Local (complete noncompact) G_2 holonomy spaces

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Local G_2-spaces from local Calabi–Yau 3-folds

Main Theorem: Foscolo–H–Nordström arXiv:1709.04904

Let $(B, g_0, \omega_0, \Omega_0)$ be an irreducible asymptotically conical Calabi–Yau 3–fold asymptotic to the Calabi–Yau cone $C(\Sigma)$ over a smooth Sasaki–Einstein 5–manifold Σ. Let $M^7 \to B^6$ be a principal $U(1)$–bundle such that $c_1(M) \neq 0$ but $c_1(M) \cup [\omega_0] = 0 \in H^4(B)$.

Then there exists $\epsilon_0 > 0$ such that for every $\epsilon \in (0, \epsilon_0)$ the noncompact 7–manifold M carries an $\mathbb{S}^1$–invariant torsion-free G_2–structure φ_ϵ such that the Riemannian metric g_ϵ on M induced by φ_ϵ is complete with (restricted) holonomy equal to G_2. Moreover g_ϵ collapses (with bounded curvature) as $\epsilon \to 0$ to (B, g_0).

The asymptotic geometry of these metrics will be described in more detail later in the talk; it generalises the asymptotic of ALF hyperKähler spaces like (multi)-Taub-NUT and Atiyah–Hitchin metrics.

Physically our Theorem gives a very general existence result for M-theory lifts of Type IIA supergravity solutions with only Ramond–Ramond flux (but no $D6$-branes).

Why the Main Theorem is so powerful

- It works for *any* asymptotically conical (AC) Calabi–Yau 3-fold B (independent of the details of the method used to find it).
- It does not rely on understanding any *submanifold geometry* of B, e.g. *special Lagrangian submanifolds* of B. Currently we lack tools to construct special Lagrangian 3-folds in general AC Calabi–Yau 3-folds.
- A single (relatively simple) condition to verify to guarantee a solution.
  - In some cases (any small resolution) the condition is automatic;
  - otherwise it places nontrivial constraints on the permitted Kähler classes within the Kähler cone.
  - In concrete cases we can work out those constraints explicitly, e.g. $B = K_{\mathbb{P}^1 \times \mathbb{P}^1}$ (see later slide).
- Complex Monge-Ampère methods are now able to construct many asymptotically conical Calabi–Yau 3-folds.
  - The most delicate part is in fact understanding *Calabi–Yau cone metrics*, or equivalently *Sasaki–Einstein metrics*; recently a suitable notion of *stability* has been proven necessary and sufficient.

Asymptotic geometry: from ALF to ALC

Key feature: the metric on an ALF space M^4 like Taub–NUT is asymptotically a circle bundle over an exterior domain in $\mathbb{R}^3$ (or in $\mathbb{R}^3/\mathbb{Z}_2$) where the circle fibre has asymptotically constant length.

In higher dimensional generalisations, often called ALC spaces (asymptotically locally conical), we:

- replace the base $\mathbb{R}^3$ with a Riemannian cone $C = C(\Sigma)$
- we take the cone $C(\Sigma)$ to be Ricci-flat if we want to consider Ricci-flat ALC spaces
- $C(\Sigma)$ should be a Calabi–Yau cone if we want to consider ALC G_2 spaces
- $C(\Sigma)$ is Calabi–Yau iff Σ is Sasaki–Einstein.
- Existence of Sasaki–Einstein metrics has close connections to existence of KE metrics with positive scalar curvature. Many Sasaki–Einstein metrics are now known to exist.

The G_2–holonomy metrics produced by our Main Theorem are all ALC metrics in this sense.

Application I: cohomogeneity 1 G_2–spaces $M_{m,n}$

We already find **infinitely many diffeomorphism types** of *cohomogeneity one* simply connected ALC G_2–spaces using a single AC CY 3-fold $K_{\mathbb{P}^1\times\mathbb{P}^1}$.

Theorem (FHN). For any pair of coprime integers m, n satisfying $mn > 0$, the simply connected noncompact $7-$manifold $M_{m,n}$ which is the total space of the principal circle bundle over $K_{\mathbb{P}^1\times\mathbb{P}^1}$ with first Chern class $(m, -n)$ carries a 1-parameter family $g_{\epsilon,m,n}$ of ALC G_2–metrics.

