On Mirror Symmetry, Fukaya categories, and Bridgeland stability, with a view towards Lagrangian Mean Curvature Flow

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Based on *'Conjectures on Bridgeland stability for Fukaya categories of Calabi–Yau manifolds, special Lagrangians, and Lagrangian mean curvature flow'*, EMS Surv. Math. Sci. 2 (2015), 1-62. arXiv:1401.4949.
See also Thomas and Yau math.DG/0104196, math.DG/0104197.

These slides available at `http://people.maths.ox.ac.uk/~joyce/`.

Plan of today's talk:

1. Homological Mirror Symmetry

A *Calabi–Yau m-fold* (M, J, g, Ω) is a compact Ricci-flat Kähler manifold of complex dimension m, with trivial canonical bundle. The Kähler form ω of g makes (M, ω) into a symplectic manifold. String Theorists conjectured that Calabi–Yau m-folds should come in mirror pairs (M, J, g, Ω), $(\check M, \check J, \check g, \check\Omega)$, where the complex geometry of (M, J, g, Ω) is somehow equivalent to the symplectic geometry of $(\check M, \check J, \check g, \check\Omega)$, and vice versa. In 1994, Kontsevich expressed this in the *Homological Mirror Symmetry Conjecture* as equivalences of triangulated categories:

$$D^b \operatorname{coh}(M, J) \simeq D^b F(\check M, \check\omega), \quad D^b F(M, \omega) \simeq D^b \operatorname{coh}(\check M, \check J). \quad (1)$$

Here $\operatorname{coh}(M, J)$ is the abelian category of coherent sheaves on (M, J), and $D^b \operatorname{coh}(M, J)$ its derived category, and $F(M, \omega)$ is the Fukaya category of Lagrangians in (M, ω), an A_∞-category, and $D^b F(M, \omega)$ is its derived category.

In 2002, motivated by ideas of String Theorists, Tom Bridgeland invented *Bridgeland stability conditions* on triangulated categories. This gives an extension of the HMS Conjecture (folklore):

- There should be a Bridgeland stability condition $\mathcal{S}_{B+i\omega}$ on $D^b \operatorname{coh}(M, J)$, depending on the 'complexified Kähler form' $B + i\omega$.
- There should be a Bridgeland stability condition $\mathcal{S}_\Omega$ on $D^b F(M, \omega)$, depending on the 'holomorphic volume form' Ω.
- The HMS equivalences (1) should identify $\mathcal{S}_{B+i\omega} \simeq \mathcal{S}_{\check\Omega}$ and $\mathcal{S}_\Omega \simeq \mathcal{S}_{\check B + i\check\omega}$.

These are not known — it is difficult to construct Bridgeland stability conditions on CY categories, particularly in high dimensions. Bridgeland stability conditions on $D^b \operatorname{coh}(M, J)$ are known to exist in dimensions 1,2, and in some special cases in dimension 3. So far as I understand, Bridgeland stability conditions on $D^b F(M, \omega)$ are not known to exist, except via mirror symmetry.

Bridgeland stability conditions

If $\mathcal{T}$ is a triangulated category, a *Bridgeland stability condition* $\mathcal{S} = (Z, P)$ on $\mathcal{T}$ assigns a 'central charge map' $Z : K^{\rm num}(\mathcal{T}) \to \mathbb{C}$, and for each $\phi \in \mathbb{R}$, a subcategory $P(\phi) \subset \mathcal{T}$ of 'semistable objects with phase ϕ', where if $0 \neq E \in P(\phi)$ then $Z([E]) \in e^{i\pi\phi} \cdot \mathbb{R}_{>0}$, such that every object in $\mathcal{T}$ is built uniquely out of a chain of semistable objects $E_1, \ldots, E_n$ via a kind of Harder–Narasimhan filtration.

Usually it is easy to write down Z, but difficult to construct the subcategories $P(\phi) \subset \mathcal{T}$.

For the conjectural Bridgeland stability condition $\mathcal{S}_\Omega$ on $D^bF(M, \omega)$, it is expected that the subcategories $P(\phi) \subset D^bF(M, \omega)$ should consist of '(graded) special Lagrangian m-folds with phase ϕ'.

