Enumerative invariants in Algebraic Geometry and wall crossing formulae

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Report on work in progress.

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1. Introduction

At the June meeting I described a very general conjectural theory of enumerative invariants in Algebraic and Differential Geometry, and their wall-crossing formulae under change of stability condition, which we had proved for quivers without oriented cycles (see arXiv:2005.05637, joint with Jacob Gross and Yuuji Tanaka).

I can now announce considerable progress in this programme. I am writing a big paper/book which will prove the conjectures in most of the cases in Algebraic Geometry that I care about (with a few caveats, later). I hope to have finished it by the January meeting.

I also now understand how to extend the picture in several ways I didn't in June, including how to fit DT for Calabi–Yau 3-folds into the same framework, and how to include Donaldson and Seiberg–Witten invariants of surfaces when $b^2_+ > 1$. I will explain the latter in the second half of today's talk.

Warning: I may be lying. I haven't finished all the proofs yet.

Set up of the problem

Here is the situation I will consider (details omitted/simplified):

- $\mathcal{A}$ is a $\mathbb{C}$-linear abelian category satisfying a list of assumptions, in which we expect to form invariants 'counting' semistable objects in $\mathcal{A}$. Examples: $\mathcal{A} = \text{mod-}\mathbb{C}Q$ or $\text{mod-}\mathbb{C}Q/I$ representations of a quiver (with relations); $\mathcal{A} = \text{coh}(X)$ coherent sheaves on smooth projective $\mathbb{C}$-scheme X which is a curve, surface, Fano 3-fold, Calabi–Yau 3- or 4-fold. Or $\text{coh}_{\text{cs}}(X)$ for X quasiprojective.
- $\mathcal{M}$ moduli stack of objects in $\mathcal{A}$, as Artin $\mathbb{C}$-stack.
- $\mathcal{M}^{\text{pl}}$ 'projective linear' moduli stack of objects in $\mathcal{A}$ modulo multiples of the identity. $\Pi^{\text{pl}} : \mathcal{M} \to \mathcal{M}^{\text{pl}}$ fibration, fibre $[*/\mathbb{G}_m]$.
- $K(\mathcal{A})$ a quotient of Grothendieck group $K_0(\mathcal{A})$ such that $\mathcal{M} = \coprod_{\alpha \in K(\mathcal{A})} \mathcal{M}_\alpha$, $\mathcal{M}^{\text{pl}} = \coprod_{\alpha \in K(\mathcal{A})} \mathcal{M}^{\text{pl}}_\alpha$, with $\mathcal{M}_\alpha, \mathcal{M}^{\text{pl}}_\alpha$ open and closed. Examples $K(\mathcal{A}) = \mathbb{Z}^{Q_0}$ for quivers, $K(\mathcal{A}) \subset H^*(X, \mathbb{Q})$ lattice of Chern characters for X smooth (quasi-)projective.
- $C(\mathcal{A}) = \{[\![E]\!] : 0 \neq E \in \mathcal{A}\} \subset K(\mathcal{A})$ 'positive cone'.
- Some functors $F_k : \mathcal{A} \to \text{Vect}_{\mathbb{C}}$ used for 'stable pairs'.

- $\mathcal{E}^\bullet_{\alpha,\beta} = (\mathcal{E}xt^\bullet)^\vee$ dual of Ext complex on $\mathcal{M}_\alpha \times \mathcal{M}_\beta$, perfect complex.
- $\chi : K(\mathcal{A}) \times K(\mathcal{A}) \to \mathbb{Z}$ biadditive, $\chi(\alpha,\beta) = \mathrm{rank}\,\mathcal{E}^\bullet_{\alpha,\beta}$.
- $\phi : \mathcal{F}^\bullet \to \mathbb{L}_{\mathcal{M}^{\rm pl}}$ Behrend–Fantechi perfect obstruction theory on $\mathcal{M}^{\rm pl}$ with $[(\Pi^{\rm pl})^*(\mathcal{F}^\bullet)] = [\mathcal{O}_{\mathcal{M}}] - [\Delta^*_{\mathcal{M}}(\mathcal{E}^\bullet)]$ in $K_0(\mathrm{Perf}(\mathcal{M}))$. (Variants on this for Fano 3-folds and Calabi–Yau 3- or 4-folds).
- $\mathscr{S}$ set of (weak) stability conditions $(\tau, T, \leqslant)$ on $\mathcal{A}$. That is, $(T, \leqslant)$ total order, e.g. $(\mathbb{R}, \leqslant)$, $\tau : C(\mathcal{A}) \to T$ map satisfying conditions. Write $\mathcal{M}^{\rm st}_\alpha(\tau) \subseteq \mathcal{M}^{\rm ss}_\alpha(\tau) \subseteq \mathcal{M}^{\rm pl}_\alpha$ for the open substack of τ-(semi)stable objects in $\mathcal{A}$ in class $\alpha \in C(\mathcal{A}) \subset K(\mathcal{A})$.
- **Important assumptions on** $\mathscr{S}$: the $\mathcal{M}^{\rm ss}_\alpha(\tau)$ have a properness property. Implies that $\mathcal{M}^{\rm ss}_\alpha(\tau)$ is a proper Deligne–Mumford stack if $\mathcal{M}^{\rm st}_\alpha(\tau) = \mathcal{M}^{\rm ss}_\alpha(\tau)$. Then obstruction theory $\phi : \mathcal{F}^\bullet \to \mathbb{L}_{\mathcal{M}^{\rm pl}}$ defines a virtual class $[\mathcal{M}^{\rm ss}_\alpha(\tau)]_{\rm virt}$ in $H_{2-2\chi(\alpha,\alpha)}(\mathcal{M}^{\rm pl}_\alpha, \mathbb{Q})$.
- Can connect any $(\tau_0, T_0, \leqslant), (\tau_1, T_1, \leqslant)$ in $\mathscr{S}$ by a 'continuous path' $(\tau_t, T_t, \leqslant)_{t\in[0,1]}$ in $\mathscr{S}$.
- A finiteness assumption which ensures finitely many terms in wall-crossing formula. (Cf Bogomolov vanishing; tricky $\dim X \geqslant 3$.)

