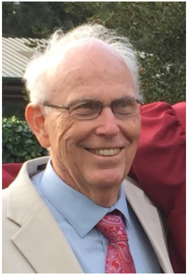
PSEUDO-CONVEXITY FOR THE SPECIAL LAGRANGIAN POTENTIAL EQUATION

with Reese Harvey



SOME HISTORY

Calibrations

A **Calibration** on a manifold X is a smooth k-form $\varphi \in \mathcal{E}^k(X)$ with $d\varphi = 0$ such that

$$\varphi\big|_P \ \leq\ d\mathrm{vol}_P$$

for all oriented tangent k-planes P on X.

$$\mathbf{G}(\varphi) \ =\ \{P \in \mathrm{Grass}^{\mathrm{or}}(k, X) : \varphi\big|_P \ =\ d\mathrm{vol}_P\}$$

An oriented k-dimensional submanifold $M \subset X$ is a **φ-submanifold** if

$$T_x M \in \mathbf{G}(\varphi) \qquad \forall\, x \in M.$$

Calibrations

PROPOSITION. *A φ-submanifold is homologically volume minimizing.*

Proof. If M' is homologous to the φ-submanifold M, then

$$\mathrm{vol}(M') \ \geq \ \int_{M'} \varphi \ = \ \int_M \varphi \ = \ \mathrm{vol}(M).$$

This Proposition extends from submanifolds to integral currents.

Kähler Manifolds

Herb Federer: (using Wirtinger's inequaity)

If ω is the Kähler form on a Kähler manifold X, then

$$\frac{1}{k!}\omega^k \text{ is a calibration.}$$

The $\frac{1}{k!}\omega^k$-submanifolds are **complex submanifolds**.

The (locally closed) $\frac{1}{k!}\omega^k$-currents are **positive holomorphic chains**.

Jim King

The Special Lagrangian Calibration

In the late seventies Reese Harvey and I looked for more calibrations with large sets of φ-submanifolds.

One which we found:

$$\varphi \equiv \mathrm{Re}\{\mathbf{dz_1} \wedge \cdots \wedge \mathbf{dz_n}\} \qquad \text{in} \ \ \mathbb{C}^n$$

I will work in $\mathbb{C}^n$,

but, as we knew, everything carries over to **Calabi-Yau** manifolds.

The Special Lagrangian Calibration

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One which we found:

$$\varphi \equiv \mathrm{Re}\{\mathbf{dz_1} \wedge \cdots \wedge \mathbf{dz_n}\} \qquad \text{in } \mathbb{C}^n$$

This is a sum of 2^{n-1} orthogonal simple vectors.

Nevertheless, for a **unit simple** vector $\xi = e_1 \wedge \cdots \wedge e_n$

$$|\{dz_1 \wedge \cdots \wedge dz_n\}(\xi)| \leq 1$$

and $|\{dz_1 \wedge \cdots \wedge dz_n\}(\xi)| = 1 \iff \xi$ is **Lagrangian**.

The Special Lagrangian Calibration

Thus for a **unit simple** vector $\xi = e_1 \wedge \cdots \wedge e_n$

$$\mathrm{Re}\{dz_1 \wedge \cdots \wedge dz_n\}(\xi) \ = \ \pm 1 \qquad \iff$$

$$\xi \ \text{ is Lagrangian} \quad \text{and} \quad \mathrm{Im}\{\mathbf{dz_1} \wedge \cdots \wedge \mathbf{dz_n}\}(\xi) \ = \ \mathbf{0}$$

The Special Lagrangian Calibration

Otherwise said, for an oriented real n-plane P

$$\mathrm{Re}\{dz_1 \wedge \cdots \wedge dz_n\}\big|_P \;=\; \pm d\mathrm{vol}_P \qquad \iff$$

$$\mathbf{P} \text{ is Lagrangian} \quad \text{and} \quad \mathrm{Im}\{\mathbf{dz_1} \wedge \cdots \wedge \mathbf{dz_n}\}\big|_{\mathbf{P}} \;=\; \mathbf{0}$$

Lagrangian Graphs

Let $z = x + iy$, and consider a submanifold of the form

$$M^n \equiv \{(x, y) : y = F(x), x \in \Omega\}.$$

where Ω is a simply-connected domain.