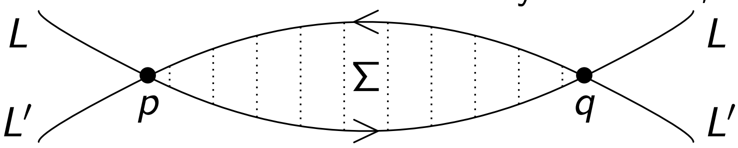
More about $D^b \operatorname{coh}(M, J)$ and $D^bF(M, \omega)$

Objects of $D^b \operatorname{coh}(M, J)$ are (complexes of) coherent sheaves on (M, J). Think of a coherent sheaf as a (singular) holomorphic vector bundle on a (singular) compact complex submanifold of M. They are algebraic geometry objects, and form a nice category $\operatorname{coh}(X)$ in a straightforward way: the morphisms in $\operatorname{coh}(X)$ are basically bundle-linear holomorphic maps of vector bundles. The moduli spaces $\mathfrak{M}_{\operatorname{coh}(X)}, \mathfrak{M}_{D^b \operatorname{coh}(X)}$ of objects in $\operatorname{coh}(X)$ and $D^b \operatorname{coh}(X)$ are *singular* schemes or stacks, not manifolds.

Objects of $D^bF(M, \omega)$ are complexes of (Maslov zero, graded) Lagrangians L in (M, ω), plus some extra data b I'll explain shortly. The morphisms in $D^bF(M, \omega)$ are *Lagrangian Floer cohomology*, that is, $\operatorname{Hom}_{D^bF(M,\omega)}(L, L'[i]) = HF^i(L, L')$. Hamiltonian isotopic Lagrangians L, L' are isomorphic in $D^bF(M, \omega)$, so we can think of the moduli space $\mathfrak{M}_{D^bF(M,\omega)}$ of objects in $D^bF(M, \omega)$ as parametrizing Hamiltonian isotopy classes of Lagrangians.

Lagrangian Floer cohomology for dummies

Let (M,ω) be a symplectic manifold, and fix an almost complex structure J on M compatible with ω. Let L, L' be compact graded Lagrangians in M, and suppose for simplicity they intersect transversely. Then (roughly) the *Lagrangian Floer cohomology* $HF^*(L,L')$ is the cohomology of a complex $\bigl(CF^*(L,L'),\mathrm{d}\bigr)$, where $CF^*(L,L')$ has basis points $p \in L \cap L'$, and differential $\mathrm{d}(p) = \sum_{q\in L\cap L'} N_{p,q}\cdot q$ where $N_{p,q}$ in $\mathbb{Z}$ or $\mathbb{Q}$ is the 'number' of J-holomorphic discs Σ in M with boundary in $L \cup L'$, of this kind:



J-holomorphic disc Σ with boundary in $L \cup L'$.

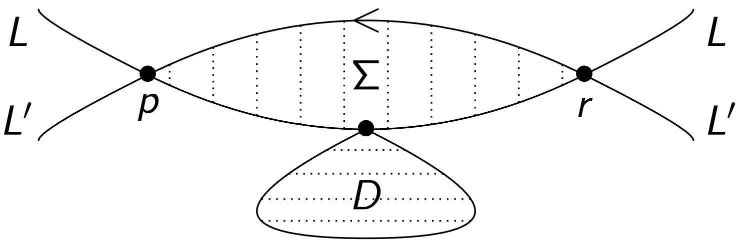
Write $\bar{\mathcal{M}}(p,q)$ for the moduli space of such J-holomorphic discs. Under good conditions, $\bar{\mathcal{M}}(p,q)$ is a compact, oriented manifold with corners of dimension $\mu(q)-\mu(p)-1$, where $\mu(p)$ is the 'Maslov index' of p, and $N_{p,q}$ counts dimension 0 moduli spaces only.

In good cases, in oriented manifolds with corners we have

$$\partial\bar{\mathcal{M}}(p,r) \cong \textstyle\coprod_{q\in L\cap L'} \bar{\mathcal{M}}(p,q)\times\bar{\mathcal{M}}(q,r). \qquad (2)$$

Take p, r with $\mu(r)=\mu(p)+2$, so $\dim\bar{\mathcal{M}}(p,r)=1$. Then the number of boundary points of $\bar{\mathcal{M}}(p,r)$, counted with signs, is 0, so $\sum_{q\in L\cap L'} \#(\bar{\mathcal{M}}(p,q)\times\bar{\mathcal{M}}(q,r)) = \sum_{q\in L\cap L'} N_{p,q}\cdot N_{q,r} = 0$. This is what we need to show that $\mathrm{d}^2=0$ in $CF^*(L,L')$.