Vertex algebras and Lie algebras

As I have told you in previous conferences, under these assumptions, I define the structure of a graded vertex algebra on $H_*(\mathcal{M},\mathbb{Q})$ with shifted grading $\hat{H}_n(\mathcal{M}_\alpha,\mathbb{Q}) = H_{n-2\chi(\alpha,\alpha)}(\mathcal{M}_\alpha,\mathbb{Q})$, given for $u \in H_*(\mathcal{M}_\alpha)$ and $v \in H_*(\mathcal{M}_\beta)$, $\alpha,\beta \in K(\mathcal{A})$, by

$$Y(u,z)v = Y(z)(u \otimes v) = (-1)^{\chi(\alpha,\beta)} \sum\nolimits_{i \geqslant 0} z^{\chi(\alpha,\beta)+\chi(\beta,\alpha)-i} \cdot$$
$$H_*(\Phi_{\alpha,\beta}) \circ (e^{zD} \otimes \mathrm{id})((u \boxtimes v) \cap c_i(\mathcal{E}^\bullet_{\alpha,\beta} \oplus \sigma^*_{\alpha,\beta}(\mathcal{E}^\bullet_{\beta,\alpha})^\vee)),$$

with $\Phi_{\alpha,\beta} : \mathcal{M}_\alpha \times \mathcal{M}_\beta \to \mathcal{M}_{\alpha+\beta}$ the direct sum map. This then induces the structure of a graded Lie algebra on $H_*(\mathcal{M}^{\mathrm{pl}},\mathbb{Q})$ with shifted grading $\check{H}_n(\mathcal{M}^{\mathrm{pl}}_\alpha,\mathbb{Q}) = H_{n+2-2\chi(\alpha,\alpha)}(\mathcal{M}^{\mathrm{pl}}_\alpha,\mathbb{Q})$. Thus $\check{H}_0(\mathcal{M}^{\mathrm{pl}}) = \bigoplus_{\alpha \in K(\mathcal{A})} H_{2-2\chi(\alpha,\alpha)}(\mathcal{M}^{\mathrm{pl}}_\alpha,\mathbb{Q})$ is a Lie algebra.
Note that virtual classes $[\mathcal{M}^{\mathrm{ss}}_\alpha(\tau)]_{\mathrm{virt}}$ lie in $H_{2-2\chi(\alpha,\alpha)}(\mathcal{M}^{\mathrm{pl}}_\alpha,\mathbb{Q})$.

The problems I want to solve

Here are my main goals:

- As above, if $\mathcal{M}_\alpha^{\rm st}(\tau)=\mathcal{M}_\alpha^{\rm ss}(\tau)$ then $\mathcal{M}_\alpha^{\rm ss}(\tau)$ is a proper D–M stack with virtual class $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}$ in $H_{2-2\chi(\alpha,\alpha)}(\mathcal{M}_\alpha^{\rm pl},\mathbb{Q})$.