Lemma.

$$M^n \text{ is Lagrangian} \quad \iff \quad \exists\, u : \Omega \to \mathbb{R} \text{ s.t. } F = \nabla u.$$

The tangent space to $M^n = \mathrm{graph}(F)$ at x is just the graph of $(DF)_x$.

W.r.t. coordinates $(x_1, ..., x_n, y_1, ..., y_n)$ the matrix is just

$$\left(\left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)\right)$$

Lagrangian Graphs

This matrix is symmetric, so under a change of variables (gx, gy) we get

$$\left(\left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)\right) = \begin{pmatrix} \lambda_1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \lambda_n \end{pmatrix}$$

The graph of this in $\mathbb{C}^n = \mathbb{R}^n \oplus J\,\mathbb{R}^n$ is spanned by

$$e_1 + \lambda_1 J e_1,\ \ldots,\ e_n + \lambda_n J e_n$$

The corresponding simple vector is

$$\xi = (e_1 + \lambda_1 J e_1) \wedge \cdots \wedge (e_n + \lambda_n J e_n).$$

The SL Potential Equation

$$\xi \;=\; (e_1 + \lambda_1 J e_1) \wedge \;\cdots\; \wedge (e_n + \lambda_n J e_n).$$

$$\{\mathbf{dz_1} \wedge \cdots \wedge \mathbf{dz_n}\}(\xi) \;=\; \prod_{\mathbf{k=1}}^{\mathbf{n}} (\mathbf{1} + \mathbf{i}\,\lambda_{\mathbf{k}})$$

$$\mathrm{Im}\big[\{dz_1 \wedge \cdots \wedge dz_n\}(\xi)\big] \;=\; \sum_{k\geq 0} (-1)^{k+1}\, \sigma_{2k+1}(\lambda_1, \ldots \lambda_n)$$

THE SPECIAL LAGRANGIAN POTENTIAL EQUATION

$$\sum_{\mathbf{k}\geq \mathbf{0}} (-\mathbf{1})^{\mathbf{k+1}}\, \sigma_{\mathbf{2k+1}}(\mathbf{D^2\,u}) \;=\; \mathbf{0}$$

Example $(n = 3)$ $\quad \mathrm{tr}(D^2 u) = det(D^2 u) \quad$ i.e. $\quad \Delta u = MA(u).$

Circular Symmetry

Note that there is an S^1**-symmetry**

If we set

$$\varphi \ \equiv \ \mathrm{Re}\{e^{-i\theta} dz_1 \wedge \cdots \wedge dz_n\},$$

Then we get the equation

$$\mathrm{Im}\big[\{e^{-i\theta} dz_1 \wedge \cdots \wedge dz_n\}(\xi)\big] \ = \ 0, \quad \text{i.e.,}$$

$$\mathrm{Im}\left\{ \mathbf{e}^{-\mathbf{i}\theta} \prod_{\mathbf{k}=1}^{\mathbf{n}} (\mathbf{1} + \mathbf{i}\,\lambda_{\mathbf{k}}) \right\} \ = \ \mathbf{0}$$

A Different Way to Write the Equation

Caffarelli, Nirenberg and Spruck

$$\mathrm{Im}\left\{e^{-i\theta}\prod_{k=1}^{n}(1+i\,\lambda_k)\right\} \;=\; 0$$

Consider

$$1+i\lambda_k \in \mathbb{C}.$$

Then

$$\lambda_k \;=\; \tan\theta_k \qquad \text{where} \qquad -\frac{\pi}{2} \;<\; \theta_k \;<\; \frac{\pi}{2}$$