In general the boundary formula (2) for $\bar{\mathcal{M}}(p,r)$ is incomplete. There are two extra terms, the first from curves like this:



Bubbling off a disc D with boundary in L.

in which a J-holomorphic disc D in M with $\partial D \subset L$ 'bubbles off' from Σ, and the second from discs D' with $\partial D' \subset L'$ bubbling off in the same way. These extra terms cause $\mathrm{d}^2 \neq 0$, so HF^* is undefined.

Fukaya–Oh–Ohta–Ono define a notion of *bounding cochains* b, b' for L, L', which are chains $b \in C_{m-1}(L;\mathbb{F})$ in homology, satisfying

$$\partial b = \bigcup\nolimits_{\Sigma\ J\text{-holomorphic disc},\ \partial\Sigma\subset L} \partial\Sigma + \text{higher order terms.} \qquad (3)$$

They modify $\mathrm{d} : CF^k(L,L') \to CF^{k+1}(L,L')$ using b, b' to get $\mathrm{d}^{b,b'}$ with $(\mathrm{d}^{b,b'})^2 = 0$. Such b need not exist. We say that L has *unobstructed* HF^* if some such b exists, and *obstructed* HF^* otherwise. I expect this will be an important condition in LMCF.
Thus Lagrangian Floer cohomology is $HF^*\big((L,b),(L',b')\big)$, and objects of $D^bF(M,\omega)$ which are single Lagrangians should be pairs (L,b) where L has unobstructed HF^*.
Lagrangians L with obstructed HF^* do not appear as objects in $D^bF(M,\omega)$, and String Theory does not know about them.
This also resolves a paradox in Mirror Symmetry: we expect $\mathfrak{M}_{D^b\mathrm{coh}(M)} \cong \mathfrak{M}_{D^bF(\check M)}$, with $\mathfrak{M}_{D^b\mathrm{coh}(M)}$ a singular scheme/stack. The moduli space of Lagrangians up to Hamiltonian isotopy is a manifold, but $\mathfrak{M}_{D^bF(\check M)}$ is the moduli space of unobstructed Lagrangians (L,b), solutions of (3), which may be singular.

2. LMCF and the Thomas–Yau Conjecture

Let (M,g) be a Riemannian manifold, and $L_0 \subset M$ a compact submanifold. Then one can consider the *Mean Curvature Flow* (*MCF*) $L_t : t \in [0,\epsilon)$ of L_0, moving it in the direction of its mean curvature, decreasing its volume. Stationary points of the flow are minimal submanifolds. Finite time singularities can occur in the flow.
If (M,J,g,Ω) is a (Ricci-flat) Calabi–Yau m-fold and L_0 is a Lagrangian, then the L_t for $t \in [0,\epsilon)$ remain Lagrangian (Smoczyk). This is *Lagrangian Mean Curvature Flow* (*LMCF*). If L_0 is *graded*, or *Maslov zero*, then the flow stays in a fixed Hamiltonian isotopy class of Lagrangians. Stationary points of the flow are *special Lagrangian m-folds* (*SL m-folds*).
Lagrangian MCF also works in Kähler–Einstein manifolds.

The Thomas–Yau Conjecture, first attempt

In 2001, motivated by Mirror Symmetry, Thomas and Yau proposed:

Conjecture (Thomas–Yau Conjecture, informal version)

Let (M, J, g, Ω) be a Calabi–Yau m-fold, and L_0 a compact graded Lagrangian in (M, ω). There should be a notion of when L_0 is **stable**, *which Thomas and Yau attempt to define explicitly.*
If L_0 is stable then the LMCF $L_t : t \in [0, \infty)$ of L_0 exists for all time, and $L_t \to L_\infty$ as $t \to \infty$ for an SL m-fold L_∞, which is the unique SL m-fold in the Hamiltonian isotopy class of L_0.