But if $\mathcal{M}_\alpha^{\rm st}(\tau)\neq\mathcal{M}_\alpha^{\rm ss}(\tau)$ then $\mathcal{M}_\alpha^{\rm ss}(\tau)$ is an Artin stack, and the Behrend–Fantechi virtual class is not defined. I want to define an enumerative invariant $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm inv}$ in $H_{2-2\chi(\alpha,\alpha)}(\mathcal{M}_\alpha^{\rm pl},\mathbb{Q})$ for all $\alpha\in C(\mathcal{A})$, with $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm inv}=[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}$ if $\mathcal{M}_\alpha^{\rm st}(\tau)=\mathcal{M}_\alpha^{\rm ss}(\tau)$.
That is, I want to define invariants counting *strictly semistables*.

- I want an explicit wall-crossing formula for the $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm inv}$ under change of stability condition $(\tau,T,\leqslant)$.

All this is known for Donaldson–Thomas invariants of Calabi–Yau 3-folds (Joyce–Song, Kontsevich–Soibelman) and other motivic invariants (Joyce). I want to extend to virtual class invariants.
We regard the class in $H_{2-2\chi(\alpha,\alpha)}(\mathcal{M}_\alpha^{\rm pl},\mathbb{Q})$ as the primary invariant – can get numbers by integrating universal classes in $H^*(\mathcal{M}^{\rm pl})$.
Note that $H_*(\mathcal{M}^{\rm pl})$ is often explicitly computable (for $D^b\,{\rm coh}(X)$).

2. The main results, basic case

Theorem 1 (Work in progress, proof ongoing.)

In the situation above, with other assumptions I haven't given, I can define invariants $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm inv}$ *in* $H_{2-2\chi(\alpha,\alpha)}(\mathcal{M}_\alpha^{\rm pl},\mathbb{Q})$ *for all* $\alpha\in C(\mathcal{A})$ *satisfying:*

(a) *If* $\mathcal{M}_\alpha^{\rm st}(\tau)=\mathcal{M}_\alpha^{\rm ss}(\tau)$ *then* $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm inv}=[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}$.

(b) *The* $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm inv}$ *have an explicit inductive definition via B–F virtual classes of moduli spaces of 'stable pairs' in an auxiliary abelian category* $\bar{\mathcal{A}}$ *with* $0\to\mathcal{A}\to\bar{\mathcal{A}}\to{\rm Vect}_{\mathbb{C}}\to 0$ *exact.*

(c) *If* $(\tau,T,\leqslant),(\tilde\tau,\tilde T,\leqslant)\in\mathscr{S}$ *and* $\alpha\in C(\mathcal{A})$ *then*

$$[\mathcal{M}_\alpha^{\rm ss}(\tilde\tau)]_{\rm inv}=\sum_{\substack{n\geqslant 1,\ \alpha_1,\ldots,\alpha_n\in C(\mathcal{A}):\\ \alpha_1+\cdots+\alpha_n=\alpha}}\tilde U(\alpha_1,\ldots,\alpha_n;\tau,\tilde\tau)\cdot\bigl[\bigl[\ldots\bigl[[\mathcal{M}_{\alpha_1}^{\rm ss}(\tau)]_{\rm inv},[\mathcal{M}_{\alpha_2}^{\rm ss}(\tau)]_{\rm inv}\bigr],\ldots\bigr],[\mathcal{M}_{\alpha_n}^{\rm ss}(\tau)]_{\rm inv}\bigr],\quad (1)$$

where $\tilde U(-)$ *are explicit(ish) combinatorial coefficients, and* $[\,,\,]$ *is the Lie bracket on* $\check H_0(\mathcal{M}^{\rm pl})$.

Very brief sketch of proof

The proof of Theorem 1 will be huge and complicated, probably > 100 pages total (document currently ~ 120 pages). Rough idea:

- First prove (1) for *simple wall-crossings*, in which all moduli spaces in the formula have stable=semistable, and on the wall strictly semistables have only two stable factors in classes β, γ, with $\tau(\beta) > \tau(\gamma)$ and $\tilde\tau(\beta) < \tilde\tau(\gamma)$, so (1) becomes

$$[\mathcal{M}_\alpha^{\rm ss}(\tilde\tau)]_{\rm virt} = [\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt} + \big[[\mathcal{M}_\beta^{\rm ss}(\tau)]_{\rm virt}, [\mathcal{M}_\gamma^{\rm ss}(\tau)]_{\rm virt}\big]. \qquad (2)$$

Prove this using $\mathbb{G}_m$-localization on a master space.