- Each metric admits a cohomogeneity one action of $SU(2) \times SU(2) \times U(1)$.
- As $\epsilon \to 0$ $g_{\epsilon,m,n}$ collapses to $K_{\mathbb{P}^1\times\mathbb{P}^1}$ endowed with the unique AC Calabi–Yau metric $g_{0,m,n}$ with Kähler class $n\,[\omega_1] + m\,[\omega_2]$, where $[\omega_i]$ denote the classes of the Fubini–Study metrics on each factor of $\mathbb{P}^1 \times \mathbb{P}^1$. $g_{0,m,n}$ admits a cohomogeneity one action of $SU(2) \times SU(2)$.
- $M_{m,n}$ is diffeomorphic to $H^{2(n+m)} \times S^3$, where $H^{2(n+m)}$ is the total space of the $\mathbb{R}^2$-bundle on S^2 with Euler class $2(n+m)$. In particular the diffeomorphism type depends only on $m+n$.
- However, for different choices of (m, n) the metrics can never be isometric; in particular we obtain finitely many different families of G_2–metrics on the same underlying smooth manifold.

AC CY 3-folds via crepant resolutions of cones

Theorem. Let $\pi : B \to \mathsf{C}(\Sigma)$ be a crepant resolution of the CY cone $(\mathsf{C}(\Sigma), \omega_{\mathsf{C}}, \Omega_{\mathsf{C}})$ with complex volume form Ω_0 extending $\pi^*\Omega_{\mathsf{C}}$. Then in every cohomology class on B containing Kähler metrics there exists a unique AC Kähler Ricci-flat metric ω_0 on B with $\frac{1}{6}\omega_0^3 = \frac{1}{4}\mathrm{Re}\Omega_0 \wedge \mathrm{Im}\Omega_0$. Moreover, (B, ω_0, Ω_0) is asymptotic to the Calabi–Yau cone $\mathsf{C}(\Sigma)$ with rate -6 if the Kähler class $[\omega_0]$ is compactly supported and with rate -2 otherwise.

(A resolution of singularity is *crepant* if it has trivial canonical bundle).

Existence proof in various cases: Joyce (ALE), van Coevering, Goto.
Optimal uniqueness by Conlon–Hein.

- Theorem reduces problem of constructing AC CY metrics to
  - construction of CY cones;
  - classification of their crepant resolutions.
- Toric setting:
  - Calabi–Yau cone metrics completely understood by Futaki–Ono–Wang
  - toric crepant resolutions correspond to nonsingular subdivisions of the fan of the singular toric variety
  - Leads to existence of infinitely many toric AC CY 3-folds.

Calabi–Yau cones & Sasaki–Einstein metrics

- Existence of Calabi–Yau cone metrics, or equivalently, Sasaki–Einstein manifolds, is a difficult problem. In the regular & quasi-regular cases, it is equivalent to the existence of Kähler–Einstein (orbifold) metrics with positive scalar curvature. Many examples known by work of Tian, Kollár, Boyer–Galicki and many others; mostly related to the α-invariant.
- Explicit irregular examples were first constructed by physicists, Gauntlett–Martelli–Sparks–Waldram.
- Recently Collins–Székelyhidi proved existence of a CY cone metric on $\mathsf{C}(\Sigma)$ is implied by *K-stability*. In the conical setting, K-stability is an algebro-geometric notion for the affine variety $\mathsf{C}(\Sigma) \subset \mathbb{C}^N$ with an isolated singularity at the origin together with a holomorphic $(\mathbb{C}^*)^m$–action generated by a vector field ξ which acts with positive weights on the coordinate functions of $\mathbb{C}^N$.
- K-stability involves all possible degenerations of $(\mathsf{C}(\Sigma), \xi)$; so difficult to check in practice. If a large automorphism group exists only *equivariant* degenerations need be considered; checking K-stability can then be reduced to combinatorial calculations. e.g. for *complexity one* actions.

