mathrm{GL}(8,\mathbb{R})/\mathrm{Spin}(7)$ with dimension $64 - 21 = 43$, but $\Lambda^4T^*X \to X$ has fibre of dimension $\binom{8}{4} = 70$, so $\mathcal{A}X$ has codimension $70 - 43 = 27$ in Λ^4T^*X.
So to construct holonomy $\mathrm{Spin}(7)$ metrics we need to find sections Ω of the nonlinear subbundle $\mathcal{A}X \subset \Lambda^4T^*M$ satisfying $\mathrm{d}\Omega = 0$, which is superficially a more complicated problem than the G_2 case.
Let (X, Ω, g) be a $\mathrm{Spin}(7)$-manifold. Then we have natural decompositions of exterior forms

$$\begin{aligned} &\Lambda^1T^*X = \Lambda^1_8, && \Lambda^2T^*X = \Lambda^2_7 \oplus \Lambda^2_{21}, && \Lambda^3T^*X = \Lambda^3_8 \oplus \Lambda^3_{48}, \\ &\Lambda^4T^*X = \Lambda^4_+ \oplus \Lambda^4_-, && \Lambda^4_+ = \Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^2_{27}, && \Lambda^4_- = \Lambda^4_{35}, \\ &\Lambda^5T^*X = \Lambda^5_8 \oplus \Lambda^5_{48}, && \Lambda^6T^*X = \Lambda^6_7 \oplus \Lambda^6_{21}, && \Lambda^7T^*X = \Lambda^7_8, \end{aligned}$$

where Λ^k_l is a vector bundle of rank l.

We have inclusions of holonomy groups

$$\mathrm{SU}(2)\times\mathrm{SU}(2)\subset\mathrm{Sp}(2)\subset\mathrm{SU}(4)\subset\mathrm{Spin}(7),\ \ \mathrm{SU}(3)\subset G_2\subset\mathrm{Spin}(7).$$

Thus Calabi–Yau 4-folds and hyperkähler 8-manifolds are torsion-free $\mathrm{Spin}(7)$-manifolds. If Y is a G_2-manifold, and Z is a Calabi–Yau 3-fold, then $Y\times\mathcal{S}^1$, $Y\times\mathbb{R}$, $Z\times T^2$, $Z\times\mathbb{R}^2$ are torsion-free $\mathrm{Spin}(7)$-manifolds.
Suppose (X,Ω,g) is a compact, torsion-free $\mathrm{Spin}(7)$-manifold. Then the $\hat{A}$-*genus* of X, a characteristic class of X, satisfies

$$24\hat{A}(X)=-1+b^1-b^2+b^3+b^4_+-2b^4_-.$$

If X is simply-connected then there are four cases: (i) $\mathrm{Hol}(g)=\mathrm{Spin}(7)$ and $\hat{A}(X)=1$; (ii) $\mathrm{Hol}(g)=\mathrm{SU}(4)$ and $\hat{A}(X)=2$; (iii) $\mathrm{Hol}(g)=\mathrm{Sp}(2)$ and $\hat{A}(X)=3$; and (iv) $\mathrm{Hol}(g)=\mathrm{SU}(2)\times\mathrm{SU}(2)$ and $\hat{A}(X)=4$. Conversely, if g has any of these holonomy groups then X is simply-connected. Thus, $\mathrm{Hol}(g)=\mathrm{Spin}(7)$ if and only if $\pi_1(X)=\{1\}$ and $\hat{A}(X)=1$.

The moduli space of torsion-free $\mathrm{Spin}(7)$-structures

Theorem 1 (Bryant–Harvey?; Joyce)

Let X be a compact, oriented 8-manifold. Then the moduli space $\mathcal{M}$ of oriented, torsion-free $\mathrm{Spin}(7)$-structures (Ω,g) on X, modulo diffeomorphisms isotopic to the identity, is a smooth manifold of dimension $\dim\mathcal{M}=\hat{A}(X)+b^1(X)+b^4_-(X)$, so that $\dim\mathcal{M}=1+b^4_-(X)$ if X admits metrics of holonomy $\mathrm{Spin}(7)$. The map $\iota:\mathcal{M}\to H^4(X;\mathbb{R})$ taking $\iota:[(\Omega,g)]\mapsto[\Omega]$ is an immersion.

