Spinors and instantons

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1. The Horrocks bundle over $\mathbb{CP}^5$
   Twistor geometry with symmetry

2. Cohomogeneity-one actions of $SU(3)$
   Tautological tensors on domains of $\mathbb{HP}^2$

3. Invariant $Spin(7)$ structures
   Closed 4-forms and $Spin(7)$ holonomy

Joint work with Udhav Fowdar
1.1 Complex projective space $\mathbb{CP}^{2n+1}$

The choice of a symplectic form $\omega$ on $\mathbb{C}^{2n+2}$ determines an indecomposable 'null-correlation' bundle $E$ of rank $2n$ over $\mathbb{CP}^{2n+1}$.

Set $T = T_{\mathbb{CP}^{2n+1}}$, and let $L = \mathcal{O}(-1)$ denote the tautological line bundle. Then $E$ is defined as $L_\omega / L$, and there are short exact sequences

$$
0 \rightarrow \mathcal{O}(-1) \rightarrow \mathbb{C}^{2n+2} \rightarrow T(-1) \rightarrow 0
$$

$$
0 \rightarrow E \rightarrow T(-1) \rightarrow \mathcal{O}(1) \rightarrow 0
$$

$$
0 \rightarrow E(1) \rightarrow T \rightarrow \mathcal{O}(2) \rightarrow 0
$$

The distribution $E(1)$ defines a contact 1-form $\theta \in H^0(\mathbb{CP}^{2m+1}, T^*(2))$:

$$
0 \neq \theta \wedge (d\theta)^n \in H^0(\mathbb{CP}^{2n+1}, K(2+2n)) = \mathbb{C}.
$$

Such a holomorphic contact structure is a typical feature of the twistor space of an Einstein manifold, in this case $\mathbb{HP}^n$. 
1.2 Low rank bundles

Indecomposable vector bundles over $\mathbb{CP}^N$ with rank $r < N$ are rare. Examples:

[1972] Horrocks-Mumford: $r = 2$ and $N = 4$ with 15,000 symmetries, giving rise to fibred Calabi-Yau 3-folds.

[1976] Tango: $r = N - 1$ for any $N$, determined by a subspace $W \subset \Lambda^2 \mathbb{C}^{N+1}$ of dimension $\binom{N-1}{2}$, disjoint from $\mathbb{Gr}_2(\mathbb{C}^{N+1})$.

[1978] Horrocks: $r = 3$ and $N = 5$ using a monad

$$
\mathcal{O}(-1) \xleftarrow{\alpha} \Lambda^2_0 E \xrightarrow{\beta} \mathcal{O}(1),
$$

where $\Lambda^2_0 E = \Lambda^2 E / \mathcal{O}$. Set $Y = \ker \beta / \im \alpha$. Assuming $\omega = e^{12} + e^{34} + e^{56}$, the linear maps

$$
\alpha = e^{135} + e^{246}, \quad \beta = e^{135} - e^{426}
$$

are defined by stable elements of $H^0(\mathbb{CP}^5, \Lambda^2_0 E(1)) \cong \Lambda^3_0 \mathbb{C}^6$. 
1.3 A real structure on $\mathbb{CP}^{2n+1}$

Identify $\mathbb{C}^{2n+2}$ with $\mathbb{H}^{n+1}$ by means of the anti-holomorphic involution $j$. This determines a reduction to $\text{Sp}(2n, \mathbb{C}) \cap \text{SL}(n, \mathbb{H}) = \text{Sp}(n)$, and a fibration

$$\pi: \mathbb{CP}^{2n+1} \longrightarrow \mathbb{HP}^n \subset \text{Gr}_2(\mathbb{C}^{2n+2}),$$

whose fibres are the ‘real’ (i.e. $j$-invariant) projective lines.

It is well known that $E$ can be defined as the pullback of a complex vector bundle (also denoted $E$) with an ‘instanton’ connection over $\mathbb{HP}^n$. Naïve generalizations of the ADHM construction yield families of instantons with gauge group $\text{Sp}(n)$.

