# Enumerative invariants: what we did, and where to go next

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Fix some interesting kind of space $X$ in Differential or Algebraic Geometry, for example a $G_2$-manifold, Spin(7)-manifold, complex surface, Fano 3-fold, Calabi–Yau 3-fold, or Calabi–Yau 4-fold. Consider geometric objects $E$ living on $X$, for example associative 3-folds, $G_2$-instantons, $\mathrm{Spin}(7)$-instantons, Gromov–Witten curves, vector bundles and coherent sheaves. The families of objects $E$ with fixed topological invariants $\alpha$ form a moduli space $\mathcal{M}_\alpha^{\mathrm{ss}}$, with some geometric structure (e.g. manifold, scheme, or stack).

The idea of *enumerative invariants* is to 'count' moduli spaces $\mathcal{M}_\alpha^{\mathrm{ss}}$ to get a number $I(\alpha)$ (or homology class, etc.). To count as interesting, it is expected as a minimum that $I(\alpha)$ is unchanged under deformations of $X$. You get extra points if $I(\alpha)$ does an interesting job (e.g. distinguishes smooth structures on $X$), or satisfies interesting identities (e.g. wall-crossing formulae), or fits in a generating function with nice properties (e.g. modular), or if you give it an exciting name involving quantum, black holes, etc.

Typically (especially in algebraic geometry) $\mathcal{M}_\alpha^{\rm ss}$ is a rather singular space, and may also be noncompact.

**Principle**

To do enumerative invariants, we need $\mathcal{M}_\alpha^{\rm ss}$ to behave like a *compact, oriented manifold* (or *orbifold*) of known dimension $d$: it must have a *fundamental class* $[M_\alpha^{\rm ss}]_{\rm fund}\in H_d(\mathcal{M}_\alpha^{\rm ss},\mathbb{Q})$. Then we define the invariant by $I(\alpha)=\int_{[\mathcal{M}_\alpha^{\rm ss}]_{\rm fund}}\Psi$, for $\Psi$ some universal cohomology class on $\mathcal{M}_\alpha^{\rm ss}$ (there are usually plenty of $\Psi$).

This raises several problems:

- **Behaving like a manifold.** We need $\mathcal{M}_\alpha^{\rm ss}$ to be a manifold or *derived smooth manifold/orbifold* in Differential Geometry, or a *quasi-smooth derived scheme/Deligne–Mumford stack* in Algebraic Geometry (or a classical scheme/D–M stack with obstruction theory). To get a derived smooth manifold in Differential Geometry, you need objects $E$ to be solutions of a *nonlinear elliptic p.d.e.* To get a quasi-smooth derived scheme you need dimension restrictions on $X$ (e.g. complex surface, Fano 3-fold, C–Y 3-fold).

• **Bad points.** At reducible connections, or strictly semistable sheaves, the manifold-like structure breaks.
Much of my research has involved working out how to correctly 'count' moduli spaces including strictly semistable sheaves.
• **Compactifying moduli spaces.** In Differential Geometry $\mathcal{M}_\alpha^{\rm ss}$ is usually noncompact, and you compactify to $\bar{\mathcal{M}}_\alpha^{\rm ss}$ by adding singular solutions. This is difficult, but showing that the 'manifold-like' structure extends over $\bar{\mathcal{M}}_\alpha^{\rm ss}\setminus\mathcal{M}_\alpha^{\rm ss}$ is practically impossible in $\dim>4$. So conjectures about counting associatives, $G_2$-instantons, etc. are likely to remain conjectures until we all retire.
In Algebraic Geometry we usually get compact moduli spaces for free by considering the right kind of singular object (e.g. coherent sheaves, not vector bundles).
• **Orienting moduli spaces.** We have proved many (6) theorems showing moduli spaces $\mathcal{M}_\alpha^{\rm ss}$ are orientable, and saying what data you need on $X$ to construct a canonical orientation on $\mathcal{M}_\alpha^{\rm ss}$.
• **Actually computing invariants.** Once you know your invariants exist, can you compute them in examples? We have very good theorems for quivers, and coherent sheaves on curves and surfaces.

# Enumerative geometry of $G_2$-manifolds

I'll get this over so Differential Geometers can go to sleep. I see the high level picture (see Thomas Walpuski's talk for more) as:

**Conjecture 1**

**(a) (Donaldson–Segal 2009)** *Enumerative invariants counting $G_2$-instantons and associative 3-folds in a $G_2$-manifold might work.*
**(b) (Joyce 2016)** *Invariants counting $G_2$-instantons and associative 3-folds might not work (unless 'unobstructed'??).*

- For 4-manifolds $X^4$ with $b^2_+ = 0$, Donaldson invariants are undefined.
- For $X^4$ with $b^2_+ = 1$, Donaldson invariants are defined, satisfy WCF.
- For 4-manifolds $X^4$ with $b^2_+ > 1$, Donaldson invariants are deformation-invariant.