It seems to be difficult to compute the submanifold $\iota(\mathcal{M})\subset H^4(X;\mathbb{R})$ (the period domain) in examples. For the analogous problem for Calabi–Yau manifolds, there is a deep theory which allows you to do this.

In 1987, Robert Bryant proved the local existence of many metrics with holonomy $\mathrm{Spin}(7)$, using EDS. In 1989, Robert Bryant and Simon Salamon produced explicit, complete examples of holonomy $\mathrm{Spin}(7)$ manifolds. In 1996 and 2000 I gave two constructions of compact 8-manifolds with holonomy $\mathrm{Spin}(7)$, which I will explain today. Alexei Kovalev has a third construction (this conference).
Let (X,Ω,g) be a torsion-free $\mathrm{Spin}(7)$-manifold. A *Cayley 4-fold* C in X is a 4-submanifold $C\subset X$ calibrated w.r.t. Ω. They are minimal submanifolds. McLean (1998) studied the deformation theory of compact Cayley 4-folds C, and found it is elliptic, and obstructed, and the Cayley moduli space $\mathcal{M}_C$ has virtual dimension $b^0(C)-b^1(C)+b^2_+(C)-[C]\cdot[C]$. I gave examples of compact Cayley 4-folds in compact 8-manifolds with holonomy $\mathrm{Spin}(7)$, as fixed points of involutions.

If (X,Ω,g) is a $\mathrm{Spin}(7)$-manifold, there is a natural splitting $\Lambda^2T^*X=\Lambda^2_7\oplus\Lambda^2_{21}$ into vector subbundles of rank 7,21. Let $P\rightarrow X$ be a principal bundle, and A a connection on P with curvature F_A. We call (P,A) a $\mathrm{Spin}(7)$-*instanton* if $\pi^2_7(F_A)=0$, where π^2_7 is the projection to $\mathrm{ad}(P)\otimes\Lambda^2_7\subset\mathrm{ad}(P)\otimes\Lambda^2T^*X$.
$\mathrm{Spin}(7)$-instantons have elliptic deformation theory, and so form well-behaved moduli spaces. These should fit into the Donaldson–Thomas / Donaldson–Segal programme for defining invariants, though I'm not sure if there are particular conjectures.
Borisov–Joyce use Derived Algebraic Geometry to prove existence of deformation-invariant Donaldson–Thomas style invariants 'counting' stable coherent sheaves on a Calabi–Yau 4-fold. Morally speaking, the B–J construction treats moduli spaces of Hermitian–Yang–Mills connections as moduli spaces of $\mathrm{Spin}(7)$-instantons in order to define the virtual cycle. So this is some evidence that there may be interesting deformation-invariant counting information in $\mathrm{Spin}(7)$-instanton moduli spaces.

2. Spin(7)-manifolds from resolutions of T^8/Γ

My first construction of compact 8-manifolds with holonomy $\mathrm{Spin}(7)$ (1996, 2000) worked, as in the G_2 case, by starting with a torus $T^8 = \mathbb{R}^8/\Lambda$ with flat $\mathrm{Spin}(7)$-structure (Ω_0, g_0), choosing a finite group Γ of automorphisms of (T^8, Ω_0, g_0), taking the quotient orbifold T^8/Γ, and resolving the singularities by gluing in ALE and Quasi-ALE Calabi–Yau 4-folds.

This is more difficult than the G_2 case for several reasons:

- All orbifold singularities of C–Y 3-folds admit crepant resolutions. But many C–Y 4-fold orbifold singularities do not, $\mathbb{C}^4/\{\pm 1\}$ for instance. So, it is a lot harder to find orbifolds T^8/Γ all of whose singularities can be resolved.
- In the G_2 case, we can work with orbifolds T^7/Γ whose only singularities look like $T^3 \times \mathbb{C}^2/G$ or $\mathcal{S}^1 \times \mathbb{C}^3/G$. In the $\mathrm{Spin}(7)$ case, because of the $\hat{A}$-genus, it is necessary that T^8/Γ has orbifold strata intersecting in points.