Calabi–Yau cones with small resolutions

- $B \to \mathrm{C}(\Sigma)$ is a *small* resolution, if the exceptional set contains no divisors (only curves). Then $H^4(B) = (0)$ and the condition from our Main Theorem $c_1(M) \cup [\omega_0] = 0 \in H^4(B)$ is automatic.
- $C(\Sigma)$ admits a small resolution $\Rightarrow$ singularity is *terminal* and hence so-called compound Du Val (cDV):
  a 3-fold hypersurface singularity $\{f(x, y, z) + tg(x, y, z, t) = 0\} \subset \mathbb{C}^4$, where $\{f = 0\} \subset \mathbb{C}^3$ defines a Du Val (ADE) singularity.
- Can use deformations of partial resolutions of DuVal singularities to construct small resolutions of cDV singularities (Brieskorn, Pinkham, Friedman, Katz, Morrison).
- For $p \geq 1$ consider the compound A_p singularity $X_p \subset \mathbb{C}^4$ defined by

$$x^2 + y^2 + z^{p+1} - w^{p+1} = 0.$$

  X_p admits Kähler small resolutions B_p. The exceptional set is a chain of p rational curves meeting transversely. $\pi_1(B_p) = 0$ and its nonzero Betti numbers are $b_0(B_p) = 1$, $b_2(B_p) = p$.

Local G_2–spaces via small resolutions

- As an application of their K-stability result for CY cones Collins–Székelyhidi proved: *X_p admits a CY cone metric with Reeb vector field ξ acting on $\mathbb{C}^4$ with weights $\frac{3}{4}(p+1, p+1, 2, 2)$.*
- Thus $X_p = \mathrm{C}(\Sigma)$ is the Calabi–Yau cone over a quasi-regular Sasaki–Einstein structure on $\Sigma \simeq \#_p \left(S^2 \times S^3\right)$.
- In particular any small resolution B_p admits a p-parameter family of Calabi–Yau structures (ω_0, Ω_0) asymptotic to $X_p = \mathrm{C}(\Sigma)$ with rate -2.

Hence our Main Theorem gives

Theorem *Let $M^7 \to B_p$ be a principal circle bundle over a small resolution B_p. (By passing to a finite cover we can assume that $c_1(M)$ is a primitive element in the lattice $H^2(B;\mathbb{Z})$ so that M is simply connected.) Then M carries a p-dimensional family of complete ALC G_2–metrics up to scale. Moreover, for $p, p' \geq 2$ with $p \neq p'$ the ALC G_2–manifolds M and M' constructed in this way are not diffeomorphic.*

In particular, there exist families of ALC G_2–manifolds of arbitrarily high dimension.

The main theorem again

Main Theorem: Foscolo–H–Nordström arXiv:1709.04904

Let $(B, g_0, \omega_0, \Omega_0)$ *be an irreducible asymptotically conical Calabi–Yau 3–fold asymptotic to the Calabi–Yau cone* $\mathrm{C}(\Sigma)$ *over a smooth Sasaki–Einstein 5–manifold* Σ. *Let* $M^7 \to B^6$ *be a principal* $U(1)$*–bundle such that* $c_1(M) \neq 0$ *but* $c_1(M) \cup [\omega_0] = 0 \in H^4(B)$.

Then there exists $\epsilon_0 > 0$ *such that for every* $\epsilon \in (0, \epsilon_0)$ *the noncompact 7–manifold* M *carries an* $\mathbb{S}^1$*–invariant torsion-free* G_2*–structure* φ_ϵ *such that the Riemannian metric* g_ϵ *on* M *induced by* φ_ϵ *is complete with (restricted) holonomy equal to* G_2 *and has asymptotically locally conical (ALC) geometry at infinity . Moreover* g_ϵ *collapses (with bounded curvature) as* $\epsilon \to 0$ *to* (B, g_0).

We need to look in detail at the equations for *circle-invariant torsion-free* G_2*–structures on principal circle bundles* $M \to B$ as first considered by Apostolov–Salamon (maths) and various physicists, e.g. Kaste–Minasian–Petrini–Tomasiello.