This cannot be true in the precise form they stated it (which doesn't really make sense, because of mistakes inserted by Yau), but that is not the point. Their conjecture was prescient, as it pre-dates both Bridgeland stability (2002), and the definition of $D^bF(M, \omega)$ (2030?). They knew their conjecture was only a first approximation. Our mission, should we choose to accept it, is to work out the correct conjecture, and then prove that (!).

What Richard Thomas really meant to say . . .

I want to explain a revised version of the Thomas–Yau Conjecture. It is fiendishly difficult, with difficulty increasing with dimension — the 3-d version may be about as hard as the Poincaré Conjecture, recently solved by Perelman. But the 2-d version may be feasible, and the big picture suggests smaller, more accessible problems. Here are the main changes we make to the T–Y Conjecture:

- We should work in the derived Fukaya category $D^bF(M, \omega)$. Objects of $D^bF(M, \omega)$ are pairs (L, b), where L is a (graded) Lagrangian with *unobstructed Lagrangian Floer cohomology*, and b is a *bounding cochain* for L. We should restrict our attention to Lagrangians with unobstructed HF^*.
- 'Stability of Lagrangians' is a Bridgeland stability condition $\mathcal{S}_\Omega$ on $D^bF(M, \omega)$, as in the extended HMS Conjecture. We should not define $\mathcal{S}_\Omega$ explicitly, as Thomas–Yau tried to do; the existence of $\mathcal{S}_\Omega$ is difficult, part of the conjecture, but may be provable by Fukaya category techniques and HMS.

Revising the T–Y Conjecture: finite time singularities

- Finite time singularities of LMCF are unavoidable, as examples of Neves 2010 show. So our conjecture should concern long-time unique existence of LMCF $L_t : t \in [0, \infty)$ *with surgeries* at times $0 < T_1 < T_2 < \cdots$, in a similar way to Perelman's proof of the Poincaré Conjecture. That is, L_{T_n} is singular, and L_t for $t \in (T_n - \epsilon, T_n)$ and $t \in (T_n, T_n + \epsilon)$ may not be in the same Hamiltonian isotopy class, or even be diffeomorphic. However, L_t must remain in a fixed isomorphism class in $D^b F(M, \omega)$ for all t in $[0, \infty) \setminus \{T_1, T_2, \ldots\}$.
- For finite time singularities L_T of LMCF starting from L_0 with HF^* unobstructed, i.e. $(L_0, b_0) \in D^b F(M, \omega)$, I am suggesting one can continue LMCF past the singularity after a surgery. However, I expect that for L_0 with HF^* obstructed, so $L_0 \notin D^b F(M, \omega)$, there may be finite time singularities of LMCF which one cannot continue past, even after a surgery.

Revising the T–Y Conjecture: enlarging $D^b F(M, \omega)$

- Under LMCF in dimension $\geqslant 2$, embedded Lagrangians can turn into immersed Lagrangians. So we have to enlarge our definition of Fukaya category to $D^b \tilde{F}(M, \omega)$ including immersed Lagrangians as objects, as in Akaho–Joyce JDG 2010. This will probably not change $D^b F(M, \omega)$ up to equivalence, so symplectic geometers might not care, but it is essential for our conjecture, which concerns actual geometric Lagrangians.
- I expect that in dimension $\geqslant 3$, LMCF can turn nonsingular Lagrangians by a surgery into singular Lagrangians, but with 'stable SL singularities' for which LMCF has short time existence (T. Behrndt). So we need to include such singular Lagrangians as objects in $D^b \tilde{F}(M, \omega)$ too.
- Under LMCF of *immersed* Lagrangians L_t, $t \in (T - \epsilon, T + \epsilon)$ it can happen that L_t has HF^* unobstructed for $t < T$ and HF^* obstructed for $t \geqslant T$, even though all L_t are nonsingular. Then we should do a surgery at an immersed point of L_T.