When $n = 2$, we can realize the Horrocks (parent) bundle $Y$ as the pullback of a subbundle of $\Lambda_0^2 E$, by further reducing the symmetry group from $\text{Sp}(3)$ to $\text{SU}(3)$. Today’s aim is to explain this setup in a way that relates to $\text{Spin}(7)$, and the construction of metrics with exceptional holonomy.
1.4 Generalized instantons

Suppose that $M^d$ has an $\tilde{G}$-structure, where $\tilde{G} \subset SO(d)$ is the normalizer of some subgroup $G$ with Lie algebra $\mathfrak{g}$. Examples arise from special holonomy:

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<tr>
<th>$d$</th>
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<th>$\tilde{G}$</th>
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<tbody>
<tr>
<td>4</td>
<td>$\text{SU}(2)_+$</td>
<td>$\text{SU}(2)<em>+\text{SU}(2)</em>- = SO(4)$</td>
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<td>$4n$</td>
<td>$\text{Sp}(n)$</td>
<td>$\text{Sp}(n)\text{Sp}(1), \ n \geq 2$</td>
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<td>$2n$</td>
<td>$\text{SU}(n)$</td>
<td>$\text{U}(n), \ n \geq 2$</td>
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<td>7</td>
<td>$G_2$</td>
<td>$G_2$</td>
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<td>8</td>
<td>$\text{Spin}(7)$</td>
<td>$\text{Spin}(7)$</td>
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**Definition.** In this context, a connection on a bundle $W$ over $M^d$ is an **instanton** if its curvature $F$ lies in $\mathfrak{g} \otimes \text{End} W$, where $\mathfrak{g} \subset \text{so}(d) \subset \Lambda^2 T^*_m M$.

Such connections yield absolute minima for the Yang-Mills functional $\int_M \|F\|^2 d\nu$. Deformations are governed by an elliptic complex under a weak torsion condition [Reyes-Carrión]. For example, $d \ast \varphi = 0$ suffices for $G_2$. 
1.5 Quaternionic projective plane

\[ \mathbb{HP}^2 = \frac{\mathbb{H}^3 \setminus \{0\}}{\mathbb{H}^*} \cong \frac{\text{Sp}(3)}{\text{Sp}(2) \times \text{Sp}(1)} \]

Let \( H \) be the tautological line bundle with fibre \( \mathbb{C}^2 \), and \( E = H^\perp \) its orthogonal complement with fibre \( \mathbb{C}^4 \) (an instanton for \( G = \text{Sp}(2) \)). We have

\[ \mathbb{C}^6 = E \oplus H, \quad T\mathbb{HP}^2 \cong \text{Hom}(H, H^\perp) \cong E \otimes H. \]

Constant sections of \( \bigotimes^k \mathbb{C}^6 \) distinguish tensors that encode holomorphic data:

- For \( k = 1 \) \( u \in \mathbb{C}^6 \) reduces the symmetry to \( \text{Sp}(2) \times \text{Sp}(1) \) and projects to sections of \( E \) and \( H \) that describe the geometry of the spinor bundle \( \mathbb{HP}^2 \setminus \{o\} \rightarrow \mathbb{HP}^1 = S^4 \).
- For \( k = 2 \) an invariant \( \zeta \in S^2 \mathbb{C}^6 \) arises from the action of \( \text{U}(1) \subset \text{U}(3) \subset \text{Sp}(3) \).
- For \( k = 3 \) an invariant \( \xi \in \Lambda^3 \mathbb{C}^6 \) further reduces the isometry group to \( \text{SU}(3) \).

Both \( \text{Sp}(2) \times \text{Sp}(1) \) and \( \text{SU}(3) \) act with cohomogeneity one on \( \mathbb{HP}^2 \) and provide two model geometries well known in the context of exceptional holonomy.
1.6 Adjoint orbits of $G_2$ (digression)

These are the Kähler manifolds

$$G_2/T^2$$

$$Q^5 = G_2/U(2)^-$$

$$G_2/U(2)^+ = Z^5$$

that also occur in the study of closed $G_2$-structures on 7-manifolds [Ball].

$Q^5 \cong \text{Gr}_2(\mathbb{R}^7)$ is the complex quadric. It possesses a horizontal holomorphic rank 2 vector bundle $L_+$, used to characterize almost complex curves in $S^6$ [Bryant]. No indecomposable bundles on $Q^N$ with $r < N$ and $5 < N$ are known.

In characteristic 2, there is a map $f: \mathbb{CP}^5 \to Q^5$ such that $Y \cong f^*L_+ \oplus \mathbb{C}$ [Faenzi].