- For $(\tau, T, \leqslant) \in \mathscr{S}$ and $t \in T$, write $\mathcal{A}_{\tau,t}^{\rm ss} \subset \mathcal{A}$ for the abelian subcategory of τ-semistable objects $E \in \mathcal{A}$ with $\tau(\llbracket E \rrbracket) = t$. Introduce a class of auxiliary abelian categories $\bar{\mathcal{A}}$ with $0 \to \mathcal{A}_{\tau,t}^{\rm ss} \to \bar{\mathcal{A}} \to \text{mod-}\mathbb{C}Q \to 0$ exact for certain quivers Q. Can define invariants $[\bar{\mathcal{M}}_{(\alpha,\boldsymbol{d})}^{\rm ss}(\mu)]_{\rm inv}$ for $\bar{\mathcal{A}}$, $\alpha \in C(\mathcal{A})_{\tau=t}$, $\boldsymbol{d} \in \mathbb{N}^{Q_0}$. When $Q = \bullet$ and $\text{mod-}\mathbb{C}Q = \mathrm{Vect}_{\mathbb{C}}$ this gives the stable pair invariants in Theorem 1(b).

- At this stage we have defined invariants $[\mathcal{M}^{\rm ss}_\alpha(\tau)]_{\rm inv}$ and $[\bar{\mathcal{M}}^{\rm ss}_{(\alpha,\boldsymbol{d})}(\mu)]_{\rm inv}$ by 'stable pairs' in both $\mathcal{A}$ and auxiliary categories $\bar{\mathcal{A}}$, with $[\cdots]_{\rm inv}=[\cdots]_{\rm virt}$ if stable=semistable, and we can prove they satisfy (2) for simple wall-crossings.
We can also prove some relations between the invariants for different categories, e.g. if $\bar{\mathcal{M}}^{\rm ss}_{(\alpha,\boldsymbol{d})}(\mu)$ has a smooth morphism to $\mathcal{M}^{\rm ss}_\alpha(\tau)$ with fibre $\mathbb{CP}^n$ then $[\bar{\mathcal{M}}^{\rm ss}_{(\alpha,\boldsymbol{d})}(\mu)]_{\rm inv}$ determines $[\mathcal{M}^{\rm ss}_\alpha(\tau)]_{\rm inv}$.
The idea is to get enough of these two relations to prove (1).
- Prove that if $(\tau,T,\leqslant),(\tilde\tau,\tilde T,\leqslant)$ are 'sufficiently close' in $\mathscr{S}$, can reduce the complicated WCF (1) in $\mathcal{A}$ to a finite sequence of simple WCFs (2) and smooth fibrations in auxiliary categories $\bar{\mathcal{A}}$.
- For the general case of $(\tau,T,\leqslant),(\tilde\tau,\tilde T,\leqslant)$ joined by a 'continuous path' $(\tau_t,T_t,\leqslant)_{t\in[0,1]}$ in $\mathscr{S}$, prove (1) by considering the WCF $\tau\Rightarrow\tau_t$ as t deforms from 0 to 1, and composing WCFs $\tau\Rightarrow\tau_s\Rightarrow\tau_t$ for $s<t$ 'sufficiently close'. Need a strong finiteness condition to ensure WCFS have finitely many nonzero terms.

Some cases in which the main results apply

Theorem 1 applies in the form above to examples including:

(a) $\mathcal{A} = \text{mod-}\mathbb{C}Q$ for Q a quiver with no oriented cycles (to get proper moduli spaces), and slope stability conditions $(\mu, \mathbb{R}, \leqslant)$.

(b) $\mathcal{A} = \text{mod-}\mathbb{C}Q/I$ for Q/I a quiver with relations, and slope stability conditions $(\mu, \mathbb{R}, \leqslant)$, provided have properness condition on $\mathcal{M}^{\text{ss}}_{\boldsymbol{d}}(\mu)$ (no oriented cycles is sufficient).

(c) $\text{coh}(X)$ for X a projective curve (boring as only 1 stability condition, but at least can count strictly semistables).

(d) $\text{coh}(X)$ for X a projective surface with geometric genus $p_g = 0$, for both Gieseker and slope stability conditions defined using real Kähler forms. (Here $p_g = h^{0,2}(X) = b^2_+ - 1$. Will explain case $p_g > 0$ later.)

(e) $\text{coh}(X)$ for X a Fano 3-fold, with caveats: have to exclude dim 0 sheaves, and restrict to Kahler forms in small open ball in Kähler cone to make finiteness condition hold.

Aside on finiteness conditions

In general it is not obvious that the WCF (1) should have finitely many nonzero terms, but I don't see a good notion of convergence, so if there are infinitely many nonzero terms the theory breaks.

- For quivers and $\mathrm{coh}(X)$ for X a curve this is not a problem.
- For $\mathrm{coh}(X)$ for X a surface, if $\mathrm{rank}\,\alpha > 0$ then Bogomolov's theorem shows that $\mathcal{M}^{\mathrm{ss}}_\alpha(\tau) \neq \emptyset$ implies $\Delta(\alpha) \geqslant 0$, where Δ is the discriminant. Then can show (1) has finitely many nonzero terms.
- If X is a smooth projective m-fold, $m > 2$, in general I cannot prove (1) has only finitely many terms with $\tilde{U}(\alpha_1, \ldots, \alpha_n; \tau, \tilde{\tau}) \neq 0$ and $\mathcal{M}^{\mathrm{ss}}_{\alpha_i}(\tau) \neq \emptyset$ for all i. However, I can prove this if $\tau, \tilde{\tau}$ are Gieseker or slope stability conditions from real Kähler forms which are sufficiently close in the Kähler cone.
- Using this, we can transform between any two such stability conditions in finitely many steps, where each step has WCFs with only finitely many terms.