Now

$$\log\prod_{k=1}^{n}(1+i\,\lambda_k) \;=\; \sum_{k=1}^{n}\log(1+i\lambda_k) \;=\; R+i\Theta$$

$$\Theta \;=\; \sum_{k=1}^{n}\arg(1+i\lambda_k) \;=\; \sum_{k=1}^{n}\theta_k \;=\; \sum_{k=1}^{n}\arctan(\lambda_k).$$

The Special Lagrangian Potential Operator

$$\mathrm{Im}\left\{e^{-i\theta}\prod_{k=1}^{n}(1+i\,\lambda_k)\right\} \;=\; \mathrm{Im}\left\{e^{-i\theta}e^{R+i\Theta}\right\} \;=\; e^{R}\mathrm{Im}\left\{e^{-i\theta}e^{i\Theta}\right\} \;=\; 0$$

So we have

$$\Theta \;=\; \sum_{k=1}^{n}\arctan\lambda_k \;=\; \theta \pmod{\pi}$$

Definition. For $A \in \mathrm{Sym}^2(\mathbb{R}^n)$ we set

$$\mathbf{SL(A)} \;\equiv\; \mathrm{tr}\,\{\arctan(\mathbf{A})\}$$

$SL(A)$ takes values in $(-n\frac{\pi}{2}, n\frac{\pi}{2})$.

Some Things We Knew in the CG Paper

There exist many solutions to these equations

The linearization at a solution is elliptic

Given a smooth solution to the Dirichlet Problem, there are solutions for all nearby boundary values.

A Lagrangian submanifold has mean curvature 0

$\iff$ it is Special Lagrangian for some phase θ

The Dirichlet Problem

Caffarelli, Nirenberg and Spruck (1985)

$$\mathrm{tr}\left\{\arctan\left(D^2 u\right)\right\} = \theta$$

$$\theta \equiv \text{phase} \in \left(-n\frac{\pi}{2},\ n\frac{\pi}{2}\right).$$

$$-n\frac{\pi}{2} \quad -(n-1)\frac{\pi}{2} \qquad\qquad 0 \qquad\qquad (n-1)\frac{\pi}{2} \quad n\frac{\pi}{2}$$

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The operator is concave on $\{A : \mathrm{tr}(\arctan, A) \geq (n-1)\frac{\pi}{2}\}$.

The operator is convex on $\{A : \mathrm{tr}(\arctan, A) \leq -(n-1)\frac{\pi}{2}\}$.

The Dirichlet Problem

THEOREM. (CNS) *Let* $|\theta| \in ((n-1)\frac{\pi}{2}, n\frac{\pi}{2})$ *and consider a domain* $\Omega \subset \mathbb{R}^n$ *with a smooth, strictly convex boundary. Then for any smooth function* $\varphi \in C^\infty(\partial\Omega)$ *there exists a unique solution* $u \in C^\infty(\overline{\Omega})$ *to the Dirichlet Problem for*

$$\mathrm{tr}\left\{\arctan\left(D^2 u\right)\right\} = \theta$$

with $u\big|_{\partial\Omega} = \varphi$.

What about solutions for other phases?

Viscosity Solutions

Consider the subequation

$$\mathbf{F}_\theta \ \equiv\ \{A \in \mathrm{Sym}^2(\mathbb{R}^n) : \mathrm{tr}(\arctan A) \geq \theta\}.$$

The dual subequation is

$$\widetilde{\mathbf{F}}_\theta \ \equiv \sim \{-\mathrm{Int}\mathbf{F}_\theta\} \ =\ \mathbf{F}_{-\theta}$$

For $\Omega \subset \mathbb{R}^n$ let $\mathrm{USC}(\Omega)$ denote the u.s.c. functions $u : \Omega \to [-\infty, \infty)$.