$Z^5$ is the twistor space of $M^8 = G_2/\text{SO}(4)$. It has a holomorphic rank 3 vector bundle pulled back from an instanton on $M^8$ [Nagatomo-Nitta].
2.1 Cohomogeneity-one actions by SU(3)

The following symmetric spaces of dimension 8 have such actions with principal orbit the Aloff-Wallach space \( N_{1,0} \cong SU(3)/U(1)_{1,0,-1} \), and singular orbits chosen from \( \{ S^5, \mathbb{C}P^2, L \} \), where \( L = SU(3)/SO(3) \):

\[
\begin{array}{ccc}
S^5 & \mathbb{H}P^2 & \mathbb{C}P^2 \\
\mathbb{C}P^2 & Q^4 & \mathbb{C}P^2 \\
L & G_2/\text{SO}(4) & \mathbb{C}P^2 \\
L & SU(3) & S^5
\end{array}
\]

In the first two cases, SU(3) extends to a global action by U(3). The Lie group SU(3) acts on itself by \( A \mapsto X^{-1}AX \), and the map \( A \mapsto A\overline{A} \) defines a singular but equivariant fibration from SU(3) onto the hypersurface \( \{ B \in SU(3) : \text{tr} \, B \in \mathbb{R} \} \).

All these compact spaces have reduced holonomy. They also admit Spin(7) structures (since \( 4p_2 - p_1^2 = 8\chi \)), but not of course Spin(7) holonomy (\( \hat{A} = 0 \)). The aim of part 3 is to describe explicit Spin(7) structures over \( \mathbb{H}P^2 \) [Gray-Green].
2.2 The circle action on $\mathbb{H}P^2$

\[ S^5 = \frac{\text{SU}(3)}{\text{SU}(2)} \leftrightarrow N_{0,1} \times (0, b) \rightarrow \frac{\text{SU}(3)}{\text{U}(2)} = \mathbb{C}P^2 \]

The SU(3) orbits are preserved by U(1), whose fixed point set is the $\mathbb{C}P^2$, and

\[ \mathbb{H}P^2 / U(1) \cong S^7 \subset \mathfrak{su}(3). \]

$\mathbb{H}P^2 \setminus \mathbb{C}P^2$ is diffeomorphic to the total space of a circle bundle over $\Lambda^2 T^* \mathbb{C}P^2^*$, a manifold that admits a complete Ricci-flat metric with holonomy $G_2$ [Atiyah-Witten].

The principal orbits are parametrized by $\|\zeta_H\|^2$, where $\eta_H$ is a section of the vector bundle $S^2 H$ spanned by \{I, J, K\}, used to define the QK quotient $S^5 / U(1) \cong \mathbb{C}P^2^*$ for the action of U(1) [Galicki-Lawson, Battaglia].

We’ll define $\zeta_H$ and related tensors in a tautological fashion next.
2.3 Degree 2 tensors

The action of $U(1)$ determines a constant splitting of the trivial bundle

$$\mathbb{C}^6 = \mathbb{C}^3 \oplus j\mathbb{C}^3 = \langle e^1, e^3, e^5 \rangle \oplus \langle e^2, e^4, e^6 \rangle \quad (= E \oplus H)$$

over $\mathbb{HP}^2$ and a $U(3)$-invariant section $\zeta = e^1 e^2 + e^3 e^4 + e^5 e^6$ of

$$S^2\mathbb{C}^6 \cong (E \otimes H) \oplus S^2H \oplus S^2E$$

$$\zeta = X + \zeta_H + \zeta_E.$$

**Lemmas.** Let $\nabla$ denote the Levi-Civita connection on $\mathbb{HP}^2$.

- $X$ is the Killing vector field associated to the action of $U(1)$.
- $\nabla X$ can be identified with $\zeta_H + \zeta_E$ (in the holonomy algebra $\mathfrak{sp}(1) + \mathfrak{sp}(2)$).