Under analogy $b^2_+(X^4) \approx b^2_7(X^7)$, expect:

- For holonomy $G_2$, invariants counting $G_2$-instantons are undefined.
- For CY3$\times\mathcal{S}^1$, invariants counting $G_2$-instantons may be defined, and satisfy WCF — compare D–T invariants of CY3.
- For K3$\times T^3$, invariants counting $G_2$-instantons may be defined, and be deformation-invariant — compare Donaldson invariants of K3.

# Orientations on moduli spaces

We have a pretty complete answer in all the cases we care about:

**Theorem 1**

**(a) (Joyce 2016)** *Moduli spaces of associative 3-folds are orientable. Canonical orientations are induced by a 'flag structure'.*
**(b) (Joyce–Tanaka–Upmeier 2018.)** *General theory of orientations for moduli spaces of connections in gauge theory.*
**(c) (Joyce–Upmeier 2018.)** *Moduli spaces of $G_2$-instantons are orientable. Canonical orientations are induced by a 'flag structure'.*
**(d) (Cao–Gross–Joyce 2018.)** *Moduli of* $\mathrm{Spin}(7)$*-instantons, and coherent sheaves on a compact CY 4-fold, are orientable.*
**(e) (Bojko PhD 2020.)** *Moduli spaces of coherent sheaves on a noncompact CY 4-fold are orientable.*
**(f) (Joyce–Upmeier 2020.)** *Compact Calabi–Yau 3-folds have 'orientation data' [an important unsolved problem in D–T theory].*

The proofs of (b)–(f) involve understanding moduli spaces of *all* connections on $X$ using homotopy theory, classifying spaces, etc.

# A universal picture of enumerative invariants — set up

Let us restrict to enumerative invariant problems that are $\mathbb{C}$-*linear*, with a good notion of *direct sum* $\oplus$, such that reducibles / strictly semistables come from direct sums of smaller objects. Examples are gauge theory of $\mathrm{U}(m)$-connections, and vector bundles / coherent sheaves in algebraic geometry. Write $\mathcal{A}$ for the category of all the objects (e.g. all $\mathrm{U}(m)$ connections on $X$, or $\mathrm{coh}(X)$).

There are two ways of forming a moduli stack of all objects in $\mathcal{A}$: the full moduli stack $\mathcal{M}$, and the 'projective linear' moduli stack $\mathcal{M}^{\mathrm{pl}}$ which quotients out by multiples of identity morphisms. There is a $B\mathbb{G}_m$-fibration $\mathcal{M} \to \mathcal{M}^{\mathrm{pl}}$.

The moduli spaces of 'semistable' objects $\mathcal{M}_\alpha^{\mathrm{ss}}$ we want to 'count' are generally open subspaces of $\mathcal{M}^{\mathrm{pl}}$. Thus, if we can form a 'fundamental class' $[\mathcal{M}_\alpha^{\mathrm{ss}}]_{\mathrm{fund}}$ we can push it forward to $H_*(\mathcal{M}^{\mathrm{pl}}, \mathbb{Q})$. The universal cohomology classes $\Psi$ used to define invariants $I(\alpha) = \int_{[\mathcal{M}_\alpha^{\mathrm{ss}}]_{\mathrm{fund}}} \Psi$ always come from $H^*(\mathcal{M}^{\mathrm{pl}}, \mathbb{Q})$.

We take the point of view that the enumerative invariant *is* the class $[\mathcal{M}_\alpha^{\mathrm{ss}}]_{\mathrm{fund}}$ in $H_*(\mathcal{M}^{\mathrm{pl}}, \mathbb{Q})$.

**Theorem 2 (Joyce 2018)**

*Given almost any $\mathbb{C}$-linear additive category $\mathcal{A}$ of geometric origin, with moduli spaces $\mathcal{M}, \mathcal{M}^{\mathrm{pl}}$, we can define the structure of a vertex algebra on $H_*(\mathcal{M}, \mathbb{Q})$, which induces the structure of a Lie algebra on $H_*(\mathcal{M}^{\mathrm{pl}}, \mathbb{Q})$ (lying a little bit).*

I discovered this (and reinvented vertex algebras by mistake) while trying to understand wall-crossing formulae for DT4 invariants of C–Y 4-folds, but then realized it was far more general. Vertex algebras are horribly complicated objects.