Def. Fix a domain $\Omega \subset \mathbb{R}^n$ and $u \in USC(\Omega)$. By a **test function** for u at $x \in \Omega$ we mean a C^2-function ϕ in a neighborhood of x with

$$u \leq \phi \ \text{ near } x \qquad \text{and} \qquad u(x) \ =\ \phi(x).$$

Viscosity Solutions

$$\mathbf{F}_\theta \;\equiv\; \{A \in \mathrm{Sym}^2(\mathbb{R}^n) : \mathrm{tr}(\arctan, A) \geq \theta\}.$$

$$\widetilde{\mathbf{F}}_\theta \;\equiv \sim \{-\mathrm{Int}\mathbf{F}_\theta\} \;=\; \mathbf{F}_{-\theta}$$

Def. Fix a domain $\Omega \subset \mathbb{R}^n$ and $u \in USC(\Omega)$. Then u is **$\mathbf{F}_\theta$-subharmonic** if for every test function ϕ for u at any point $x \in \Omega$, we have

$$\mathbf{D}^2_{\mathbf{x}}\phi \in \mathbf{F}_\theta.$$

Def. $u \in C(\Omega)$ is **$\mathbf{F}_\theta$-harmonic**, i.e., a **viscosity solution of our equation**,

u is $\mathbf{F}_\theta$-subharmonic and $-u$ is $\widetilde{\mathbf{F}}_\theta$-subharmonic

Note : $\quad \partial\mathbf{F}_\theta \;=\; \mathbf{F}_\theta \cap (-\widetilde{\mathbf{F}}_\theta)$

Work of Wang and Yuan

These viscosity solutions are interesting.

Dake Wang and Yu Yuan found viscosity solutions in $\mathbb{R}^3$ to the SL potential equation,

for each θ in the interior interval $(-\frac{\pi}{2}, \frac{\pi}{2})$.

These solutions are C^∞ outside the origin, but only $C^{1,\alpha}$, and no more, across the origin.

Graphing the gradients gives infinitely many Special Lagrangian Varieties each with an isolated singularity

Continuous Solutions to the (DP)

Definitions hold for any non-empty closed $\mathbf{F} \subset \mathrm{Sym}^2(\mathbb{R}^n)$ satisfying

$$\mathbf{F} + \{A \geq 0\} \subset \mathbf{F}$$

Theorem. (Harvey-L. 2009) Uniqueness for the (DP). *For all bounded domains $\Omega \subset \mathbb{R}^n$ and $u, v \in \mathrm{USC}(\overline{\Omega})$ with*

$$u\big|_{\Omega} \;\; \mathbf{F} - \text{subharmonic} \qquad \text{and} \qquad v\big|_{\Omega} \;\; \widetilde{\mathbf{F}} - \text{subharmonic}$$

one has that

$$u + v \;\leq\; 0 \quad \text{on} \;\; \partial\Omega \qquad \Rightarrow \qquad u + v \;\leq\; 0 \quad \text{on} \;\; \Omega.$$

Theorem. (Harvey-L. 2009) Existence for the (DP). *Let $\Omega \subset\subset \mathbb{R}^n$ be a domain with smooth boundary $\partial\Omega$, If at each point $x \in \partial\Omega$ the boundary is both*

strictly $\mathbf{F}$*-convex* *and* *strictly* $\widetilde{\mathbf{F}}$*-convex*

then Perron existence holds for the (DP).

Strict **F**-Convexity of $\partial\Omega$

Def. The **asymptotic interior** of **F** is

$$\mathrm{Int}\,\overrightarrow{\mathbf{F}} \;\equiv\; \{A \in \mathrm{Sym}^2(\mathbb{R}^n) : \exists\, \epsilon > 0 \text{ and } t_0 \;\; s.t.\; t(A - \epsilon I) \in \mathbf{F}\; \forall\, t \geq t_0\}$$

This is an **open cone** with 0 as vertex.