Fixed points of $U(1)$ occur when the fibres of $\mathbb{C}^3 \cap H$ are non-zero, defining $\mathbb{CP}^2$. But $\zeta_H$ vanishes at points where the fibre of $H$ is $\zeta$-isotropic, defining $S^5 \to \mathbb{CP}^2^*$. 
2.4 Degree 3 tensors

Fix a unit stable 3-form $e^{135} + e^{246}$; it defines a constant section $\eta$ of

$$\Lambda^3_0 \mathbb{C}^6 = \Lambda^3_0 (E \oplus H) \cong E \oplus (\Lambda^2_0 E \otimes H)$$

$$\eta = \eta_E + \eta_H.$$

**Lemmas** [Fowdar-S].

- The section $\eta_E$ is (like $X$) nowhere zero on $\mathbb{HP}^2 \setminus \mathbb{CP}^2$.
- The rank of $\eta_H$ is everywhere 2.
- $\nabla \eta_E$ can be identified with $\eta_H$, and $\nabla \eta_H$ can be identified with $\eta_E$.

Recall that $\Lambda^2_0 E$ is an instanton on $\mathbb{HP}^2$ (meaning $F^i_j \in \mathfrak{sp}(2)$). The same is true of the induced connection on the kernel $V$ of $\eta_H$: $\Lambda^2_0 E \to H$.

**Corollary** [MamoneCapria-S]. $V$ is a vector bundle on $\mathbb{HP}^2$ that possesses an instanton connection with gauge group $SU(3)$, and $\pi^* V \cong Y$.

The ‘pre Horrocks bundle’ $V$ has Chern class $c(V) = c(\Lambda^2_0 E - H) = 1 + 3x^2$. 
2.5 Geometry of the Horrocks bundle (digression)

This has been studied by [Ancona-Ottaviani, Decker-Manolache-Schreyer]. It leads one to seek to a real interpretation of properties of $Y$, such as

**Theorem** [DMS]. The zero set of a generic section $s \in H^0(\mathbb{CP}^5, Y(2)) \cong \mathfrak{su}(3)$ is a reducible variety of degree 14 consisting of the disjoint planes $P(C^3), P(jC^3)$, three quadrics, and one del Pezzo surface $dP_6$, meeting in an octahedron of lines.

The octahedredal graph projects to three points and six 2-spheres in $\mathbb{HP}^2$. The points are joined by three quaternionic lines $m_i = S_i^4$, $i = 1, 2, 3$.

Each $m_i \cap S^5$ is a circle in $\mathbb{HP}^2$ that determines a real quadric in the twistor space $\mathbb{CP}^3_i$.

$dP_6$ will be determined by the eigenvalues of $s$, and is invariant by a maximal torus of SU(3).
3.1 Spinors on $\mathbb{H}P^2$

The spin bundles over $\mathbb{H}P^n$ satisfy $\Delta_+ - \Delta_- = \Lambda_0^n(E - H)$. For $n = 2$,

$$\Delta_+ \cong S^2H \oplus \Lambda_0^2E, \quad \Delta_- \cong E \otimes H \cong T_{\mathbb{H}P^2}.$$

A section of $S^2H$ defines an almost complex structure, one of $\Lambda_0^2E$ defines a reduction to Sp$(1)^3/\mathbb{Z}_2$, splitting each tangent space into two Cayley 4-planes. Neither can exist globally over $\mathbb{H}P^2$.

**Proposition.** Let $G$ be Sp$(2) \times$ Sp$(1)$ or U$(3)$. Then $\mathbb{H}P^2$ possesses $G$-invariant Spin$(7)$ structures.

The proof uses the tensors of degrees 1,2,3. For U$(3)$, we use the sections

- $\zeta_H$ of $S^2H$, vanishing only on $S^5$,
- $(\eta_E \wedge j\eta_E)_0 \sim (X \otimes X)_5$ of $\Lambda_0^2E$, vanishing only on $\mathbb{C}P^2$.

Let $t = ||\zeta_H||^2 \in [0, b]$. Choose $\delta = f(t)\phi_E + g(t)\zeta_H$, where $f(0)$ and $g(b)$ are non-zero. This defines an Sp$(1)^2U(1)$ structure at generic points of $\mathbb{H}P^2$. 
3.2 Spinors, forms, and metrics

Given \((M^8, g)\) with a unit spinor \(\delta \in \Delta_+\), one can project its square

\[
\delta \otimes \delta \in S^2\Delta_+ \cong \Lambda^0 \oplus \Lambda^2 \oplus \Lambda^4_+
\]

to obtain a 4-form \(\Psi\) in \(\Lambda^4_+\). The holonomy of \(g\) reduces to Spin(7) iff \(d\Psi = 0\). However, to build up a full stock of metrics that are not conformal to a given one, we need more general 4-forms including ASD ones.