Who ordered the vertex algebras? They clearly come from String Theory. The collaboration String Theorists cannot explain them, and should all be fired.

Recent work by Bojko–Lim–Moreira 2022 relates the vertex algebras to Virasoro constraints on invariants. Again, String Theorists should explain this.

# Explicit description of the homology of moduli spaces

To apply Theorem 2 in examples, it is useful to have an explicit description of $H_*(\mathcal{M},\mathbb{Q})$, and $H_*(\mathcal{M}^{\rm pl},\mathbb{Q})$.

**Theorem 3 (Simons PhD student Jacob Gross arXiv:1907.03269)**

*Let $X$ be a connected complex projective surface. Write $\mathcal{M}$ for the moduli stack of objects in $D^b\operatorname{coh}(X)$ and $K^0_{\rm sst}(X)$ for the* **semi-topological K-theory** *of $X$ (equal to $\operatorname{Image}(K^0(\operatorname{coh}(X))\to K^0_{\rm top}(X))$ for $X$ a surface). Then $\mathcal{M}=\coprod_{\kappa\in K^0_{\rm sst}(X)}\mathcal{M}_\kappa$ with $\mathcal{M}_\kappa$ connected, and*

$$H_*(\mathcal{M}_\kappa,\mathbb{Q})\cong \operatorname{Sym}^*\bigl(H^{\rm even}(X,\mathbb{Q})\otimes_{\mathbb{Q}} t^2\mathbb{Q}[t^2]\bigr)\otimes_{\mathbb{Q}} \textstyle\bigwedge^*\bigl(H^{\rm odd}(X,\mathbb{Q})\otimes_{\mathbb{Q}} t\mathbb{Q}[t^2]\bigr). \tag{0.1}$$

*A similar equation holds for cohomology $H^*(\mathcal{M}_\kappa,\mathbb{Q})$.*

This also holds if $X$ is a curve, projective toric manifold, and a few other cases. Jacob also described the vertex algebra structure on $H_*(\mathcal{M},\mathbb{Q})$ (it is basically a super-lattice vertex algebra). This yields (more-or-less) a description of $H_*(\mathcal{M}^{\rm pl},\mathbb{Q})$ and its Lie bracket.

## Conjecture 2 (Gross–Joyce–Tanaka arXiv:2005.05637, 2020.)

*Let there be given a $\mathbb{C}$-linear enumerative invariant problem in Differential or Algebraic Geometry, with additive category $\mathcal{A}$, lattice of topological charges $K(\mathcal{A})$, stacks $\mathcal{M}$ and $\mathcal{M}^{\mathrm{pl}}$, and nice stability conditions $\tau$, so we get $\tau$-(semi)stable moduli stacks $\mathcal{M}_\alpha^{\mathrm{st}} \subseteq \mathcal{M}_\alpha^{\mathrm{ss}} \subseteq \mathcal{M}_\alpha^{\mathrm{pl}} \subset \mathcal{M}^{\mathrm{pl}}$ for $\alpha \in K(\mathcal{A})$. Then*

(i) *There is a systematic way to define invariants $[\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)]_{\mathrm{inv}} \in H_*(\mathcal{M}_\alpha^{\mathrm{pl}}, \mathbb{Q})$ for all $\alpha \in K(\mathcal{A})$. If $\mathcal{M}_\alpha^{\mathrm{st}} = \mathcal{M}_\alpha^{\mathrm{ss}}$ then $[\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)]_{\mathrm{inv}} = [\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)]_{\mathrm{virt}}$ is the 'virtual class'. If $\mathcal{M}_\alpha^{\mathrm{st}} \neq \mathcal{M}_\alpha^{\mathrm{ss}}$ then the virtual class is not defined, but we can still construct $[\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)]_{\mathrm{inv}}$ in a more complicated way using 'pair invariants', and get an answer independent of the pairs.*

(ii) *The $[\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)]_{\mathrm{inv}}$ satisfy an explicit wall-crossing formula (WCF) under change of stability condition $\tau \rightsquigarrow \tilde{\tau}$. This writes $[\mathcal{M}_\alpha^{\mathrm{ss}}(\tilde{\tau})]_{\mathrm{inv}}$ as a $\mathbb{Q}$-linear combination of repeated Lie brackets*

$$\Bigl[\bigl[\cdots\bigl[[\mathcal{M}_{\alpha_1}^{\mathrm{ss}}(\tau)]_{\mathrm{inv}}, [\mathcal{M}_{\alpha_1}^{\mathrm{ss}}(\tau)]_{\mathrm{inv}}\bigr], \ldots\bigr], [\mathcal{M}_{\alpha_k}^{\mathrm{ss}}(\tau)]_{\mathrm{inv}}\Bigr]$$