Def. Let II_x be the second fundamental form of $\partial\Omega$ with respect to the inward-pointing normal at x. Then $\partial\Omega$ is strictly **F-convex** if for all $x \in \partial\Omega$

$$\begin{pmatrix} II_x & 0 \\ 0 & t \end{pmatrix} \in \;\mathrm{Int}\,\overrightarrow{\mathbf{F}} \qquad \forall\, t >> 1.$$

Equivalently: $\exists\, \rho \in C^\infty(\overline{\Omega})$ with $\Omega = \{\rho < 0\}$, $\rho = 0$ and $\nabla\rho \neq 0$ on $\partial\Omega$ with

$$D^2_x\rho \in \;\mathrm{Int}\,\overrightarrow{\mathbf{F}} \qquad \forall\, x \in \overline{\Omega}.$$

Compute $\mathrm{Int}\,\overrightarrow{\mathbf{F}}_\theta$

$$-n\frac{\pi}{2} \quad -(n-2)\frac{\pi}{2} \qquad\qquad 0 \qquad\qquad (n-2)\frac{\pi}{2} \quad n\frac{\pi}{2}$$

$$|-----|-------------|-------------|-----|$$

$$\mathbf{F}_{-\theta} \;=\; \widetilde{\mathbf{F}}_\theta \qquad \text{and} \qquad \mathbf{F}_\theta \subset \widetilde{\mathbf{F}}_{-\theta} = \widetilde{\mathbf{F}}_\theta \text{ for } \theta > 0$$

Yu Yuan showed $\quad \mathbf{F}_\theta$ is convex $\iff \theta \geq (n-2)\frac{\pi}{2}$.

Special Phases: $\theta_k = (n-2k)\frac{\pi}{2}$

Phase Intervals: $I_k = (\theta_k, \theta_{k-1}) \quad$ for $k = 1, ..., n-1$.

Compute $\mathrm{Int}\,\overrightarrow{\mathbf{F}}_\theta$

THEOREM. *$\overrightarrow{\mathbf{F}}_\theta$ (the closure of $\mathrm{Int}\,\overrightarrow{\mathbf{F}}_\theta$) for $\theta \in (-n\frac{\pi}{2}, n\frac{\pi}{2})$ is:*

(1) If $\theta \in I_k$, $k = 1, ..., n-1$,

$$\overrightarrow{\mathbf{F}}_\theta \;=\; \{A : \lambda_k(A) \geq 0\}$$

where $\lambda_1(A) \leq \lambda_2(A) \leq \cdots$ are the ordered eigenvalues of A.

(2) If $\theta = \theta_k$, $k = 1, ..., n-1$,

$$\overrightarrow{\mathbf{F}}_{\theta_k} \;=\; \{A : \lambda_k(\sigma_{n-1}(A)) \geq 0\}.$$

where $\lambda_k(\sigma_{n-1}(A))$ are the ordered eigenvalues of the Gårding polynomial σ_{n-1} on A. These eigenvalues are the negatives of the roots of $\sigma_{n-1}(tI + A)$.

Compute $\operatorname{Int} \overrightarrow{\mathbf{F}}_\theta$

THEOREM. $\overrightarrow{\mathbf{F}}_\theta$ *(the closure of* $\operatorname{Int} \overrightarrow{\mathbf{F}}_\theta$*) for* $\theta \in (-n\frac{\pi}{2}, n\frac{\pi}{2})$ *is:*

(1) If $\theta \in I_k, \quad k = 1, ..., n-1,$

$$\overrightarrow{\mathbf{F}}_\theta \ = \ \{A : \lambda_k(A) \geq 0\}$$

where $\lambda_1(A) \leq \lambda_2(A) \leq \cdots$ *are the ordered eigenvalues of A.*

(2) If $\theta = \theta_k, \quad k = 1, ..., n-1,$

$$\overrightarrow{\mathbf{F}}_{\theta_k} \ = \ \{A : \lambda_k(\sigma_{n-1}(A)) \geq 0\}.$$

where $\lambda_k(\sigma_{n-1}(A))$ *are the ordered eigenvalues of the Gårding polynomial* σ_{n-1} *on A.*

$$-n\frac{\pi}{2} \qquad -(n-2)\frac{\pi}{2} \qquad\qquad 0 \qquad\qquad (n-2)\frac{\pi}{2} \qquad n\frac{\pi}{2}$$

$$|----|-----------|-----------|----|$$

A Note

$\overrightarrow{\mathbf{F}}_{|\theta|}$ gets bigger as $|\theta| \searrow 0$.