Expected extension to $\mathrm{coh}(X)$ for X a Calabi–Yau 3-fold

The theory explained so far *does not include Calabi–Yau 3-folds*, as although 3-Calabi–Yau moduli spaces do have a perfect (symmetric) obstruction theory $\mathcal{F}^\bullet$ on $\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)$ (at least when $\mathcal{M}_\alpha^{\mathrm{st}}(\tau)=\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)$), this does not satisfy the condition $[(\Pi^{\mathrm{pl}})^*(\mathcal{F}^\bullet)]=[\mathcal{O}_{\mathcal{M}}]-[\Delta_{\mathcal{M}}^*(\mathcal{E}^\bullet)]$ linking the obstruction theory to the Lie bracket on $H_*(\mathcal{M}^{\mathrm{pl}})$.

In fact, because of Serre duality, the Lie bracket on $H_*(\mathcal{M}^{\mathrm{pl}})$ is zero.

Here is how to fix this: we introduce a degree -2 formal parameter y into the vertex algebra construction, making $\hat{H}_*(\mathcal{M})[[y]]$ into a (completed) graded vertex algebra, and $\check{H}_*(\mathcal{M}^{\mathrm{pl}})[[y]]$ into a (completed) graded Lie algebra, with Lie bracket $[,]=\sum_{i=0}^\infty y^i[\,,\,]_i$. The leading term $[\,,\,]_0$ is the previous Lie bracket, and so is zero. Hence the next term $[\,,\,]_1$ induces a degree 2 Lie bracket on $\check{H}_*(\mathcal{M}^{\mathrm{pl}})[[y]]|_{y=0}=\check{H}_*(\mathcal{M}^{\mathrm{pl}})$. We can then do a version of the previous construction using the Lie bracket $[\,,\,]_1$ on $\check{H}_{*-2}(\mathcal{M}^{\mathrm{pl}})$. This will (I hope) re-prove much of Donaldson–Thomas theory without using Behrend, PTVV, BBJ, DAG, or critical loci.

Conjectural extension to $\mathrm{coh}(X)$ for X a Calabi–Yau 4-fold

Borisov–Joyce defined virtual classes for proper oriented 4-Calabi–Yau moduli schemes, with the intention of developing Donaldson–Thomas type invariants for Calabi–Yau 4-folds. The definition uses real derived differential geometry, and everybody hates it. Oh–Thomas (arXiv:2009.05542 and in progress) are doing the job properly using Behrend–Fantechi style algebraic geometry. I hope to be able to prove the analogue of Theorem 1 for $\mathrm{coh}(X)$ a Calabi–Yau 4-fold, using Oh–Thomas virtual cycles. I haven't worked out the details yet, and it is likely I will need properties of Oh–Thomas virtual cycles or auxiliary moduli spaces that aren't proved yet, e.g. defined for Deligne–Mumford stacks not schemes, pull back of the obstruction theories under smooth morphisms of classical stacks, and a pushforward theorem for virtual cycles under smooth morphisms.

3. Extension to surfaces with $p_g > 0$ (speculative/lies?)

Let X be a projective surface with $p_g > 0$ (i.e. $h^{0,2}(X) > 0$, $b^2_+(X) > 1$). Then the theory of §2 does apply to $\mathcal{A} = \mathrm{coh}(X)$. However, if $\mathrm{rank}\,\alpha > 0$ then the obstruction theory $\phi : \mathcal{F}^\bullet \to \mathbb{L}_{\mathcal{M}^{\mathrm{pl}}}$ on $\mathcal{M}^{\mathrm{ss}}_\alpha(\tau)$ has a constant factor $H^{0,2}(X)^*$ in $h^{-1}(\mathcal{F}^\bullet)$, which forces $[\mathcal{M}^{\mathrm{ss}}_\alpha(\tau)]_{\mathrm{inv}} = 0$, so the theory is boring.
You can define a 'reduced' obstruction theory $\phi : \mathcal{F}^\bullet_{\mathrm{red}} \to \mathbb{L}_{\mathcal{M}^{\mathrm{pl}}}$ (at least on the $\mathrm{rank} > 0$ part of $\mathcal{M}^{\mathrm{pl}}$) by deleting the $H^{0,2}(X)^*$ factor. Then you get virtual classes $[\mathcal{M}^{\mathrm{ss}}_\alpha(\tau)]^{\mathrm{red}}_{\mathrm{virt}}$ when $\mathcal{M}^{\mathrm{st}}_\alpha(\tau) = \mathcal{M}^{\mathrm{ss}}_\alpha(\tau)$, which may be nonzero.
Think of these as *algebraic* $\mathrm{U}(n)$ *Donaldson invariants*, as roughly $\mathrm{U}(n)$-instantons $\Leftrightarrow$ holo. rank n vector bundles with H–E conns $\Leftrightarrow$ semistable algebraic rank n vector bundles.
So it seems natural to try to extend our theory to include invariants defined using 'reduced' obstruction theories.

