Pseudoholomorphic Curves in the 6-sphere

Geometry, examples, singularities

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Some references (ordered by date):

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Background

The unit sphere $S^6 \subset \operatorname{Im} \mathbb{O}$ is homogeneous under the action of $\mathrm{G}_2 = \operatorname{Aut}(\mathbb{O})$, which preserves a (non-integrable) almost-complex structure J on S^6. [Kirchhoff]

$$\begin{array}{ccc} \mathrm{SU}(3) & \longrightarrow & \mathrm{G}_2 \\ & & \downarrow \\ & & S^6 \end{array}$$

Curves: Any real-analytic mapping $u : \mathbb{R} \to S^6$ extends to a pseudo-holomorphic map $z : D \to S^6$ on some open $\mathbb{R}$-neighborhood $D \subset \mathbb{C}$.

Higher dimensions?: There are no almost-complex 4-dimensional submanifolds $X \subset S^6$. [Gray]

$\mathrm{G}_2 = \operatorname{Aut}(S^6, J)$: If $U \subset S^6$ is open, connected, and $f : U \to S^6$ is almost-complex, then either f is constant or f is the restriction to U of left action by an element of G_2.

Some properties and examples of pseudoholomorphic curves $C \subset S^6$:

- $C \subset S^6$ is a (branched) minimal surface. (They can't be all calibrated, even locally.)
- The cone $[0,\infty)\cdot C \subset \operatorname{Im}\mathbb{O} = \mathbb{R}^7$ is associative (and hence minimizing [Harvey-Lawson]).
- Conversely, for any associative cone $[0,\infty)\cdot C \subset \operatorname{Im}\mathbb{O}$, the link $C \subset S^6$ is a pseudoholomorphic curve. (Models for singularities of associative 'varieties'.)
- $C = E \cap S^6$ is almost complex, where $E \in \mathrm{Assoc} \subset \mathrm{Gr}(3, \operatorname{Im}\mathbb{O})$. (A geodesic 2-sphere.)
- A surface $C \subset u^\perp \cap S^6 \simeq S^5$ for $u \in S^6$ is pseudoholomorphic if and only if the cone on C is special Lagrangian in $u^\perp \simeq \mathbb{C}^3$.
- There is a linearly full (connected) pseudoholomorphic curve $\Sigma \subset S^6$ with constant Gauss curvature $K = \frac{1}{6}$. (So Σ has area 24π.) Note: Σ is homogeneous under a maximal subgroup $\mathrm{SO}(3) \subset \mathrm{G}_2$ that acts irreducibly on $\operatorname{Im}\mathbb{O}$. [Sekigawa]

'Frenet frame' for non-linear, connected, pseudoholomorphic compact curve $\phi : C \to S^6$:

- There exists an orthogonal splitting $\phi^*(T^{(1,0)}S^6) = T_\phi \oplus N_\phi \oplus B_\phi$ into complex line bundles over C and holomorphic structures on those line bundles together with holomorphic sections

$$\tau \not\equiv 0 \text{ of } T_\phi \otimes K_C, \qquad \nu \not\equiv 0 \text{ of } N_\phi \otimes T_\phi^* \otimes K_C, \qquad \textit{and} \qquad \beta \text{ of } B_\phi \otimes N_\phi^* \otimes K_C.$$

  τ is the 'first fund. form', ν is 'second fund. form', and β is 'third fund. form'.

- $T_\phi \otimes N_\phi \otimes B_\phi \simeq \phi^*\big(\Lambda^3_C(T^{(1,0)}S^6)\big) \simeq \mathbb{C}$, so, if $\beta \not\equiv 0$, then there is an equation of divisors

$$3\,[\tau] + 2\,[\nu] + 1\,[\beta] = 6\,[K_C].$$

- $\deg(K_C) = 2g(C) - 2$ and $[\tau]$, $[\nu]$ and $[\beta]$ are effective.
  - If $g(C) = 0$, then $\beta \equiv 0$.
  - If $g(C) = 1$ and $\beta \not\equiv 0$, then $[\tau] = [\nu] = [\beta] = 0$.
  - If $g(C) > 1$ and $\beta \not\equiv 0$, then $[\tau]$, $[\nu]$, and $[\beta]$ are 'bounded'.