So for these branches where solutions can be singular,

there exist solutions on many more domains.

Compute $\operatorname{Int} \overrightarrow{\mathbf{F}}_{\theta_k}$ for Special Phases θ_k

THEOREM. *Let* $\Lambda_k \equiv \{A : \lambda_k(A) \geq 0\}$. *Then*

$$\overrightarrow{\mathbf{F}}_{\theta_k} = \Lambda_k \sqcup S_k$$

where

$$S_k = (\Lambda_{k+1} \sim \Lambda_k) \cap \{\operatorname{sgn} \sigma_{n-1}(A) \operatorname{sgn} \sigma_n(A) = -1\}.$$

Note: The subequations Λ_k are **quite large**. The first $k-1$ eigenvalues can be < 0.

This gives (probably) the best geometric condition on $\partial\Omega$ for existence to the (DP) for

$$\operatorname{tr}\{\arctan(D^2 u)\} = \theta.$$

Generalizations I: Gårding Polynomials

A homogeneous polynomial $\mathfrak{g} : \mathrm{Sym}^2(\mathbb{R}^n) \to \mathbb{R}$ of some degree $m > 0$ is

Gårding hyperbolic wrt the Identity

if for each $A \in \mathrm{Sym}^2(\mathbb{R}^n)$

all the roots of the polynomial $\quad t \ \to \ \mathfrak{g}(tI + A) \quad$ are real.

The **Gårding eigenvlaues** are the negatives of the roots:

$$\lambda_1^{\mathfrak{g}}(A) \le \cdots \le \lambda_m^{\mathfrak{g}}(A)$$

The **Gårding cone** $\Gamma \equiv \{\lambda_1^{\mathfrak{g}}(A) \ge 0\}$ is convex

and we assume that

$$\Gamma + \{A \ge 0\} \ \subset \ \Gamma$$

SL Potential Operators Associated to $\mathfrak{g}$.

Given such a $\mathfrak{g}$, there is an associated SLP operator on $\Gamma \subset \mathrm{Sym}^2(\mathbb{R}^n)$

$$SL^{\mathfrak{g}}(A) \equiv \sum_{k=1}^{m} \arctan \lambda_k^{\mathfrak{g}}(A)$$

THEOREM. The existence, uniquenes of viscosity solutions to the (DP) above hold for $SL^{\mathfrak{g}}(D^2 u)$.

Furthermore, the computation of the asymptotic interior $\overrightarrow{\mathbf{F}}_{\theta}^{\mathfrak{g}}$ is completely analogous to the one above.

Example. $\mathbb{C}^n = (\mathbb{R}^{2n}, J)$.

Let $A_{\mathbb{C}} \equiv \frac{1}{2}(A - JAJ)$ and note that $[A_{\mathbb{C}}, J] = 0$.

Define $\mathfrak{g}(A) \equiv \det_{\mathbb{C}}(A_{\mathbb{C}})$

Quaternionic case is similar.

Generalizations II: The SL Curvature Equation

$$u : \Omega \to \mathbb{R}$$

a smooth function on an open set $\Omega \subset \mathbb{R}^n$.

$M \subset \Omega \times \mathbb{R}$ its graph.

$\kappa_1, ..., \kappa_n$ the **principal curvatures** of M.

Define

$$\mathbf{SL}^{\mathrm{curv}}(\mathbf{u}) \equiv \sum_{\mathbf{k}=1}^{\mathbf{n}} \arctan(\kappa_{\mathbf{k}})$$

Theorem. All of the above applies to this equation except the uniqueness conclusion.