- As $\mathrm{Spin}(7)$ forms are not generic, we need a different method of proof to deform small torsion to zero torsion.
- The general method of proof for both G_2 and $\mathrm{Spin}(7)$ needs

$$(\text{codimension of singularities}) > \tfrac{1}{2}(\text{total dimension}). \quad (*)$$

For G_2 this is $4 > \frac{1}{2}\cdot 7$. For $\mathrm{Spin}(7)$ it is $4 \not> \frac{1}{2}\cdot 8$. As equality holds in the $\mathrm{Spin}(7)$ case, we have to work harder to define a $\mathrm{Spin}(7)$-structure with small enough torsion to deform to torsion-free.

Here $(*)$ holds as if c is the codimension of singularities and d the total dimension, and $G = G_2$ or $\mathrm{Spin}(7)$, then roughly we expect to construct a G-structure Ω with torsion τ satisfying $\|\tau\|_{L^2} = O(t^c)$, where t is the injectivity radius. Then we solve $L_\Omega(\delta\Omega) = -\tau$, where $\delta\Omega$ is an infinitesimal change in Ω and L_Ω is the linearized torsion. We find that $\|\delta\Omega\|_{C^0} \leqslant Ct^{-d/2}\cdot\|\tau\|_{L^2} = O(t^{c-d/2})$. So if $c > d/2$ then taking t small is enough. But if $c = d/2$ then we need to add in extra correction terms to make $\|\tau\|_{L^2} = o(t^c)$.

An example

Let $T^8 = \mathbb{R}^8/\mathbb{Z}^8$ with standard $\mathrm{Spin}(7)$-structure (Ω_0, g_0). Let $\Gamma = \langle \alpha, \beta, \gamma, \delta \rangle \cong \mathbb{Z}_2^4$, where $\alpha, \beta, \gamma, \delta$ are involutions acting by

$$\alpha(x_1, \ldots, x_8) = (-x_1, -x_2, -x_3, -x_4, x_5, x_6, x_7, x_8),$$
$$\beta(x_1, \ldots, x_8) = (x_1, x_2, x_3, x_4, -x_5, -x_6, -x_7, -x_8),$$
$$\gamma(x_1, \ldots, x_8) = (\tfrac{1}{2} - x_1, \tfrac{1}{2} - x_2, x_3, x_4, \tfrac{1}{2} - x_5, \tfrac{1}{2} - x_6, x_7, x_8),$$
$$\delta(x_1, \ldots, x_8) = (-x_1, x_2, \tfrac{1}{2} - x_3, x_4, \tfrac{1}{2} - x_5, x_6, \tfrac{1}{2} - x_7, x_8).$$

The singular set of T^8/Γ consists of:

(a) 4 copies of $T^4/\{\pm 1\}$ from the fixed points of α.

(b) 4 copies of $T^4/\{\pm 1\}$ from the fixed points of β.

(c) 2 copies of T^4 from the fixed points of γ.

(d) 2 copies of T^4 from the fixed points of δ.

Here (a),(b) intersect in 64 points, from the fixed points of $\alpha\beta$.

Let Y be the Eguchi–Hanson space, ALE C–Y 2-fold asymptotic to $\mathbb{C}^2/\{\pm 1\}$. We desingularize T^8/Γ by gluing in $T^4/\{\pm 1\} \times Y$ at each $T^4/\{\pm 1\}$ from (a),(b), and gluing in $T^4 \times Y$ at each T^4 from (c),(d), and gluing in $Y \times Y$ at each point of intersection of (a),(b). This gives a compact, simply-connected 8-manifold X with $b^2(X) = 12$, $b^3(X) = 16$, $b^4_+(X) = 107$ and $b^4_-(X) = 43$. We then write down a family of $\mathrm{Spin}(7)$-structures (Ω_t, g_t) on X for $t \in (0, \epsilon]$, which have small torsion when t is small. We do this by shrinking the C–Y 2 structure on Y by a factor $t > 0$, using this to give $\mathrm{Spin}(7)$ structures on $T^4/\{\pm 1\} \times Y$, $T^4 \times Y$, $Y \times Y$, and gluing these to (Ω_0, g_0) with a partition of unity. We glue the forms either as $\mathrm{Spin}(7)$-structures (Ω_t, g_t), or as closed 4-forms $\tilde{\Omega}_t$. Then $\phi_t = \tilde{\Omega}_t - \Omega_t$ is a small 4-form. The obvious definitions would give $\|\phi_t\|_{L^2} = O(t^4)$, which is not small enough. But by solving an equation on T^8/Γ to cancel out the leading order errors, we can achieve $\|\phi_t\|_{L^2} = O(t^{9/2})$, which is small enough.

