Spinors and instantons

Simon Salamon
King's College London

Simons Collaboration on
Special Holonomy in Geometry, Analysis, and Physics

Sixth Annual Meeting, September 2022

Exceptional bundles and special holonomy

1. The Horrocks bundle over $\mathbb{CP}^5$

Twistor geometry with symmetry

$\Uparrow$

2. Cohomogeneity-one actions of $\mathrm{SU}(3)$

Tautological tensors on domains of $\mathbb{HP}^2$

$\Downarrow$

3. Invariant $\mathrm{Spin}(7)$ **structures**

Closed 4-forms and Spin(7) holonomy

Joint work with Udhav Fowdar

1.1 Complex projective space $\mathbb{CP}^{2n+1}$

The choice of a symplectic form ω on $\mathbb{C}^{2n+2}$ determines an an indecomposable 'null-correlation' bundle E of rank $2n$ over $\mathbb{CP}^{2n+1}$.

Set $T = T\mathbb{CP}^{2n+1}$, and let $L = \mathcal{O}(-1)$ denote the tautological line bundle. Then E is defined as $L_\omega^\perp/L$, and there are short exact sequences

$$\begin{array}{ccccccccc}
0 \longrightarrow \mathcal{O}(-1) \longrightarrow \underline{\mathbb{C}}^{2n+2} \longrightarrow T(-1) \longrightarrow 0 \\
0 \longrightarrow E \longrightarrow T(-1) \longrightarrow \mathcal{O}(1) \to 0 \\
0 \longrightarrow E(1) \longrightarrow T \longrightarrow \mathcal{O}(2) \to 0
\end{array}$$

The distribution $E(1)$ defines a contact 1-form $\theta \in H^0(\mathbb{CP}^{2m+1}, T^*(2))$:

$$0 \neq \theta \wedge (d\theta)^n \in H^0(\mathbb{CP}^{2n+1}, K(2+2n)) = \mathbb{C}.$$

Such a holomorphic contact structure is a typical feature of the twistor space of an Einstein manifold, in this case $\mathbb{HP}^n$.

1.2 Low rank bundles

Indecomposable vector bundles over $\mathbb{CP}^N$ with rank $r < N$ are rare. Examples:

[1972] Horrocks-Mumford: $r = 2$ and $N = 4$ with 15,000 symmetries, giving rise to fibred Calabi-Yau 3-folds.

[1976] Tango: $r = N - 1$ for *any* N, determined by a subspace $W \subset \Lambda^2\mathbb{C}^{N+1}$ of dimension $\binom{N-1}{2}$, disjoint from $\mathbb{G}\mathrm{r}_2(\mathbb{C}^{N+1})$.

[1978] Horrocks: $r = 3$ and $N = 5$ using a monad

$$\mathcal{O}(-1) \xhookrightarrow{\alpha} \Lambda^2_0 E \overset{\beta}{\twoheadrightarrow} \mathcal{O}(1),$$

where $\Lambda^2_0 E = \Lambda^2 E/\mathcal{O}$. Set $Y = \ker\beta/\operatorname{im}\alpha$. Assuming $\omega = e^{12} + e^{34} + e^{56}$, the linear maps

$$\alpha = e^{135} + e^{246}, \qquad \beta = e^{135} - e^{426}$$

are defined by stable elements of $H^0(\mathbb{CP}^5, \Lambda^2_0 E(1)) \cong \Lambda^3_0\mathbb{C}^6$.

1.3 A real structure on $\mathbb{CP}^{2n+1}$

Identify $\mathbb{C}^{2n+2}$ with $\mathbb{H}^{n+1}$ by means of the anti-holomorphic involution j. This determines a reduction to $\mathrm{Sp}(2n,\mathbb{C})\cap\mathrm{SL}(n,\mathbb{H})=\mathrm{Sp}(n)$, and a fibration

$$\pi\colon \mathbb{CP}^{2n+1}\longrightarrow \mathbb{HP}^n\subset \mathbb{G}\mathrm{r}_2(\mathbb{C}^{2n+2}),$$

whose fibres are the 'real' (i.e. j-invariant) projective lines.

It is well known that E can be defined as the pullback of a complex vector bundle (also denoted E) with an 'instanton' connection over $\mathbb{HP}^n$. Naïve generalizations of the ADHM construction yield families of instantons with gauge group $\mathrm{Sp}(n)$.