On the face of it, we should hope to define 'reduced' invariants $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm red}$ even when $\mathcal{M}_\alpha^{\rm st}(\tau) \neq \mathcal{M}_\alpha^{\rm ss}(\tau)$, by the pair invariant method, with $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm red} = [\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}^{\rm red}$ when $\mathcal{M}_\alpha^{\rm st}(\tau) = \mathcal{M}_\alpha^{\rm ss}(\tau)$.
However, for $b^2_+(X) > 1$ Donaldson invariants are independent of the stability condition (which corresponds to the splitting $H^2(X) = H^2_+(X) \oplus H^2_-(X)$, since $H^2_+(X) = H^{(2,0)+(0,2)}(X) \oplus \langle[\omega]\rangle$, and the stability condition is determined by the Kähler class $[\omega]$).
So naïvely we would not expect an interesting wall crossing formula. This would be a pity, as WCFs can be powerful tools.
I will propose a more complex set-up involving an abelian category $\mathcal{A}$ which can have nontrivial reduced and non-reduced invariants at the same time, and nontrivial WCFs for both.
When $\mathcal{A} = \mathrm{coh}(X)$ the non-reduced invariants are zero in $\mathrm{rank} > 0$, so the WCF is trivial. But for $\mathcal{A}$ a category of 'L-Bradlow pairs', we get both (non-reduced) Seiberg-Witten type invariants and (reduced) Donaldson type invariants, and the WCF can be used to compute Donaldson invariants in terms of Seiberg–Witten invariants.

Set up of the 'reduced' problem

Like a lot of this project, I found Mochizuki's 2009 monograph on invariants counting coherent sheaves on surfaces very helpful here. Assume all the previous data, plus the following:

- We are given an open substack $\dot{\mathcal{M}}^{\rm pl} \subseteq \mathcal{M}^{\rm pl}$, a positive integer d (will take $d = p_g$), and a 'reduced' $\phi_{\rm red} : \mathcal{F}^\bullet_{\rm red} \to \mathbb{L}_{\dot{\mathcal{M}}^{\rm pl}}$ obstruction theory on $\dot{\mathcal{M}}^{\rm pl}$ with $[(\Pi^{\rm pl})^*(\mathcal{F}^\bullet)] = (d+1)[\mathcal{O}_{\dot{\mathcal{M}}}] - [\Delta^*_{\dot{\mathcal{M}}}(\mathcal{E}^\bullet)]$ in $K_0(\mathrm{Perf}(\dot{\mathcal{M}}))$, where $\dot{\mathcal{M}} = (\Pi^{\rm pl})^{-1}(\dot{\mathcal{M}}^{\rm pl}) \subseteq \mathcal{M}$.
- We are given a subset $C(\mathcal{A})_{\rm rp} \subseteq C(\mathcal{A})$ of 'reduced permissible classes'. If $(\tau, T, \leqslant) \in \mathscr{S}$ and $\alpha \in C(\mathcal{A})_{\rm rp}$ then $\mathcal{M}^{\rm ss}_\alpha(\tau) \subseteq \dot{\mathcal{M}}^{\rm pl}_\alpha$, so if $\mathcal{M}^{\rm st}_\alpha(\tau) = \mathcal{M}^{\rm ss}_\alpha(\tau)$ we have a reduced virtual class $[\mathcal{M}^{\rm ss}_\alpha(\tau)]^{\rm red}_{\rm virt}$ in $\check{H}_{2d}(\mathcal{M}^{\rm pl}_\alpha)$, defined using $\phi_{\rm red} : \mathcal{F}^\bullet_{\rm red} \to \mathbb{L}_{\dot{\mathcal{M}}^{\rm pl}}$.
- Conditions on these, which roughly include that if $\alpha = \beta + \gamma$ with $\alpha \in C(\mathcal{A})_{\rm rp}$ and $\beta, \gamma \in C(\mathcal{A})$ then at least one of β, γ lies in $C(\mathcal{A})_{\rm rp}$. (Basically, $C(\mathcal{A})_{\rm rp}$ is an ideal in the monoid $C(\mathcal{A})$.)