Revising the T–Y Conjecture: starting from any Lagrangian

- Once we include singularities and surgeries, we don't need to start LMCF from a stable Lagrangian: we can start from any object (L_0, b_0) in $D^bF(M,\omega)$. As $t \to \infty$, I expect (L_t, b_t) to converge to a finite union $(L^1_\infty, b^1_\infty) \cup \cdots \cup (L^k_\infty, b^k_\infty)$ with L^i_∞ a (singular) special Lagrangian of phase ϕ^i, $\phi^1 < \cdots < \phi^k$, and the objects $(L^1_\infty, b^1_\infty), \ldots, (L^k_\infty, b^k_\infty)$ are the semistable factors in the decomposition of (L_0, b_0) under the Bridgeland stability condition $\mathcal{S}_\Omega$ on $D^bF(M,\omega)$; this constructs $\mathcal{S}_\Omega$.

Conjecture (Thomas–Yau 2.0, still informal)

Let (L_0, b_0) be an object in $D^b\tilde{F}(M,\omega)$. Then there exists a unique family L_t, $t \in [0,\infty)$ satisfying LMCF with surgeries at singular times $0 < T_1 < T_2 < T_3 < \cdots$, and bounding cochains b_t for L_t for $t \in [0,\infty) \setminus \{T_1, T_2, \ldots\}$ unique up to equivalence such that $(L_t, b_t) \cong (L_0, b_0)$ in $D^b\tilde{F}(M,\omega)$.
Taking the limit of (L_t, b_t) as $t \to \infty$ enables us to construct the Bridgeland stability condition $\mathcal{S}_\Omega$ on $D^b\tilde{F}(M,\omega)$.

3. Finite time singularities of Lagrangian MCF

Finite time singularities of MCF L_t, $t \in [0,T)$ as $t \to T_-$ are divided into Type I (quickly forming) and Type II (slowly forming). In a Type I singularity, part of the submanifold L_t shrinks homothetically to a point x in M with rate $(T-t)^{1/2}$, and the flow near x is modelled on an *MCF shrinker* in $\mathbb{R}^n \cong T_xM$. Type II singularities are more difficult to describe, and less well understood.

An oriented Lagrangian L_0 in a Calabi–Yau m-fold (M,J,g,Ω) is called *almost calibrated* if the phase function $\Phi : L_0 \to \mathrm{U}(1)$ has $\mathrm{Re}(e^{-i\pi\phi_0}\Phi) > 0$ for some $\phi_0 \in \mathbb{R}$, that is, the phase variation of L_0 is less than π. Wang 2001 proved that LMCF starting from an almost calibrated Lagrangian L_0 remains almost calibrated, and does not develop a Type I singularity. Neves 2006 proved that LMCF starting from a graded Lagrangian L_0 does not develop a Type I singularity. Basically this is because there are no graded LMCF shrinkers in $\mathbb{C}^m$.

An important result for any Thomas–Yau type programme is:

Theorem (Neves 2010)

Let (M, J, g, Ω) be a Calabi–Yau m-fold, and L_0 a compact Lagrangian in (M, ω). Then there exists a Hamiltonian perturbation $\tilde{L}_0$ of L_0 such that the Lagrangian MCF $\tilde{L}_t$, $t \in [0, T)$ starting from $\tilde{L}_0$ develops a finite time singularity at $t = T$.

In particular, no notion of 'stability' of Lagrangians L_0 which depends only on the Hamiltonian isotopy class can ensure that LMCF L_t, $t \in [0, \infty)$ exists for all time. So any revision of the Thomas–Yau Conjecture must cope with finite time singularities of LMCF, presumably by continuing the flow after a surgery.

Possible surgeries during the flow

In my paper I describe (without proof) some of the surgeries I think are possible in LMCF at singular times T_i, in a feeble attempt to make Thomas–Yau 2.0 sound more credible.
I will explain three of these:

(a) 'Neck pinch' by shrinking a Lawlor neck, giving an immersed Lagrangian for $t > T_i$.

(b) 'Opening a neck' by gluing in a Joyce–Lee–Tsui expander at an immersed point – roughly, the inverse to (a).

(c) 'Collapsing a zero object', when a connected component L' of L shrinks to a point, but $(L', b') \cong 0$ in $D^b\tilde{F}(M, \omega)$, so the isomorphism class of (L, b) is not changed by deleting (L', b').