Generalizations III: On Riemannian Manifolds

This equation can be viewed on a riemannian manifold
by replacing $D^2 u$ with the riemannian Hessian, Hess(u).

If one assumes that $\exists$ a strictly convex function on a neighborhood of $\overline{\Omega}$,
the basic theorems go through.

The Inhomogeneous Dirichlet Problem

$$\mathrm{tr}\left\{\arctan(D_x^2 u)\right\} \;=\; \psi(x) \qquad (*)$$

THEOREM. (T. Collins, S. Picard and X. Wu). *Let $\Omega \subset\subset \mathbb{R}^n$ have a C^4 boundary $\partial\Omega$. Let*

$$\varphi \in C^4(\partial\Omega)$$

$$\psi : \overline{\Omega} \to \left[(n-2)\frac{\pi}{2} + \delta,\ n\frac{\pi}{2}\right) \ \text{ for } \ \delta > 0 \ \text{ be } \ C^2.$$

Suppose $\exists\, \underline{u} \in C^4(\overline{\Omega})$ s.t.

$$\mathrm{tr}\left\{\arctan(D_x^2 \underline{u})\right\} \;\geq\; \psi(x) \qquad \text{and} \qquad \underline{u}\big|_{\partial\Omega} \;=\; \varphi.$$

Then the (DP) for (*) admits a unique $C^{3,\alpha}(\overline{\Omega})$-solution.

If all data are smooth, so is the solution.

Note 1. They also prove a complex version, replacing $D^2 u$ by $(D^2 u)_{\mathbb{C}}$

Note 2. S. Dinew, H. Do and T. D. Tô have proved the continuous viscosity version of this result.

The Inhomogeneous Dirichlet Problem

We have a general theorem about the (IDP) for a constant coefficient operator $\mathbf{f} \in C(\mathbf{F})$ on a subequation $\mathbf{F}$ with $\partial \mathbf{F} = \{\mathbf{f}(J) = c_0\}$. Consider

$$\mathbf{f}(D_x^2 u) \ = \ \psi(x) \qquad (*)$$

on a domain $\Omega \subset\subset \mathbb{R}^n$ where

$$\psi(\overline{\Omega}) \ \subset \ \mathbf{f}(\mathbf{F}).$$

Then **this Dirichlet problem is uniquely solvable for all $\varphi \in C(\partial\Omega)$** provided

(1) $\partial\Omega$ satisfies the boundary convexity condition,

(2) $\mathbf{f}$ is tamable.

Note 1. $\mathbf{f}$ is tamable if $\exists$ a strictly increasing function $\chi : \mathbf{f}(\mathbf{F}) \to \mathbb{R}$ such that $\mathbf{g} \equiv \chi \circ \mathbf{f}$ is tame, where tame means that for all $t > 0$, $\exists\, c(t) > 0$ s.t.

$$\mathbf{g}(A + tI) - \mathbf{g}(A) \ \geq \ c(t) \qquad \forall\, A \in \mathbf{F}.$$

Note 2. The SLP operator is tamable on $\mathbf{F}_\theta$ for $\theta > (n-2)\frac{\pi}{2}$.

The Geometry of the IDP for SL

Let $L \subset \mathbb{R}^n \times \mathbb{R}^n = \mathbb{C}^n$ be a Lagrangian submanifold

which is a graph over a domain in $\mathbb{R}^n \times \{0\}$.

Fix standard coordinates $z = (x, y)$ on $\mathbb{C}^n$ and set $dz = dz_1 \wedge \cdots \wedge dz_n$.

$$dz\big|_L \;=\; e^{i\theta} d\mathrm{vol}_L.$$

Then

$$\nabla\theta \;=\; -\mathbf{JH}$$

where H is the mean curvature vector field of L and J is the cx. str.

θ is the phase angle of the tangent spaces of L.