We then prove the following analytic theorem:

Theorem 2

Let $\lambda,\mu,\nu>0$. Then there exist $\kappa,K>0$ depending only on λ,μ,ν such that whenever $0<t\leqslant\kappa$, the following holds. Suppose (X,Ω_t,g_t) is a compact $\mathrm{Spin}(7)$-manifold, and ϕ_t is a 4-form on X with $\mathrm{d}\Omega_t+\mathrm{d}\phi_t=0$, and: (i) $\|\phi_t\|_{L^2}\leqslant\lambda t^{13/3}$,
(ii) $\|\mathrm{d}\phi_t\|_{L^2}\leqslant\lambda t^{7/5}$,
(iii) the injectivity radius $\delta(g_t)$ satisfies $\delta(g_t)\geqslant\mu t$, and
(iv) the Riemann curvature $R(g_t)$ satisfies $\|R(g_t)\|_{C^0}\leqslant\nu t^{-2}$.
Then there exists a torsion-free $\mathrm{Spin}(7)$-structure $(\hat\Omega_t,\hat g_t)$ on X with $\|\hat\Omega_t-\Omega_t\|_{C^0}\leqslant Kt^{1/3}$.

We show (i)–(iv) hold in our examples for some $A_1,\ldots,A_4>0$ independent of $t\in(0,\epsilon]$, and thus Theorem 2 shows that we can deform (Ω_t,g_t) to a torsion-free $\mathrm{Spin}(7)$-structure $(\hat\Omega_t,\hat g_t)$ for small $t>0$. As in §1, a topological criterion, that $\pi_1(X)=\{1\}$ and $\hat A(X)=1$, then implies that $\mathrm{Hol}(\hat g_t)=\mathrm{Spin}(7)$.

Note that we do not know the cohomology class $[\hat\Omega_t]\in H^4(X;\mathbb{R})$ (in particular, it need not be $[\Omega_t+\phi_t]$). This is necessary, since as in Theorem 1 we don't know the possible cohomology classes of torsion-free $\mathrm{Spin}(7)$-structures in advance.
To prove Theorem 2, we first show that for any C^0-small 4-form η_t on X which lies in $\Lambda^4_- T^*X$ w.r.t. g_t, there is a unique decomposition

$$\Omega_t+\eta_t=\hat\Omega_t+\chi_t$$

for $\hat\Omega_t$ a $\mathrm{Spin}(7)$-form which is C^0-close to Ω_t, and χ_t a C^0-small 4-form which lies in Λ^4_{27} w.r.t. $\hat\Omega_t$. We regard $\hat\Omega_t,\chi_t$ as smooth nonlinear functions of η_t. So we can write $\mathrm{d}\hat\Omega_t=0$ as a nonlinear p.d.e. in η_t, and we show that it has a unique solution η_t which is L^2-orthogonal to the harmonic anti-self-dual 4-forms $\mathcal{H}^4_-$. This works as our equation involves a small nonlinear perturbation of the isomorphism

$$d:\bigl\{\text{anti-self-dual 4-forms } L^2\text{-orthogonal to } \mathcal{H}^4_-\bigr\}\xrightarrow{\cong}\bigl\{\text{exact 5-forms}\bigr\}.$$

3. Spin(7)-manifolds from Calabi–Yau 4-orbifolds

In 2000 I gave a second construction of compact 8-manifolds with holonomy $\mathrm{Spin}(7)$, which starts not with T^8/Γ, but with a Calabi–Yau 4-orbifold Y.