(a) 'Neck pinch' by shrinking a Lawlor neck

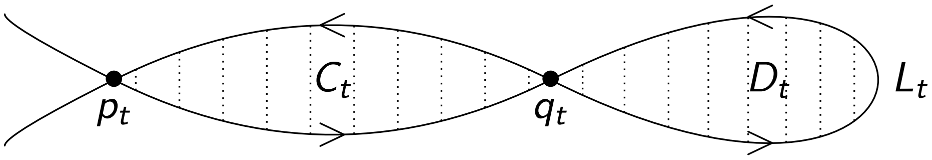
Let Π_0, Π_1 be special Lagrangian planes in $\mathbb{C}^m$ of the same phase, intersecting transversely at 0, and satisfying an angle condition. Lawlor 1989 defined an explicit SL m-fold N in $\mathbb{C}^m$ diffeomorphic to $\mathcal{S}^{m-1} \times \mathbb{R}$ and asymptotic to $\Pi_0 \cup \Pi_1$ at infinity – a 'Lawlor neck'. As a manifold it is the connect sum of Π_0 and Π_1 at 0.
I claim that a possible Type II finite time singularity of LMCF L_t, $t \in [0, T)$ in a Calabi–Yau m-fold (M, J, g, Ω) is when, near some $x \in M$, L_t in M looks like $c_t \cdot N$ in $T_xM \cong \mathbb{C}^m$ for some $c_t \in (0, \infty)$ with $c_t \to 0$ as $t \to T_-$. Since $\lim_{c\to 0} c \cdot N = \Pi_0 \cup \Pi_1$, the limit $L_T = \lim_{t\to T_-}$ is actually a *nonsingular*, *immersed* Lagrangian, topologically different to L_t for $t \in [0, T)$.
I claim this is a *generic* singularity, in that if LMCF starting from L_0 has such a neck pinch, then so does LMCF starting from $\tilde{L}_0$ for any sufficiently small Hamiltonian perturbation $\tilde{L}_0$ of L_0.

Work in progress (?) with Yng-Ing Lee shows that such neck pinches happen in examples of $\mathrm{SO}(m)$-equivariant Lagrangian MCF in $\mathbb{C}^m$. Since L_T is a compact, nonsingular, immersed Lagrangian, we can continue the flow L_t, $t \in [T, T')$ by LMCF in immersed Lagrangians. This neck pinch process can cut one connected component of L_t for $t < T$ into two components for $t > T$. This is important for the Bridgeland stability condition picture. As in Thomas–Yau 2.0, we hope to construct LMCF with surgeries L_t, $t \in [0, \infty)$ such that $\lim_{t\to\infty} L_t = L_\infty = L^{\phi_1} \cup \cdots \cup L^{\phi_n}$ is a union of special Lagrangian components of different phases. Thus, if L_0 is connected, but (L_0, b_0) is not semistable, then the flow L_t, $t \in [0, \infty)$ has to cut L_0 into $n > 1$ components for $t \gg 0$. I believe this 'neck pinch' mechanism is how this happens.

(b) 'Opening a neck'

LMCF of *immersed* Lagrangians $L_t : t \in [0, T)$ only changes L_t by Hamiltonian isotopy in a weak, local sense: the flow can slide two sheets of L_t over one another, introduce extra self-intersection points, etc. In the Akaho–Joyce immersed HF^* theory, this kind of weak Hamiltonian isotopy can move you from Lagrangians with HF^* unobstructed to Lagrangians with HF^* obstructed.

The typical problem is if we have J-holomorphic curves C_t, D_t like this:



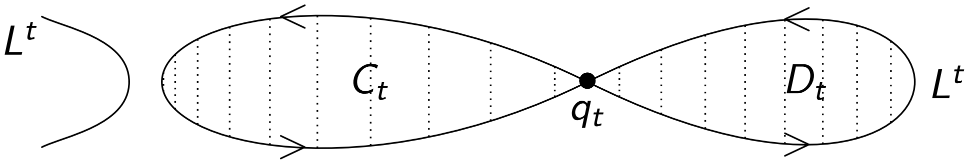
Wall-crossing for immersed HF^* unobstructed/obstructed.

then HF^* is unobstructed when $\mathrm{area}(C_t) < \mathrm{area}(D_t)$ and obstructed when $\mathrm{area}(C_t) > \mathrm{area}(D_t)$. But flowing to obstructed Lagrangians L_t is bad, as no bounding cochain b_t exists.