Circle-invariant torsion-free G_2–structures

Any circle-invariant G_2–structure φ on $M^7 \to B^6$ a principal circle bundle can be written as

$$\begin{cases} \varphi = \theta \wedge \omega + h^{\frac{3}{4}} \mathrm{Re}\Omega, \\ *_\varphi \varphi = -h^{1/4} \theta \wedge \mathrm{Im}\Omega + \dfrac{1}{2} h\, \omega^2, \\ g_\varphi = \sqrt{h}\, g_B + h^{-1} \theta^2 \end{cases}$$

where (ω, Ω) is an SU(3)–structure on B, g_B metric induced by (ω, Ω), h is a positive function on B and θ is a connection 1-form on $M \to B$.

φ is *torsion-free*, i.e $d\varphi = 0$, $d * \varphi = 0$, iff

$$\begin{cases} d\omega = 0, & d\big(h^{\frac{3}{4}} \mathrm{Re}\Omega\big) = -d\theta \wedge \omega, \\ d\big(h^{\frac{1}{4}} \mathrm{Im}\Omega\big) = 0, & \frac{1}{2} dh \wedge \omega^2 = h^{\frac{1}{4}} d\theta \wedge \mathrm{Im}\Omega. \end{cases} \qquad \text{(AS)}$$

- Some freedom in choice of conformal factor in front of g_B. Our choice is motivated by fact that $d\varphi = 0 \Rightarrow d\omega = 0$, so B is symplectic.
- Row 1 of (AS) $\Rightarrow [d\theta] \cup [\omega] = c_1(M) \cup [\omega] = 0 \in H^4(B)$.

Intrinsic torsion of solutions to (AS) eqns

Instrinsic torsion of SU(3)–structures. For any an SU(3)–structure (ω, Ω) on B^6 there exist functions $w_1, \hat{w}_1$, primitive $(1,1)$–forms $w_2, \hat{w}_2$, a 3-form $w_3 \in \Omega^3_{12}(B)$ and 1-forms w_4, w_5 on B such that

$$
\begin{aligned}
d\omega &= 3w_1 \mathrm{Re}\Omega + 3\hat{w}_1 \mathrm{Im}\Omega + w_3 + w_4 \wedge \omega, \\
d\mathrm{Re}\Omega &= 2\hat{w}_1 \omega^2 + w_5 \wedge \mathrm{Re}\Omega + w_2 \wedge \omega, \\
d\mathrm{Im}\Omega &= -2w_1 \omega^2 + w_5 \wedge \mathrm{Im}\Omega + \hat{w}_2 \wedge \omega.
\end{aligned}
$$

The only nonzero torsion components of a solution of (AS) are

$$
w_5 = -\tfrac{1}{4} h^{-1} dh, \qquad w_2 = -h^{-\frac{3}{4}} \kappa_0,
$$

where κ_0 is the projection of the curvature $d\theta$ of θ onto the space of primitive $(1,1)$–forms. Moreover, $\big(h, \theta\big)$ satisfies

$$
d\left(\tfrac{4}{3} h^{\frac{3}{4}}\right) = *\left(d\theta \wedge \mathrm{Re}\Omega\right), \qquad d\theta \wedge \omega^2 = 0.
$$

The associated almost complex structure is *non-integrable* whenever $w_2 \neq 0$.

The adiabatic limit: collapsing the circle fibres

Let φ_ϵ be a family of S^1–invariant torsion-free G_2–structures on $M \to B$ with *circle fibres shrinking to zero length* as $\epsilon \to 0$. By rescaling along the fibres we write

$$\varphi_\epsilon = \epsilon\,\theta_\epsilon \wedge \omega_\epsilon + (h_\epsilon)^{\frac{3}{4}}\mathrm{Re}\Omega_\epsilon, \qquad g_{\varphi_\epsilon} = \sqrt{h_\epsilon}\,g_\epsilon + \epsilon^2 h_\epsilon^{-1}\theta_\epsilon^2,$$

where g_ϵ is the metric on B induced by $(\omega_\epsilon, \Omega_\epsilon)$. (AS) system $\iff$