The main results, reduced case

Theorem 2 (Work in progress, proof ongoing.)

In the situation above, as well as $[\mathcal{M}^{\rm ss}_\alpha(\tau)]_{\rm inv}$ *in Theorem 1, I can define invariants* $[\mathcal{M}^{\rm ss}_\alpha(\tau)]_{\rm red}$ *in* $H_{2d+2-2\chi(\alpha,\alpha)}(\mathcal{M}^{\rm pl}_\alpha,\mathbb{Q})=\check H_{2d}(\mathcal{M}^{\rm pl}_\alpha,\mathbb{Q})$ *for all* $\alpha\in C(\mathcal{A})_{\rm rp}$*, satisfying:*

(a) *If* $\mathcal{M}^{\rm st}_\alpha(\tau)=\mathcal{M}^{\rm ss}_\alpha(\tau)$ *then* $[\mathcal{M}^{\rm ss}_\alpha(\tau)]_{\rm red}=[\mathcal{M}^{\rm ss}_\alpha(\tau)]^{\rm red}_{\rm virt}$.

(b) *The* $[\mathcal{M}^{\rm ss}_\alpha(\tau)]^{\rm red}_{\rm inv}$ *have an explicit inductive definition via B–F virtual classes of moduli spaces of 'stable pairs'.*

(c) *If* $(\tau,T,\leqslant),(\tilde\tau,\tilde T,\leqslant)\in\mathscr{S}$ *and* $\alpha\in C(\mathcal{A})_{\rm rp}$ *then*

$$[\mathcal{M}^{\rm ss}_\alpha(\tilde\tau)]_{\rm red}=\sum_{\substack{n\geqslant i\geqslant 1,\ \alpha_i\in C(\mathcal{A})_{\rm rp},\\ \alpha_1,\ldots,\alpha_{i-1},\alpha_{i+1},\\ \ldots\alpha_n\in C(\mathcal{A})\setminus C(\mathcal{A})_{\rm rp}:\\ \alpha_1+\cdots+\alpha_n=\alpha}}\tilde U(\alpha_1,\ldots,\alpha_n;\tau,\tilde\tau)\cdot\bigl[\ldots\bigl[[\mathcal{M}^{\rm ss}_{\alpha_1}(\tau)]_{\rm inv},\ldots\bigr],[\mathcal{M}^{\rm ss}_{\alpha_{i-1}}(\tau)]_{\rm inv}\bigr],[\mathcal{M}^{\rm ss}_{\alpha_i}(\tau)]_{\rm red}\bigr],[\mathcal{M}^{\rm ss}_{\alpha_{i+1}}(\tau)]_{\rm inv}\bigr],\ldots\bigr],[\mathcal{M}^{\rm ss}_{\alpha_n}(\tau)]_{\rm inv}\bigr].\qquad(3)$$

(d) *If* $\alpha\in C(\mathcal{A})_{\rm rp}$ *then* $[\mathcal{M}^{\rm ss}_\alpha(\tau)]_{\rm inv}=0$ *in Theorem 1.*

Here the 'reduced' WCF (3) is like (1), but every term involves exactly one 'reduced' invariant $[\cdots]_{\rm red}$, and other 'non-reduced' invariants $[\cdots]_{\rm inv}$. Reduced invariants live in $\check H_{2d}(\mathcal{M}_\alpha^{\rm pl},\mathbb{Q})$, which is a representation of the Lie algebra $\check H_0(\mathcal{M}_\alpha^{\rm pl},\mathbb{Q})$, in which non-reduced invariants $[\cdots]_{\rm inv}$ live. Algebraically, (1) is to (3) as algebras are to representations.

To prove Theorem 2, the main new step is to replace the simple WCF (2) for $\alpha\in C(\mathcal{A})_{\rm rp}$ with $\alpha=\beta+\gamma$ by

$$[\mathcal{M}_\alpha^{\rm ss}(\tilde\tau)]_{\rm virt}^{\rm red}=[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}^{\rm red}$$
$$+\begin{cases}\bigl[[\mathcal{M}_\beta^{\rm ss}(\tau)]_{\rm virt}^{\rm red},[\mathcal{M}_\gamma^{\rm ss}(\tau)]_{\rm virt}\bigr], & \beta\in C(\mathcal{A})_{\rm rp},\ \gamma\notin C(\mathcal{A})_{\rm rp},\\ \bigl[[\mathcal{M}_\beta^{\rm ss}(\tau)]_{\rm virt},[\mathcal{M}_\gamma^{\rm ss}(\tau)]_{\rm virt}^{\rm red}\bigr], & \beta\notin C(\mathcal{A})_{\rm rp},\ \gamma\in C(\mathcal{A})_{\rm rp},\\ 0, & \beta,\gamma\in C(\mathcal{A})_{\rm rp}.\end{cases}$$

The case $\beta,\gamma\notin C(\mathcal{A})_{\rm rp}$ does not occur by the ideal property of $C(\mathcal{A})_{\rm rp}$. The rest of the proof of Theorem 1 generalizes easily(ish).