Suppose Y is a compact complex 4-orbifold with $c_1(Y)=0$, admitting Kähler metrics, such that the only singularities of Y are $p_1,\ldots,p_k$ locally modelled on $\mathbb{C}^4/\langle i\rangle$, where $\langle i\rangle=\mathbb{Z}_4$ acts on $\mathbb{C}^4$ by multiplication in the obvious way. Suppose $\sigma:Y\to Y$ is an antiholomorphic involution whose only fixed points are $p_1,\ldots,p_k$.

Then $Y/\langle\sigma\rangle$ is a compact 8-orbifold with orbifold points $p_1,\ldots,p_k$ locally modelled on $\mathbb{R}^8/G$, for G a nonabelian group of order 8, which may be thought of as $\{\pm 1,\pm i,\pm j,\pm k\}\subset\mathbb{H}$.

By the Calabi Conjecture (orbifold version) there exist Ricci-flat Kähler metrics on Y, which will be σ-invariant if the Kähler class is σ-invariant. So as $\mathrm{SU}(4)\subset\mathrm{Spin}(7)$, we get σ-invariant torsion-free $\mathrm{Spin}(7)$-structures on Y, which descend to $Y/\langle\sigma\rangle$.

Thus, we have a compact orbifold $Y/\langle\sigma\rangle$ with torsion-free $\mathrm{Spin}(7)$-structure (Ω,g). We resolve $Y/\langle\sigma\rangle$ to an 8-manifold X by gluing in an ALE $\mathrm{Spin}(7)$-manifold Z asymptotic to $\mathbb{R}^8/G$ at each of $p_1,\ldots,p_k$. Then we define a family of $\mathrm{Spin}(7)$-structures (Ω_t,g_t) on X for $t\in(0,\epsilon]$, with small torsion for small t, and use Theorem 2 to deform them to torsion-free $\mathrm{Spin}(7)$-structures $(\hat\Omega_t,\hat g_t)$ as for the T^8/Γ case, which have $\mathrm{Hol}(\hat g_t)=\mathrm{Spin}(7)$.

To define the ALE $\mathrm{Spin}(7)$-manifold Z, note that G acts on $\mathbb{C}^4=T_{p_i}Y$ as $\langle\alpha,\beta\rangle$, where

$$\alpha:(z_1,\ldots,z_4)\longmapsto(iz_1,iz_2,iz_3,iz_4),$$
$$\beta:(z_1,\ldots,z_4)\longmapsto(\bar z_2,-\bar z_1,\bar z_4,-\bar z_3),$$

where $\alpha^4=\beta^4=1$, $\alpha^2=\beta^2$, $\alpha\beta=\beta\alpha^3$.

The crepant resolution of $\mathbb{C}^4/\langle\alpha\rangle$ is the blow-up W of $\mathbb{C}^4/\langle\alpha\rangle$ at 0, the line bundle $\mathcal{O}(-4)\to\mathbb{CP}^3$, and it carries an explicit ALE C–Y 4 metric due to Calabi. The action of β on $\mathbb{C}^4/\langle\alpha\rangle$ lifts to a free action of $\langle\beta\rangle=\mathbb{Z}_2$ on W, so $W/\langle\beta\rangle$ is an ALE $\mathrm{Spin}(7)$-manifold asymptotic to $\mathbb{C}^4/G$.
Now if we did the blow-up of $\mathbb{C}^4/\langle\alpha\rangle$ using the given complex structure on Y, then the $\mathrm{Spin}(7)$-manifold X we end up with would not have $\pi_1(X)=\{1\}$ and holonomy $\mathrm{Spin}(7)$, but we would have $X=\tilde X/\langle\beta\rangle$ where $\tilde X$ is a Calabi–Yau 4-fold with a free action of $\langle\beta\rangle\cong\mathbb{Z}_2$, and metrics on X with holonomy $\mathbb{Z}_2\ltimes\mathrm{SU}(4)$.
However, the action of G on $\mathbb{R}^8$ has a kind of 'hyperkähler twist': we can use a different complex structure on $\mathbb{R}^8$ which swaps round α and β. That is, there are two different ways to glue $W/\langle\beta\rangle$ into $Y/\langle\sigma\rangle$ at each p_i. As long as we use the 'twisted' gluing for at least one $i=1,\ldots,k$, we get $\pi_1(X)=\{1\}$, and holonomy $\mathrm{Spin}(7)$ metrics on X.