$$d\omega_\epsilon = 0,\ d\mathrm{Re}\Omega_\epsilon = -\tfrac{3}{4}h_\epsilon^{-1}dh_\epsilon \wedge \mathrm{Re}\Omega_\epsilon - \epsilon(h_\epsilon)^{-\frac{3}{4}}d\theta_\epsilon \wedge \omega_\epsilon,\ \epsilon d\theta_\epsilon \wedge \omega_\epsilon^2 = 0,$$

$$d\mathrm{Im}\Omega_\epsilon = -\tfrac{1}{4}h_\epsilon^{-1}dh_\epsilon \wedge \mathrm{Im}\Omega_\epsilon, \quad \tfrac{1}{2}dh_\epsilon \wedge \omega_\epsilon^2 = \epsilon\,(h_\epsilon)^{\frac{1}{4}}d\theta_\epsilon \wedge \mathrm{Im}\Omega_\epsilon.$$

- In the formal limit $\epsilon \to 0$ final equation implies $h_0 = \lim h_\epsilon$ is constant and wlog $h_0 = 1$. Then (ω_0, Ω_0) is *Calabi–Yau*, i.e. $d\omega_0 = 0$, $d\Omega_0 = 0$.

Now want to approximate $(h_\epsilon, \omega_\epsilon, \Omega_\epsilon, \theta_\epsilon)$ for $\epsilon > 0$ but small by linearising these equations on the limiting CY 3-fold (B, ω_0, Ω_0). So we write

$$h_\epsilon = 1 + \epsilon\,h + O(\epsilon^2), \qquad \epsilon\,\theta_\epsilon = \epsilon\,\theta + O(\epsilon^2),$$

$$\omega_\epsilon = \omega_0 + \epsilon\,\sigma + O(\epsilon^2), \qquad \Omega_\epsilon = \Omega_0 + \epsilon\,(\rho + i\hat{\rho}) + O(\epsilon^2).$$

The linearisation of Hitchin's duality map

The real part $\mathrm{Re}\Omega$ of the complex volume form Ω determines uniquely the imaginary part $\mathrm{Im}\Omega$ because in real dimension 6, $\mathrm{Re}\Omega_0$ is a *stable* form, i.e. its orbit in $\Lambda^3(\mathbb{R}^6)^*$ under $\mathrm{GL}(6,\mathbb{R})$ is open. There is an explicit formula for the linearisation of the map $\mathrm{Re}\Omega \mapsto \mathrm{Im}\Omega$ in terms of the Hodge star $*$ and the decomposition of forms into types.

Lemma. Given an SU(3)–structure (ω, Ω) on B, let $\rho \in \Omega^3(B)$ be a form with small enough C^0–norm so that $\mathrm{Re}\Omega + \rho$ is still a stable form. Decomposing into types we write $\rho = \rho_6 + \rho_{1\oplus 1} + \rho_{12}$. Then the image $\hat{\rho}$ of ρ under the linearisation of Hitchin's duality map at $\mathrm{Re}\Omega$ is

$$\hat{\rho} = *(\rho_6 + \rho_{1\oplus 1}) - *\rho_{12}.$$

Recall that $\Lambda^2\mathbb{R}^6 = \Lambda^2_1 \oplus \Lambda^2_6 \oplus \Lambda^2_8$ and $\Lambda^3\mathbb{R}^6 = \Lambda^3_6 \oplus \Lambda^3_{1\oplus 1} \oplus \Lambda^3_{12}$ where $\Lambda^2_1 = \mathbb{R}\,\omega$, $\Lambda^2_6 = \{X \lrcorner \mathrm{Re}\Omega \,|\, X \in \mathbb{R}^6\}$ and Λ^2_8 are primitive forms of type $(1,1)$ $\Lambda^3_6 = \{X^\flat \wedge \omega \,|\, X \in \mathbb{R}^6\}$, $\Lambda^3_{1\oplus 1} = \mathbb{R}\mathrm{Re}\Omega \oplus \mathbb{R}\mathrm{Im}\Omega$ and Λ^3_{12} are primitive forms of type $(1,2)+(2,1)$, $\Lambda^3_{12} = \{S_*\mathrm{Re}\Omega \,|\, S \in Sym^2(\mathbb{R}^6), SJ + JS = 0\}$.