Consider the case of \(\text{Sp}(2) \times \text{Sp}(1)\) acting on \(\mathbb{HP}^2\) with singular orbits a point \(o\) and \(\mathbb{HP}^1 = S^4\), the zero sets of tautological sections \(u_E, u_H\) of \(E, H\) \((k = 1)\). Then \(\mathbb{HP}^2 \setminus \{o\}\) can be identified with the spinor bundle over \(S^4\), and admits a complete metric \(g_{BS}\) with holonomy Spin(7). Its relationship with the symmetric metric \(g_{QK}\) is well known:

\[
g_{BS} = 4(r+1)^{-2/5} g_{\text{ver}} + 5(r+1)^{3/5} g_{\text{hor}}, \quad g_{QK} = 4(r+1)^{-1} g_{\text{ver}} + 5(r+1)^{-1} g_{\text{hor}}.
\]

The \(4+4\) splitting of \(T_m\mathbb{HP}^2\) away from \(o\) is determined by \((u_E \wedge j u_E)_0\) in \(\Lambda^2_0 E\).
3.3 Closed four-forms

Away from $o$, $u_E$ determines sections $\Omega, \Omega_{14}, \Omega_{5}$ of constant QK-norm in three summands of

$$\Lambda^4 T^*_m \mathbb{HP}^2 = \underbrace{\Lambda_1 \oplus \Lambda_5 \oplus \Lambda_{14} \oplus \Lambda_{15}}_{+} \oplus \underbrace{\Lambda_5 \oplus \Lambda_{30}}_{-},$$

since $\Lambda_5 \simeq \Lambda^2_0 E$ and $\Lambda_{14} \simeq S^2_0 (\Lambda^2_0 E)$. Here $\Omega$ is the closed QK 4-form, and $d\Omega_{14} = -5dr \wedge \Omega_{14}$. It is easy to construct closed 4-forms, but not useful ones. The stabilizer of

$$\psi_{a,b,c} = a\Omega_{14} + b\Omega + c\Omega_{5}$$

is $\text{Spin}(7)$ if $(a + 8b)(3a + 4b) = 4c^2$ (with $a > 2b$ and $a + 3b > |c|$). This forces $a, b, c$ to be constant multiples of a function of the radius $r = \|u_H\|^2/(1 - \|u_H\|^2)$:

**Proposition.** If $(a, b, c) = (-\frac{56}{5}, -\frac{3}{5}, 12)(r + 1)^{16/5}$ then $\psi_{a,b,c}$ is the closed 4-form associated to $g_{BS}$. 
3.4 Spin(7) holonomy with Sp(2) (digression)

The complete AC Spin(7) metric $g_{BS}$ is asymptotic to a cone over squashed $S^7_{sq}$ and invariant by $\text{Sp}(2) \times \text{Sp}(1)$. It is the ‘limit’ of a one-parameter family ($B_8$) of complete ALC Spin(7) metrics invariant by $\text{Sp}(2) \times \text{U}(1)$, and asymptotic to a circle of fixed radius $\ell$ times a cone over $\mathbb{C}P^3_{nK}$ [Cvetič-Gibbons-Lü-Pope, Bazaïkin].

An analogous family ($C_8$) of Spin(7) metrics exists on $K_{\mathbb{C}P^3}$ in which the role of $g_{BS}$ is played by Calabi’s metric with holonomy SU(4), similarly $K_F$ [CGLP, B].

The search for such packages of Ricci-flat metrics on 8-manifolds focusses attention on circle fibrations $nP^7 \rightarrow nK^6$ of Einstein manifolds. Such fibrations occur naturally over the two self-dual Einstein 4-manifolds:

$$S^7_{sq} \rightarrow \mathbb{C}P^3_{nK} \rightarrow S^4, \quad N_{0,1} \rightarrow F_{nK} \rightarrow \mathbb{C}P^2.$$ 

Nearly parallel G$_2$ metrics come in three types because their cone can have holonomy Spin(7), SU(4), or Sp(2) if $nP^7$ is 3-Sasakian [Boyer-Galicki]. The latter can be deformed along the 3 Killing fields, giving rise to a system of 3 ODE’s.
3.5 The action of SU(3) on non-compact 8-manifolds

Metrics with holonomy Spin(7) and SU(3) symmetry were conjectured and studied by [Gukov-Sparks, G-S-Tong, Kanno-Yasui].