When $n=2$, we can realize the Horrocks (parent) bundle Y as the pullback of a subbundle of $\Lambda^2_0 E$, by further reducing the symmetry group from Sp(3) to SU(3). Today's aim is to explain this setup in a way that relates to Spin(7), and the construction of metrics with exceptional holonomy.

1.4 Generalized instantons

Suppose that M^d has an $\widetilde{G}$-structure, where $\widetilde{G} \subset SO(d)$ is the normalizer of some subgroup G with Lie algebra $\mathfrak{g}$. Examples arise from special holonomy:

d	G	$\widetilde{G}$
4	$\mathrm{SU}(2)_+$	$\mathrm{SU}(2)_+\mathrm{SU}(2)_- = SO(4)$
$4n$	$\mathrm{Sp}(n)$	$\mathrm{Sp}(n)\mathrm{Sp}(1), \quad n \geqslant 2$
$2n$	$\mathrm{SU}(n)$	$\mathrm{U}(n), \quad n \geqslant 2$
7	G_2	G_2
8	$\mathrm{Spin}(7)$	$\mathrm{Spin}(7)$

Definition. In this context, a connection on a bundle W over M^d is an **instanton** if its curvature F lies in $\mathfrak{g} \otimes \mathrm{End}\, W$, where $\mathfrak{g} \subset \mathfrak{so}(d) \subset \Lambda^2 T_m^* M$.

Such connections yield absolute minima for the Yang-Mills functional $\int_M \|F\|^2 dv$. Deformations are governed by an elliptic complex under a weak torsion condition [Reyes-Carrión]. For example, $d * \varphi = 0$ suffices for G_2.

1.5 Quaternionic projective plane

$$\mathbb{HP}^2 = \frac{\mathbb{H}^3 \setminus \{0\}}{\mathbb{H}^*} \cong \frac{\mathrm{Sp}(3)}{\mathrm{Sp}(2) \times \mathrm{Sp}(1)}$$

Let H be the tautological line bundle with fibre $\mathbb{C}^2$, and $E = H^\perp$ its orthogonal complement with fibre $\mathbb{C}^4$ (an instanton for $G = \mathrm{Sp}(2)$). We have

$$\underline{\mathbb{C}}^6 = E \oplus H, \qquad T\mathbb{HP}^2 \cong \mathrm{Hom}(H, H^\perp) \cong E \otimes H.$$

Constant sections of $\bigotimes^k \underline{\mathbb{C}}^6$ distinguish tensors that encode holomorphic data:

$\boxed{k=1}$ $u \in \mathbb{C}^6$ reduces the symmetry to $\mathrm{Sp}(2)\times\mathrm{Sp}(1)$ and projects to sections of E and H that describe the geometry of the spinor bundle $\mathbb{HP}^2 \setminus \{o\} \longrightarrow \mathbb{HP}^1 = S^4$.

$\boxed{k=2}$ An invariant $\zeta \in S^2\mathbb{C}^6$ arises from the action of $\mathrm{U}(1) \subset \mathrm{U}(3) \subset \mathrm{Sp}(3)$.

$\boxed{k=3}$ An invariant $\xi \in \Lambda^3\mathbb{C}^6$ further reduces the isometry group to $\mathrm{SU}(3)$.

Both $\mathrm{Sp}(2)\times\mathrm{Sp}(1)$ and $\mathrm{SU}(3)$ act with cohomogeneity one on $\mathbb{HP}^2$ and provide two model geometries well known in the context of exceptional holonomy.

1.6 Adjoint orbits of G_2 (digression)

These are the Kähler manifolds

$$\begin{array}{ccc} & G_2/T^2 & \\ \swarrow & & \searrow \\ Q^5 = G_2/U(2)^- & & G_2/U(2)^+ = Z^5 \end{array}$$

that also occur in the study of closed G_2-structures on 7-manifolds [Ball].

$Q^5 \cong \mathbb{G}r_2(\mathbb{R}^7)$ is the complex quadric. It possesses a horizontal holomorphic rank 2 vector bundle L_+, used to characterize almost complex curves in S^6 [Bryant].
No indecomposable bundles on Q^N with $r < N$ and $5 < N$ are known.
In characteristic 2, there is a map $f\colon \mathbb{CP}^5 \to Q^5$ such that $Y \cong f^*L_+ \oplus \underline{\mathbb{C}}$ [Faenzi].