The linearised Apostolov–Salamon equations

Then $(\sigma, \rho + i\hat{\rho}, h, \theta)$ satisfies the following system of linear equations

$$\begin{cases} d\sigma = 0, \quad d\rho = -\frac{3}{4} dh \wedge \mathrm{Re}\Omega_0 - d\theta \wedge \omega_0, \quad d\theta \wedge \omega_0^2 = 0, \\ d\hat{\rho} = -\frac{1}{4} dh \wedge \mathrm{Im}\Omega_0, \qquad \frac{1}{2} dh \wedge \omega_0^2 = d\theta \wedge \mathrm{Im}\Omega_0, \\ \omega_0 \wedge (\rho + i\hat{\rho}) + \sigma \wedge \Omega_0 = 0, \ \mathrm{Re}\Omega_0 \wedge \hat{\rho} + \rho \wedge \mathrm{Im}\Omega_0 = 2\sigma \wedge \omega_0^2. \end{cases} \tag{LAS}$$

- The last 2 equations are the linearisation of the nonlinear constraints for an SU(3)–structure, i.e. $\omega \wedge \Omega = 0$ and $\frac{1}{6}\omega^3 = \frac{1}{4}\mathrm{Re}\Omega \wedge \mathrm{Im}\Omega$.
- Assuming (B, ω_ϵ) is a *fixed* symplectic manifold simplifies (LAS) ($\sigma = 0$).
- If (h, θ) solves (LAS) then it is an *abelian Calabi–Yau monopole* on B, i.e.

$$*dh = d\theta \wedge \mathrm{Re}\,\Omega_0, \qquad d\theta \wedge \omega_0^2 = 0.$$

- Since $d\mathrm{Re}\,\Omega = 0$, h is harmonic.
  - In many cases, e.g. B complete and h bounded, h must be constant.
  - Then θ is an abelian *Hermitian Yang-Mills connection*, i.e. $d\theta$ is a primitive $(1,1)$-form.

Constructing an approximate solution

A Calabi–Yau monopole (h, θ) determines an *infinitesimal deformation* $\rho + i\hat{\rho}$ of the complex volume form Ω_0 by solving the linear inhomogeneous system

$$d\rho = -\tfrac{3}{4} dh \wedge \mathrm{Re}\Omega_0 - d\theta \wedge \omega_0, \quad d\hat{\rho} = -\tfrac{1}{4} dh \wedge \mathrm{Im}\Omega_0.$$

In the Hermitian Yang-Mills case ($h = 1$) the first equation $\Rightarrow$

- $\varphi_\epsilon = \mathrm{Re}\,\Omega_0 + \epsilon(\theta \wedge \omega_0 + \rho)$ are all *closed* G_2–structures (ϵ suff small)
- $c_1(M) \cup [\omega_0] = [d\theta] \cup [\omega_0] = 0 \in H^4(B)$.

We consider instead the inhomogeneous linear *elliptic* system

$$d\rho = -d\theta \wedge \omega_0 = *d\theta, \qquad d^*\rho = 0.$$

Because no decaying harmonic functions or 1-forms on B exist, any such 3-form ρ will be type Ω^3_{12}, and so $\hat{\rho} = -*\rho$; hence get a solution to the previous system.Now use elliptic Fredholm analysis on AC spaces to analyse obstructions to solve

$$(d + d^*)\rho = *d\theta$$

and show these obstructions vanish by the Chern class assumption. (It turns out we can find a HYM connection θ with *no* Chern class assumption.)

Proof strategy to correct approx to true solution

- Understand how to solve the linearised Apostolov–Salamon equations on AC manifold B in weighted Holder spaces for appropriate choice of weights.
- Understand appropriate gauge-fixing conditions to apply.
- Construct successive higher–order approximations $\varphi_\epsilon^{(k)}$ to torsion-free structure with torsion of order $O(\epsilon^{k+1})$.

  This requires a full understanding of the mapping properties of the linearisation of the Apostolov–Salamon equations.
- Construct a formal power series solution to the Apostolov–Salamon equations.
- Prove convergence of this formal power series solution for ϵ sufficiently small.