*for $\alpha = \alpha_1 + \cdots + \alpha_k$, using the Lie bracket on $H_*(\mathcal{M}^{\mathrm{pl}}, \mathbb{Q})$.*

This isn't a complete statement, and there are variants: some problems need a choice of orientations on $\mathcal{M}, \mathcal{M}^{\mathrm{pl}}$, for 3-Calabi-Yau problems we should use a vertex Lie algebra rather than a vertex algebra, etc. We proved the simplest case:

**Theorem 4 (Gross–Joyce–Tanaka arXiv:2005.05637, 2020.)**

*Conjecture 2 holds for invariants counting representations of quivers $Q$ with no oriented cycles.*

**Theorem 5 (Joyce 'Monster WCF paper' arXiv:2111.04694, 2021.)**

*Conjecture 2 holds for Algebraic Geometry enumerative invariant problems defined using Behrend–Fantechi virtual classes, satisfying a list of assumptions. In particular this includes coherent sheaves on projective curves, surfaces, and Fano 3-folds, and representations of quivers with relations.*

Conjecture 2 should also hold for coherent sheaves on Calabi–Yau 4-folds, but the proof is not yet complete. This is the subject of ongoing work by Arkadij Bojko, Hyeonjun Park, and others. It may also hold for $\mathrm{U}(m)$-instantons on 4-manifolds with $b^2_+ \geqslant 1$.

# Applications of the Monster WCF paper

**Theorem 6 (PhD student Chenjing Bu arXiv:2208.00927, 2022.)**

*Compute invariants of semistable vector bundles on projective curves explicitly using Monster WCF paper. Reproves Jeffrey–Kirwan–Witten.*

**Theorem 7 (Joyce in progress, 2023, see talk Jan 2023.)**

*Compute invariants of semistable coherent sheaves on projective surfaces with $b^2_+ > 1$ explicitly using Monster WCF paper, in terms of universal functions in infinitely many variables.*
*This proves at least the structural part of many conjectures in the literature.*

Example application: computation of all higher rank Donaldson invariants for projective surfaces with $b^2_+ > 1$.

# Where to go next?

- **Orientation problems:** What algebro-topological data on a spin 8-manifold $X$ (e.g. $\mathrm{Spin}(7)$-manifold, Calabi–Yau 4-fold) is needed to define canonical orientations on Cayley 4-fold moduli spaces, and $\mathrm{Spin}(7)$-instanton moduli spaces, and moduli of coherent sheaves on Calabi–Yau 4-folds? The answer is a variant of flag structures on 7-manifolds — Joyce–Upmeier, work in progress.
- **Sheaves on surfaces:** I need to finish writing the $b^2_+ > 1$ case of computing invariants counting semistable coherent sheaves on projective surfaces, and then work out the $b^2_+ = 1$ case (difficult).
- **Constraints on Seiberg–Witten invariants?** My construction of sheaf invariants for surfaces $X$ goes via pair invariants counting 'pairs' $\phi : L \to E$ for $L$ an arbitrary (very negative) line bundle, where $E$ is the semistable sheaf. But the sheaf invariants are independent of $L$. The fact that the invariants are independent of $L$ seems quite nontrivial, and requires constraints on the Seiberg–Witten invariants of $X$ — the poles in the invariant generating function can only lie in special places. I think this may work for general 4-manifolds. I don't yet know how powerful the constraints are. If you know something about this, please tell me.

- **Invariants of Fano 3-folds:** we now know that Fano 3-folds have good sheaf-counting invariants and WCF for dimension $> 0$ sheaves. But nothing has really been done on them. One obvious thing would be to extend Feyzbakhsh–Thomas to Fano 3-folds, and show that sheaf-counting invariants are determined by Gromov–Witten invariants.
- **Calabi–Yau 4-fold DT4 invariants: general theory.** Extend the Monster WCF paper to Calabi–Yau 4-folds. Most of it should be the same, but there are some issues about CY4 virtual classes, and about how to make the pair invariants / quiver-sheaf invariants work in the CY4 case. People are working on this.
- **Calabi–Yau 4-fold DT4 invariants: examples.** Understand what DT4 invariants depend on and what their structure is. Lots of work is being done on this. 3-fold counting invariants depend on Gromov–Witten invariants. It looks like DT4 invariants may depend on Gromov–Witten invariants and also invariants counting surfaces in CY4 (compare counting Cayleys in $\mathrm{Spin}(7)$-manifolds).
- **Vertex algebra questions:** Interpret my vertex algebras in String Theory. Find applications in enumerative invariants, e.g. to Virasoro constraints (Bojko–Lim–Moreira).