Z^5 is the twistor space of $M^8 = G_2/SO(4)$. It has a holomorphic rank 3 vector bundle pulled back from an instanton on M^8 [Nagatomo-Nitta].

2.1 Cohomogeneity-one actions by SU(3)

The following symmetric spaces of dimension 8 have such actions with principal orbit the Aloff-Wallach space $N_{1,0} \cong \mathrm{SU}(3)/\mathrm{U}(1)_{1,0,-1}$, and singular orbits chosen from $\{S^5, \mathbb{CP}^2, L\}$, where $L = SU(3)/SO(3)$:

S^5	$\mathbb{HP}^2$	$\mathbb{CP}^2$
$\mathbb{CP}^2$	Q^4	$\mathbb{CP}^2$
L	$\mathrm{G}_2/SO(4)$	$\mathbb{CP}^2$
L	$\mathrm{SU}(3)$	S^5

In the first two cases, SU(3) extends to a global action by U(3). The Lie group SU(3) acts on itself by $A \mapsto X^{-1}A\overline{X}$, and the map $A \mapsto A\overline{A}$ defines a singular but equivariant fibration from SU(3) onto the hypersurface $\{B \in \mathrm{SU}(3) : \mathrm{tr}\, B \in \mathbb{R}\}$.

All these compact spaces have reduced holonomy. They also admit Spin(7) structures (since $4p_2 - {p_1}^2 = 8\chi$), but not of course Spin(7) holonomy ($\widehat{A} = 0$). The aim of part 3 is to describe explicit Spin(7) structures over $\mathbb{HP}^2$ [Gray-Green].

2.2 The circle action on $\mathbb{HP}^2$

$$S^5 = \frac{\mathrm{SU}(3)}{\mathrm{SU}(2)} \dashleftarrow \boxed{N_{0,1} \times (0, b)} \dashrightarrow \frac{\mathrm{SU}(3)}{\mathrm{U}(2)} = \mathbb{CP}^2$$

The SU(3) orbits are preserved by U(1), whose fixed point set is the $\mathbb{CP}^2$, and

$$\mathbb{HP}^2/\mathrm{U}(1) \cong S^7 \subset \mathfrak{su}(3).$$

$\mathbb{HP}^2 \setminus \mathbb{CP}^2$ is diffeomorphic to the total space of a circle bundle over $\Lambda^2_- T^*\mathbb{CP}^{2*}$, a manifold that admits a complete Ricci-flat metric with holonomy G_2 [Atiyah-Witten].

The principal orbits are parametrized by $\|\zeta_H\|^2$, where η_H is a section of the vector bundle S^2H spanned by $\{I, J, K\}$, used to define the QK quotient $S^5/\mathrm{U}(1) \cong \mathbb{CP}^{2*}$ for the action of U(1) [Galicki-Lawson, Battaglia].

We'll define ζ_H and related tensors in a tautological fashion next.

2.3 Degree 2 tensors

The action of U(1) determines a *constant* splitting of the trivial bundle

$$\underline{\mathbb{C}}^6 = \underline{\mathbb{C}}^3 \oplus j\underline{\mathbb{C}}^3 = \langle e^1, e^3, e^5 \rangle \oplus \langle e^2, e^4, e^6 \rangle \qquad (= E \oplus H)$$

over $\mathbb{HP}^2$ and a U(3)-invariant section $\zeta = e^1 e^2 + e^3 e^4 + e^5 e^6$ of

$$\begin{array}{ccccccc} S^2\underline{\mathbb{C}}^6 & \cong & (E \otimes H) & \oplus & S^2 H & \oplus & S^2 E \\ \zeta & = & X & + & \zeta_H & + & \zeta_E . \end{array}$$

Lemmas. Let ∇ denote the Levi-Civita connection on $\mathbb{HP}^2$.

- X is the Killing vector field associated to the action of U(1).
- ∇X can be identified with $\zeta_H + \zeta_E$ (in the holonomy algebra $\mathfrak{sp}(1) + \mathfrak{sp}(2)$).

Fixed points of U(1) occur when the fibres of $\underline{\mathbb{C}}^3 \cap H$ are non-zero, defining $\mathbb{CP}^2$.
But ζ_H vanishes at points where the fibre of H is ζ-isotropic, defining $S^5 \to \mathbb{CP}^{2*}$.