The Geometry of the IDP for SL

Let u be a smooth solution of

$$\mathrm{tr}\left\{\arctan(\mathbf{D}^2_{\mathbf{x}})\right\} = \psi(\mathbf{x}) \qquad \text{on} \ \ \Omega \subset \mathbb{R}^n$$

Let $\mathbf{L} = \mathrm{graph}(\mathbf{D}\,\mathbf{u})$ and $\widetilde{\psi}$ the pull-back of ψ to L.

Then $\qquad \nabla\widetilde{\psi} = -JH, \qquad$ i.e.,

$$\mathbf{J}\nabla\widetilde{\psi} = \mathbf{H}.$$

So ψ is related to the mean curvature of the graph L.

Some Nice Results

Theorem. (Simon Brendle and Micah Warren). *Let* $\Omega, \widetilde{\Omega} \subset\subset \mathbb{R}^n$ *be two domains with smooth strictly convex boundaries (2nd Fund Forms* > 0*). Then there exists a diffeomorphism*

$$F : \Omega \longrightarrow \widetilde{\Omega}$$

whose graph is Special Lagrangian.

A Bernstein Theorem

Theorem. (Yu Yuan and for $n = 2$ Lei Fu).

Let u be a smooth solution, over all of $\mathbb{R}^n$, to the equation

$$\mathrm{tr}\left\{\arctan(D^2 u)\right\} \ = \ \theta \qquad \text{with} \ \ |\theta| > (n-2)\frac{\pi}{2}$$

(in the critical interval). Then u is a quadratic polynomial.

Work of: R. McLean, N. Hitchin, D. Joyce, S. Donaldson, M.-T. Wang, R. Schoen and J. Wolfson

Much work has been on this equation. For example:

McLean and Hitchin have important work on the moduli space of Special Lagrangians

Joyce has worked extensively with many important results

Donaldson gave a moment-map interpretation

The SLP equation plays a big role in mean curvature flow – See Wang's survey

Schoen and Wolfson studied the problem of minimizing mass among Lagrangians

The Degenerate Special Lagrangian Equation

In a program of Jake Solomon, this equation **governs geodesics** for a metric on the space of positive Lagrangians in Calabi-Yau manifolds.

This is analogous to programs in the complex Monge-Ampére case.

Much work has been done by Solomon, Yanir Rubinstein, Tamás Darvas, and Matt Dellatorre.

Mirror Symmetry

The SL potential equation plays a big role in mirror symmetry.

This began with the paper of A. Strominger, S.-T. Yau and E. Zaslow [SYZ] which gave a very geometric picture of how mirror manifolds are connected.

Special Lagrangians and the SL potential equation are critical in this tableau.

There are very good articles by T. Collins, A. Jacob, C. Leung, D. Xie, S.-T. Yau, and E. Zaslow.

Mirror Symmetry

A small insight comes from the following. Let (X, ω) be an n-dimensional Kähler manfold and $a \in H^{1,1}(X, \mathbb{R})$ a fixed (1,1)-homology class. One is interested in finding an element $\alpha \in a$ such that

$$\mathrm{Im}\left(e^{-i\theta}(\omega + i\alpha)^n\right) \;=\; 0.$$

The angle θ is determined topologically by

$$\theta \;=\; \arg\left\{\int_X (\omega + i\alpha)^n\right\}.$$

This gives rise to a hermitian Yang-Mills equation

$$\Theta_\omega(\alpha) = \sum_k \arctan(\lambda_k) \;\equiv\; \theta \pmod{2\pi}$$

where the λ_k's are eigenvalues of an endomorphism $K : T^{1,0}X \to T^{1,0}X$ given by contracting by α and the dual of ω. Of course the elements in a all differ from a given one α_0 by $dd^c u$ for a function u on X.

Mirror Symmetry

Collins and Yau have recently studied geodesics on an infinite dimensional manifold which is mirror to Solomon's.

This is aimed at understanding the deformed Hermitian-Yang-Mills equation which is "mirror" to the SL potential equation.