An example

Work in the weighted projective space $\mathbb{CP}^5_{1,1,1,1,4,4}$. Let Y be the hypersurface

$$Y=\bigl\{[z_0,\ldots,z_5]\in\mathbb{CP}^5_{1,1,1,1,4,4}: z_0^{12}+z_1^{12}+z_2^{12}+z_3^{12}+z_4^3+z_5^3=0\bigr\}.$$

Then Y is a Calabi–Yau 4-orbifold with three singular points

$$p_1=[0,0,0,0,1,-1],\; p_2=[0,0,0,0,1,e^{\pi i/3}],\; p_3=[0,0,0,0,1,e^{-\pi i/3}],$$

each of which is locally modelled on $\mathbb{C}^4/\langle i\rangle$. Define an antiholomorphic involution $\sigma:Y\to Y$ by

$$\sigma:[z_0,\ldots,z_5]\longmapsto[\bar z_1,-\bar z_0,\bar z_3,-\bar z_2,\bar z_5,\bar z_4].$$

Then the only fixed points of σ are p_1,p_2,p_3. The construction yields a compact 8-manifold X with holonomy $\mathrm{Spin}(7)$, which has $b^1=b^2=b^3=0$, $b^4_+=1639$ and $b^4_-=807$. The family of holonomy $\mathrm{Spin}(7)$ metrics on X has dimension 808.

4. Open problems in Spin(7) geometry
Finding compact 8-manifolds with holonomy Spin(7)

Problem 1

Use the known constructions to of compact 8-manifolds with holonomy Spin(7) *find as many new examples as you can.*

Presumably this could be done by some kind of computer search, of the kind that some String Theorists are very good at. The examples I found, I did with a pencil and paper and counting on my fingers, and when I felt I had enough, I stopped. It is very possible that there are many more examples left to find.

Problem 2

Can you find new ways to construct compact 8-manifolds with holonomy Spin(7)*?*

Problems about Cayley submanifolds

Problem 3

Study the singularities of Cayley 4-folds. Find more examples of model singularities in $\mathbb{R}^8$*. Consider how singularities form, e.g. by shrinking ALE Cayley 4-folds. Get some idea of which singularities can occur in low codimension amongst all Cayley 4-folds, e.g. codimension 1.*

Since special Lagrangian 4-folds and complex surfaces in Calabi–Yau 4-folds, and associative 3-folds $\times\mathcal{S}^1$, coassociative 4-folds $\times\{\text{point}\}$ in G_2-manifold $\times\mathcal{S}^1$, all provide examples of Cayley 4-folds, we know quite a lot of examples already; and there is some work on Cayley 4-folds as well.

Problem 4

Construct examples of compact 8-manifolds with $\mathrm{Spin}(7)$, together with a fibration by Cayley 4-folds, including singular fibres.

This is rather a nice problem, which I would like to see done during the Simons collaboration. One could hope to approach it in a similar way to a proposal by Alexei Kovalev in the G_2 case, in which the singularities of the fibration are locally modelled on singularities of holomorphic fibrations of Cayley 4-folds by surfaces.

Problems about $\mathrm{Spin}(7)$ instantons

We can ask similar questions about $\mathrm{Spin}(7)$ instantons to G_2 instantons, and others here know a lot more about this than I do.

- Study singularities of $\mathrm{Spin}(7)$ instantons, in particular, bubbling on Cayley 4-folds.
- Use to define Donaldson–Segal style counting invariants of $\mathrm{Spin}(7)$-manifolds?
- Use to define a Floer theory for G_2 instantons on a G_2-manifold X, by considering $\mathrm{Spin}(7)$ instantons on $X \times \mathbb{R}$?