2.4 Degree 3 tensors

Fix a unit stable 3-form $e^{135}+e^{246}$; it defines a constant section η of

$$\begin{array}{rcl} \Lambda^3_0\underline{\mathbb{C}}^6 = \Lambda^3_0(E\oplus H) & \cong & E \;\oplus\; (\Lambda^2_0 E\otimes H) \\ \eta & = & \eta_E + \quad \eta_H. \end{array}$$

Lemmas [Fowdar-S].

- The section η_E is (like X) nowhere zero on $\mathbb{HP}^2\setminus\mathbb{CP}^2$.
- The rank of η_H is everywhere 2.
- $\nabla\eta_E$ can be identified with η_H, and $\nabla\eta_H$ can be identified with η_E.

Recall that $\Lambda^2_0 E$ is an instanton on $\mathbb{HP}^2$ (meaning $F^i_J\in\mathfrak{sp}(2)$). The same is true of the induced connection on the kernel V of $\eta_H\colon \Lambda^2_0E\to H$.

Corollary [MamoneCapria-S]. V is a vector bundle on $\mathbb{HP}^2$ that possesses an instanton connection with gauge group SU(3), and $\pi^*V\cong Y$.

The 'pre Horrocks bundle' V has Chern class $c(V)=c(\Lambda^2_0E-H)=1+3x^2$.

2.5 Geometry of the Horrocks bundle (digression)

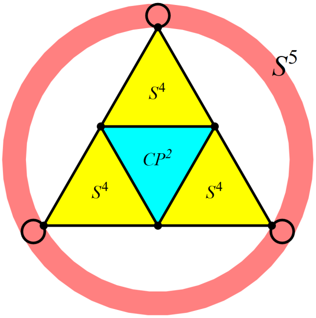
This has been studied by [Ancona-Ottaviani, Decker-Manolache-Schreyer]. It leads one to seek to a real interpretation of properties of Y, such as

Theorem [DMS]. The zero set of a generic section $s \in H^0(\mathbb{CP}^5, Y(2)) \cong \mathfrak{su}(3)$ is a reducible variety of degree 14 consisting of the disjoint planes $\mathbb{P}(\mathbb{C}^3), \mathbb{P}(j\mathbb{C}^3)$, three quadrics, and one del Pezzo surface dP_6, meeting in an octahedron of lines.

The octahedredal graph projects to three points and six 2-spheres in $\mathbb{HP}^2$. The points are joined by three quaternionic lines $m_i = S^4_i$, $i = 1, 2, 3$.

Each $m_i \cap S^5$ is a circle in $\mathbb{HP}^2$ that determines a real quadric in the twistor space $\mathbb{CP}^3_i$.

dP_6 will be determined by the eigenvalues of s, and is invariant by a maximal torus of SU(3).



3.1 Spinors on $\mathbb{HP}^2$

The spin bundles over $\mathbb{HP}^n$ satisfy $\Delta_+ - \Delta_- = \Lambda_0^n(E-H)$. For $n=2$,

$$\Delta_+ \cong S^2H \oplus \Lambda_0^2 E, \qquad \Delta_- \cong E \otimes H \cong T\mathbb{HP}^2.$$

A section of S^2H defines an almost complex structure, one of $\Lambda_0^2 E$ defines a reduction to $\mathrm{Sp}(1)^3/\mathbb{Z}_2$, splitting each tangent space into two Cayley 4-planes. Neither can exist globally over $\mathbb{HP}^2$.

Proposition. Let G be $\mathrm{Sp}(2)\times\mathrm{Sp}(1)$ or $\mathrm{U}(3)$. Then $\mathbb{HP}^2$ possesses G-invariant $\mathrm{Spin}(7)$ structures.

The proof uses the tensors of degrees 1,2,3. For U(3), we use the sections

- ζ_H of S^2H, vanishing only on S^5,
- $(\eta_E \wedge j\eta_E)_0 \sim (X\otimes X)_5$ of $\Lambda_0^2 E$, vanishing only on $\mathbb{CP}^2$.

Let $t = \|\zeta_H\|^2 \in [0,b]$. Choose $\delta = f(t)\phi_E + g(t)\zeta_H$, where $f(0)$ and $g(b)$ are non-zero. This defines an $\mathrm{Sp}(1)^2\mathrm{U}(1)$ structure at generic points of $\mathbb{HP}^2$.