The binormal lift:

- An oriented 2-plane $E = x \wedge y$ in $\operatorname{Im}\mathbb{O}$ corresponds to a unique null line $[x + iy]$ in $Q_5 \subset \mathbb{P}(\operatorname{Im}\mathbb{O} \otimes \mathbb{C}) \simeq \mathbb{CP}^6$. E is complex with respect to $\pi(E) = \dfrac{x \times y}{|x \times y|}$ in S^6.
- The fiber $\pi^{-1}(u) \subset Q_5$ for $u \in S^6$ is $\mathbb{P}(T_u S^6) \simeq \mathbb{CP}^2$.
- For $x \in C$, $B_\phi(x)$ is an oriented 2-plane in $\operatorname{Im}\mathbb{O}$, so we have a lifting $b_\phi : C \to Q_5$.

$$\begin{array}{ccc} B_\phi & \longrightarrow & Q_5 \\ \downarrow & & \downarrow \pi \\ C & \xrightarrow{\ \phi\ } & S^6 \end{array}$$

- b_ϕ is holomorphic if and only if $\beta = 0$, in which case, $b_\phi : C \to Q_5$ is tangent to a holomorphic 2-plane field $D \subset TQ_5$.
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  - If $\psi : C \to Q_5$ is a holomorphic D-curve, then $\pi \circ \psi : C \to S^6$ is pseudoholomorphic.
  - $\psi(C) \subset Q_5$ is linear if and only if $\pi \circ \psi(C) \subset S^6$ is linear.
  - Otherwise, ψ is the binormal lift of $\pi \circ \psi$.

Nonlinear D-curves:

- The set of D-lines $L \subset Q_5 = \mathrm{G}_2^{\mathbb{C}}/P_1$ is a compact homog. space $\Lambda = \mathrm{G}_2^{\mathbb{C}}/P_2$ of dim 5.
- A nonlinear D-curve $\psi : C \to Q_5$ is a non-degenerate curve in $\mathbb{CP}^6$. The ramification divisors of ψ are

$$R_1 = R_3 = R_4 = R_6 = [\tau] \qquad \text{and} \qquad R_2 = R_5 = [\nu],$$

  where τ and ν are computed for $\pi \circ \psi : C \to S^6$.
- When C is compact and $\psi : C \to Q_5$ is a nonlinear D-curve, its degree is at least 6, and the area of $\pi \circ \psi(C) \subset S^6$ is $4\pi \deg\bigl(\psi(C)\bigr)$. (See also [Fernández].)
- **Example:** The pseudoholomorphic $\phi : \Sigma \hookrightarrow S^6$ with $K \equiv \frac{1}{6}$ is homogeneous with $g = 0$, so $\beta = 0$ and $[\tau] = [\nu] = 0$. Hence, the lift $b_\phi : \Sigma \to Q_5$ must be a rational normal D-curve.
- The space of rational normal D-curves in Q_5 is $X = \mathrm{G}_2^{\mathbb{C}}/\mathrm{SO}(3,\mathbb{C})$, a non-compact complex homogeneous space of dimension 11. This $\mathrm{SO}(3,\mathbb{C}) \subset \mathrm{G}_2^{\mathbb{C}}$ acts irred. on $\mathrm{Im}\,\mathbb{O} \otimes \mathbb{C} \simeq \mathbb{C}^7$. (See also [Hashimoto, Fernández].)

Application: There exist a smooth pseudoholomorphic curve $C \subset S^6$ of genus 0 and total area 24π and an associative plane $E \subset \mathrm{Im}\,\mathbb{O}$ such that $C \cap E$ is a single point. Thus, the cone on C unioned with E is an associative cone with a single ray of singularities.

Proof:

- The generic rational normal D-curve in $X = \mathrm{G}_2^{\mathbb{C}}/\mathrm{SO}(3,\mathbb{C})$ is *not* invariant under conjugation.
- Hence its projection to $C \subset S^6$ is embedded but *not invariant under the antipodal map*. (Unlike the ones of constant Gauss curvature, a homogeneous space $X^{\mathbb{R}} = \mathrm{G}_2/\,\mathrm{SO}(3)$.)
- Choose $x \in C$ with $-x \notin C$. The 4-parameter family $F \simeq \mathbb{CP}^2$ of associative planes E that contain $\{x, -x\}$ foliates $S^6 \setminus \{x, -x\}$.
- Each $y \in C \setminus \{x\}$ lies in a unique $E_y \in F$, and there is only a 2-parameter family of such E_y.
- The generic $E \in F$ meets C only at x.

Remark: Clearly, one can arrange, in addition, that $E \cap S^6$ and C not be tangent at x.

Further results and open problems:

- Every compact Riemann surface has a non-constant pseudoholomorphic map $\phi : C \to S^6$.
- The space M_d of nonlinear rational pseudoholomorphic curves $C \subset S^6$ of total area $4\pi d$ is non-empty for $d = 6, 8, 9, 10, \ldots$ and has (complex) dimension $d + 5$ and is non-compact. [Fernández]
- The Gromov compactification of $M_6 = \mathrm{G}_2^{\mathbb{C}}/\mathrm{SO}(3, \mathbb{C})$ adds a collection of unions of linear pseudoholomorphic S^2s in S^6 of total area 24π (counting multiplicity). Exactly which 'arrangements' appear is not yet clear.
- It appears likely that that there are elements $C \in M_6$ that have exactly one self-intersection. The cone on such a curve would be an irreducible associative cone whose singlarity locus is a half-line.