Joyce–Lee–Tsui J.D.G. 84 (2010) find explicit LMCF expanders N in $\mathbb{C}^m$ asymptotic to a union of Lagrangian planes $\Pi_0 \cup \Pi_1$, very like Lawlor necks. At the time T when $\mathrm{area}(C_T) = \mathrm{area}(D_T)$, we do a surgery, gluing in a JLT expander N at p_T asymptotic to $T^+_{p_T} L_T \cup T^-_{p_T} L_T$ in $T_{p_T} M \cong \mathbb{C}^m$.

A calculation in my paper shows that the angle conditions for existence of the JLT expander hold iff $\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{area}(C_t) - \mathrm{area}(D_t)) > 0$, that is, iff we are crossing from HF^* unobstructed to obstructed.

For $t > T$ the J-holomorphic curves look like this:



$\mathrm{area}(C_t) = \mathrm{area}(D_t)$

As $\mathrm{area}(C_t) = \mathrm{area}(D_t)$, the contributions of C_t, D_t to obstructing HF^* of L_t cancel, and HF^* is unobstructed.

Begley–Moore arXiv:1501.07823 prove my conjecture that LMCF $L_t : t \in [T, T + \epsilon)$ gluing in the JLT expander at p_T exists.

(c) 'Collapsing a zero object'

Let L_0 be a compact Lagrangian in $\mathbb{C}^m$. If L_0 is contained in a ball of radius R, then LMCF $L_t : t \in [0, T)$ starting from L_0 must shrink to a point in $\mathbb{C}^m$ within time $T = R^{1/2}$, unless it becomes singular first. Similarly, any Lagrangian L_0 contained in a small ball in (M, J, g, Ω) must shrink to a point under LMCF in bounded time, unless it becomes singular first.

Now if (L_0, b_0) lies in $D^b\tilde{F}(M, \omega)$ with L_0 in a small ball in M, then L_0 is displaceable, so that $(L_0, b_0) \cong 0$ is a zero object in $D^b\tilde{F}(M, \omega)$.

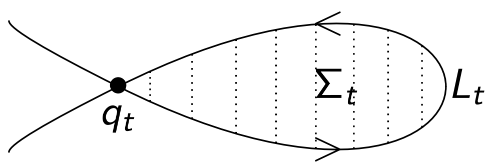
Suppose we have LMCF L_t, $t \in [0, T)$ with $L_t = L'_t \amalg L''_t$, with bounding cochains $b_t = b'_t \amalg b''_t$, where L'_t is contained in a small ball in M and shrinks to a point in M at $t = T$. Then $(L'_t, b'_t) \cong 0$, so that $(L_t, b_t) \cong (L''_t, b''_t)$ in $D^b\tilde{F}(M, \omega)$. At $t = T$ we delete (L'_t, b'_t), and continue the flow for $t > T$ by flowing L''_t. This gives an LMCF surgery which does not change the isomorphism class in $D^b\tilde{F}(M, \omega)$. Neves' 2010 examples can be explained using (a),(c).

What goes wrong in LMCF if HF^* is obstructed

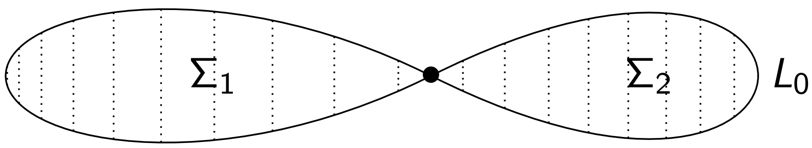
Thomas–Yau 2.0 claims long-time existence L_t, $t \in [0, \infty)$ of LMCF with surgeries starting with a Lagrangian L_0 *with HF^* unobstructed*, i.e. with an object (L_0, b_0) in $D^bF(M, \omega)$.

In contrast, I expect that for Lagrangians L_0 with HF^* obstructed, there may be finite time singularities at $t = T$ in LMCF such that *we cannot continue the flow for $t > T$*, even after a surgery.