3.2 Spinors, forms, and metrics

Given (M^8, g) with a unit spinor $\delta \in \Delta_+$, one can project its square

$$\delta\otimes\delta \in S^2\Delta_+ \cong \Lambda^0 \oplus \Lambda^2 \oplus \Lambda^4_+$$

to obtain a 4-form Ψ in Λ^4_+. The holonomy of g reduces to Spin(7) iff $d\Psi = 0$. However, to build up a full stock of metrics that are not conformal to a given one, we need more general 4-forms including ASD ones.

Consider the case of $\mathrm{Sp}(2)\times\mathrm{Sp}(1)$ acting on $\mathbb{HP}^2$ with singular orbits a point o and $\mathbb{HP}^1 = S^4$, the zero sets of tautological sections u_E, u_H of E, H $(k = 1)$. Then $\mathbb{HP}^2 \setminus \{o\}$ can be identified with the spinor bundle over S^4, and admits a complete metric g_{BS} with holonomy Spin(7). Its relationship with the symmetric metric g_{QK} is well known:

$$\begin{aligned} g_{\mathrm{BS}} &= 4(r+1)^{-2/5}g_{\mathrm{ver}} + 5(r+1)^{3/5}g_{\mathrm{hor}}, \\ g_{\mathrm{QK}} &= 4(r+1)^{-1}g_{\mathrm{ver}} + 5(r+1)^{-1}g_{\mathrm{hor}}. \end{aligned}$$

The $4+4$ splitting of $T_m\mathbb{HP}^2$ away from o is determined by $(u_E\wedge ju_E)_0$ in $\Lambda^2_0 E$.

3.3 Closed four-forms

Away from o, u_E determines sections $\Omega, \Omega_{14}, \Omega_5^-$ of constant QK-norm in three summands of

$$\Lambda^4 T_m^*\mathbb{HP}^2 = \underbrace{\boxed{\Lambda_1} \oplus \Lambda_5^+ \oplus \boxed{\Lambda_{14}} \oplus \Lambda_{15}}_{+} \oplus \underbrace{\boxed{\Lambda_5^-} \oplus \Lambda_{30}}_{-},$$

since $\Lambda_5^- \cong \Lambda_0^2 E$ and $\Lambda_{14} \cong S_0^2(\Lambda_0^2 E)$. Here Ω is the closed QK 4-form, and $d\Omega_{14} = -5dr \wedge \Omega_{14}$. It is easy to construct closed 4-forms, but not useful ones. The stabilizer of

$$\Psi_{a,b,c} = a\Omega_{14} + b\Omega + c\Omega_5^-$$

is Spin(7) if $(a+8b)(3a+4b) = 4c^2$ (with $a > 2b$ and $a+3b > |c|$). This forces a, b, c to be constant multiples of a function of the radius $r = \|u_H\|^2/(1-\|u_H\|^2)$:

Proposition. If $(a,b,c) = (-\frac{56}{5}, -\frac{3}{5}, 12)(r+1)^{16/5}$ then $\Psi_{a,b,c}$ is the closed 4-form associated to g_{BS}.

3.4 Spin(7) holonomy with Sp(2) (digression)

The complete AC Spin(7) metric $g_{\rm BS}$ is asymptotic to a cone over squashed $S^7_{\rm sq}$ and invariant by $\mathrm{Sp}(2)\times\mathrm{Sp}(1)$. It is the 'limit' of a one-parameter family ($\mathbb{B}_8$) of complete ALC Spin(7) metrics invariant by $\mathrm{Sp}(2)\times\mathrm{U}(1)$, and asymptotic to a circle of fixed radius ℓ times a cone over $\mathbb{CP}^3_{\rm nK}$ [Cvetič-Gibbons-Lü-Pope, Bazaĭkin].

An analogous family ($\mathbb{C}_8$) of Spin(7) metrics exists on $K_{\mathbb{CP}^3}$ in which the role of $g_{\rm BS}$ is played by Calabi's metric with holonomy SU(4), similarly $K_{\mathbb{F}}$ [CGLP, B].

The search for such packages of Ricci-flat metrics on 8-manifolds focusses attention on circle fibrations $\mathrm{nP}^7 \to \mathrm{nK}^6$ of Einstein manifolds. Such fibrations occur naturally over the two self-dual Einstein 4-manifolds:

$$S^7_{\rm sq} \to \mathbb{CP}^3_{\rm nK} \to S^4, \qquad N_{0,1} \to \mathbb{F}_{\rm nK} \to \mathbb{CP}^2.$$

Nearly parallel G_2 metrics come in three types because their cone can have holonomy Spin(7), SU(4), or Sp(2) if nP^7 is 3-Sasakian [Boyer-Galicki]. The latter can be deformed along the 3 Killing fields, giving rise to a system of 3 ODE's.