Let $W$ denote the normal bundle of either singular orbit $S^5$ or $\mathbb{CP}^2$ in $\mathbb{HP}^2$. Work of [Reidegeld, Bazaïkin], and [Foscolo-Haskins-Nordström] for $G_2$, has culminated in

**Theorem** [Lehmann]. $W$ admits a complete AC Spin(7) metric, invariant by $U(3)$, asymptotic to a cone over $N_{1,0}$, AND a 1-parameter family $g_\ell$ of ALC Spin(7) metrics, each asymptotic to a cone over $F = SU(3)/T^2$ times a circle of radius $\ell$. As $\ell \to 0$, the space collapses to $\Lambda^2 \mathbb{CP}^2$ with its $G_2$ metric.

More ALC/AC Spin(7) packages exist with $nP^7 = N_{k,l}$ [Chi].

The self-dual Einstein set-up can be extended to the case in which $M^4$ is an orbifold, in particular a QK quotient of $\mathbb{H}^n$ [Foscolo], but $M^4$ should itself be the base of a circle fibration for collapse with bounded curvature.
3.6 Work in progress

**Spaces associated to the Hitchin orbifolds.** Let SO(3) act irreducibly on $S^4$. There is a family of SO(3)-invariant self-dual Einstein orbifold metrics $M_k$ (with a $\mathbb{Z}_{k-2}$ singularity along $\mathbb{RP}^2$) [Hitchin, Tod]. $M_4$ can be identified with $\mathbb{CP}^2/\langle \sigma \rangle$, and its twistor space is a cubic surface in $\mathbb{CP}^4$ defined by the unique SU(2) invariant in $S^3(S^4(\mathbb{C}^2)) \cong \Lambda^3(S^6(\mathbb{C}^2))$. The ‘same’ invariant defines the 3-form on the Berger space $B^7 = SO(5)/SO(3)$, whose cone has Spin(7) holonomy.

$B^7$ is diffeomorphic to an $S^3$ bundle over $S^4$ [Goette-Kitchloo-Shankar], but more to the point, there are nearly-free $S^3$ actions giving orbifold fibrations

$$
\begin{array}{ccc}
B^7 & \xrightarrow{\sim} & S^7 \subset g_2/\mathfrak{so}(4) \\
\downarrow & & \downarrow \\
M_5 & & 
\end{array}
$$

All the 3-Sasakian spaces associated to the $M_k$ have cohomogeneous-one actions by SO(4), and are candidates for admitting a metric with positive sectional curvature [Grove-Wilking-Ziller]. This motivates a study of SO(4)-invariant nearly parallel metrics on such 7-manifolds [S-Singhal].
Closed 4-forms on 8-manifolds. There must exist $U(3)$-invariant closed 4-forms with stabilizer $Spin(7)$ on domains of $\mathbb{HP}^2$, but their components may be linear combinations of invariants occurring amongst all 6 components of $\Lambda^4 T^*\mathbb{HP}^2$.

The normal bundle of $L = SU(3)/SO(3)$ has no $U(3)$-invariant metrics with $Spin(7)$ holonomy. On the other hand, $G_2/SO(4)$ admits free families of closed non-parallel 4-forms with stabilizer $Sp(2)Sp(1)$ [Conti-Madsen-S]. The analogous statement for $\mathbb{HP}^2$ is open.

Nearly Spin(7) metrics? It is tempting to look for special classes of $Spin(7)$ metrics with non-zero $d\Psi \in \Lambda^5 = \Lambda^5_8 \oplus \Lambda^5_{48}$.

A naïve class consists of Einstein metrics with $d\Psi \in \Lambda^5_8$, including the sine cone over $nP^7$. On the other hand, any 5-form has a rank relative to the isomorphism

\[
\Lambda^5 \cong \Lambda^1 \otimes \Lambda^4_7 \quad (56 = 8 \times 7),
\]

which helps highlight degenerate $Spin(7)$ orbits.