In the Akaho–Joyce immersed HF^* theory, obstructions to HF^* of L_t can be caused by J-holomorphic 'teardrops' Σ_t with small area:

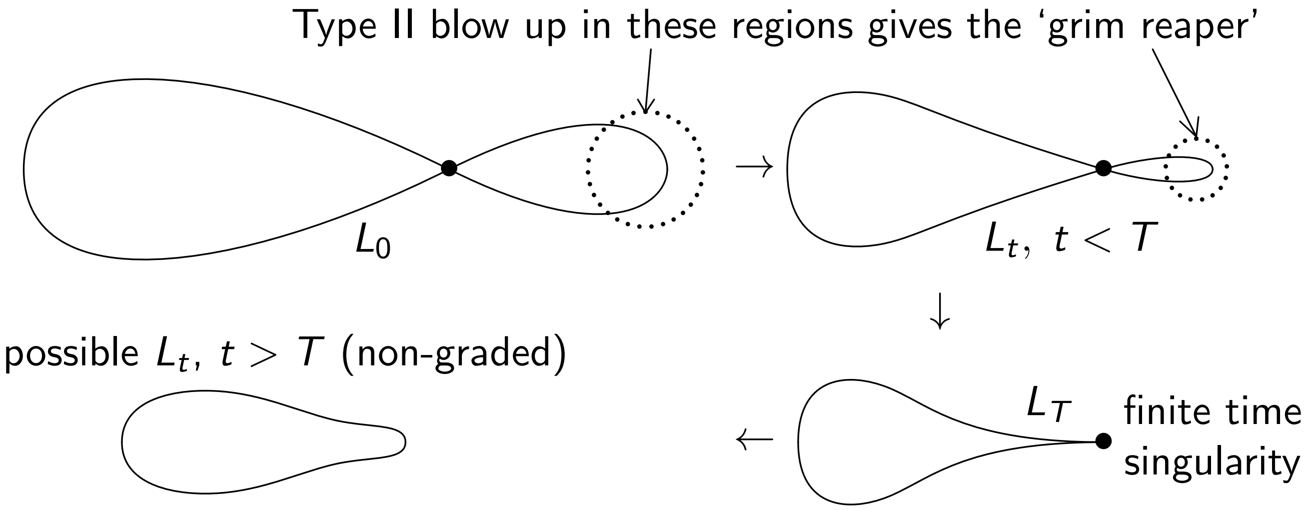


I expect that there can be Type II singularities of immersed LMCF L_t, $t \in [0, T)$ with a teardrop curve Σ_t in which $\mathrm{area}(\Sigma_t) \to 0$ as $t \to T_-$, and L_T has a singular 'cusp', after which one cannot continue the flow. For 1-dimensional LMCF in $\mathbb{C}$ we can prove this using known theorems: start with L_0 an '∞ sign' immersed graded Lagrangian in $\mathbb{C}$, with $\mathrm{area}(\Sigma_1) > \mathrm{area}(\Sigma_2)$



'∞ sign' Lagrangian L_0 in $\mathbb{C}$

Then LMCF L_t, $t \in [0, T)$ looks like this:



One can continue the flow for $t > T$, but only in *non-graded* embedded Lagrangians, which we exclude. There is no way to continue LMCF in graded Lagrangians for $t > T$. I expect similar behaviour in higher dimensions. In my paper I sketch how singularities might form with Type II blow-up a JLT LMCF translator in $\mathbb{C}^m$, shrinking a J-holomorphic teardrop.

Some interesting problems

(a) Prove existence of Bridgeland stability conditions on $D^bF(M,\omega)$ with central charge $Z=[\Omega]$ in examples, by symplectic geometry / categorical techniques / HMS.

(b) Prove that 'pinching a neck' by Type II shrinking a Lawlor neck is a generic finite time singularity of LMCF, open in Hamiltonian isotopy class of Lagrangians, not just under strong symmetry assumptions.

(c) Model 'shrinking a zero object' in examples.

(d) Model 'shrinking a holomorphic teardrop' in examples with HF^* obstructed, dim $\geqslant 2$, using JLT translators.

(e) In dimension 2, for compact immersed graded Lagrangians with HF^* unobstructed, prove only possible finite time singularities are 'pinching a neck' in (b) and 'shrinking a zero object' in (c). Deduce long time existence of LMCF with surgeries in this case. Deduce 2-d Thomas–Yau 2.0.