3.5 The action of $SU(3)$ on non-compact 8-manifolds

Metrics with holonomy Spin(7) and SU(3) symmetry were conjectured and studied by [Gukov-Sparks, G-S-Tong, Kanno-Yasui].

Let W denote the normal bundle of *either* singular orbit S^5 or $\mathbb{CP}^2$ in $\mathbb{HP}^2$. Work of [Reidegeld, Bazaĭkin], and [Foscolo-Haskins-Nordström] for G_2, has culminated in

Theorem [Lehmann]. W admits a complete AC Spin(7) metric, invariant by U(3), asymptotic to a cone over $N_{1,0}$, AND a 1-parameter family g_ℓ of ALC Spin(7) metrics, each asymptotic to a cone over $\mathbb{F} = SU(3)/T^2$ times a circle of radius ℓ. As $\ell \to 0$, the space collapses to $\Lambda^2_- \mathbb{CP}^2$ with its G_2 metric.

More ALC/AC Spin(7) packages exist with $nP^7 = N_{k,l}$ [Chi].

The self-dual Einstein set-up can be extended to the case in which M^4 is an orbifold, in particular a QK quotient of $\mathbb{H}^n$ [Foscolo], but M^4 should itself be the base of a circle fibration for collapse with bounded curvature.

3.6 Work in progress

Spaces associated to the Hitchin orbifolds. Let SO(3) act irreducibly on S^4. There is a family of $SO(3)$-invariant self-dual Einstein orbifold metrics M_k (with a $\mathbb{Z}_{k-2}$ singularity along $\mathbb{RP}^2$) [Hitchin, Tod]. M_4 can be identified with $\mathbb{CP}^2/\langle\sigma\rangle$, and its twistor space is a cubic surface in $\mathbb{CP}^4$ defined by the unique SU(2) invariant in $S^3(S^4(\mathbb{C}^2)) \cong \Lambda^3(S^6(\mathbb{C}^2))$. The 'same' invariant defines the 3-form on the Berger space $B^7 = \mathrm{SO}(5)/\mathrm{SO}(3)$, whose cone has Spin(7) holonomy.

B^7 is diffeomorphic to an S^3 bundle over S^4 [Goette-Kitchloo-Shankar], but more to the point, there are nearly-free S^3 actions giving orbifold fibrations

$$\begin{array}{ccc} B^7 & & S^7 \subset \mathfrak{g}_2/\mathfrak{so}(4) \\ + \searrow & & \swarrow - \\ & M_5 & \end{array}$$

All the 3-Sasakian spaces associated to the M_k have cohomogeneous-one actions by SO(4), and are candidates for admitting a metric with positive sectional curvature [Grove-Wilking-Ziller]. This motivates a study of SO(4)-invariant nearly parallel metrics on such 7-manifolds [S-Singhal].

Closed 4-forms on 8-manifolds. There must exist U(3)-invariant closed 4-forms with stabilizer Spin(7) on domains of $\mathbb{HP}^2$, but their components may be linear combinations of invariants occurring amongst all 6 components of $\Lambda^4 T^*\mathbb{HP}^2$.

The normal bundle of $L = SU(3)/\mathrm{SO}(3)$ has no U(3)-invariant metrics with Spin(7) holonomy. On the other hand, $\mathrm{G}_2/\mathrm{SO}(4)$ admits free families of closed non-parallel 4-forms with stabilizer Sp(2)Sp(1) [Conti-Madsen-S]. The analogous statement for $\mathbb{HP}^2$ is open.

Nearly Spin(7) metrics? It is tempting to look for special classes of Spin(7) metrics with non-zero

$$d\Psi \;\in\; \Lambda^5 = \Lambda^5_8 \oplus \Lambda^5_{48}.$$

A naïve class consists of Einstein metrics with $d\Psi \in \Lambda^5_8$, including the sine cone over nP^7. On the other hand, any 5-form has a rank relative to the isomorphism

$$\Lambda^5 \cong \Lambda^1 \otimes \Lambda^4_7 \qquad (56 = 8 \times 7),$$

which helps highlight degenerate Spin(7) orbits.