Application to algebraic Seiberg–Witten $\Rightarrow$ Donaldson

Let X be a projective surface, and L a line bundle on X. Define an abelian category $\mathcal{A}$ to have objects (E, V, ϕ) for $E \in \mathrm{coh}(X)$, V a finite-dimensional $\mathbb{C}$-vector space, and $\phi : V \to \mathrm{Hom}(L, E)$ a $\mathbb{C}$-linear map. (These are related to 'L-Bradlow pairs' in Mochizuki.) Take $K(\mathcal{A}) = K(\mathrm{coh}(X)) \oplus \mathbb{Z}$ with $[\![E, V, \phi]\!] = ([\![E]\!], \dim V)$. Then $\mathrm{coh}(X)$ embeds in $\mathcal{A}$ as the subcategory of objects $(E, 0, 0)$.

It turns out (Mochizuki) that the 'reduced' obstruction theory $\phi_{\mathrm{red}} : \mathcal{F}^\bullet_{\mathrm{red}} \to \mathbb{L}_{\dot{\mathcal{M}}^{\mathrm{pl}}}$ can be defined on an open substack $\dot{\mathcal{M}}^{\mathrm{pl}} \subset \mathcal{M}^{\mathrm{pl}}$ which includes moduli spaces $\mathcal{M}^{\mathrm{ss}}_{(\alpha,d)}(\mu)$ if either $\mathrm{rank}\,\alpha > 0$ and $d = 0$, or $\mathrm{rank}\,\alpha > 1$ and $d = 1$, so we set $C(\mathcal{A})_{\mathrm{rp}}$ to be the set of such (α, d). If $\mathrm{rank}\,\alpha = 1$ then $(\alpha, 1) \notin C(\mathcal{A})_{\mathrm{rp}}$, so $[\mathcal{M}^{\mathrm{ss}}_{(\alpha,1)}(\mu)]_{\mathrm{inv}}$ may be nonzero. Roughly these $[\mathcal{M}^{\mathrm{ss}}_{(\alpha,1)}(\mu)]_{\mathrm{inv}}$ are *algebraic Seiberg–Witten invariants*. Even though they are non-reduced, they are independent of stability condition as rank 1.

Fix a Gieseker or μ-stability condition $(\tau, T, \leqslant)$ on $\mathrm{coh}(X)$. Then we can define a 1-parameter family $(\mu_t, \tilde{T}, \leqslant)_{t\in\mathbb{R}}$ of stability conditions on $\mathbb{R}$, which all restrict to $(\tau, T, \leqslant)$ on $\mathrm{coh}(X) \subset \mathcal{A}$. Let $\alpha \in C(\mathrm{coh}(X))$ with $\mathrm{rank}\,\alpha > 1$, and take $L = \mathcal{O}_X(-N)$ to be very negative line bundle. It turns out that if $t \ll 0$ then $\mathcal{M}^{\mathrm{ss}}_{(\alpha,1)}(\mu_t) = \emptyset$, and if $t \gg 0$ then $\mathcal{M}^{\mathrm{ss}}_{(\alpha,1)}(\mu_t)$ is a stable pair moduli space over $\mathcal{M}^{\mathrm{ss}}_{\alpha}(\tau)$, so $[\mathcal{M}^{\mathrm{ss}}_{(\alpha,1)}(\mu_t)]^{\mathrm{red}}_{\mathrm{virt}}$ is determined by the algebraic Donaldson invariant $[\mathcal{M}^{\mathrm{ss}}_{\alpha}(\tau)]_{\mathrm{red}}$ and lower order terms. By considering the (nontrivial) WCF (3) for $[\mathcal{M}^{\mathrm{ss}}_{(\alpha,1)}(\mu_t)]_{\mathrm{red}}$ from $t \ll 0$ to $t \gg 0$ for $\mathrm{rank}\,\alpha = r > 1$, I expect to derive a formula

(rank r Donaldson invariants) $=$ function (rank $< r$ Donaldson invariants and Seiberg–Witten invariants).

Hence by induction on r we write rank $r > 1$ Donaldson invariants of surfaces in terms of Seiberg–Witten invariants and rank 1 Donaldson invariants, which count Hilbert schemes. This works for both $b^2_+ > 1$ and $b^2_+ = 1$, though with different formulae.