Codimension one collapse of G_2–holonomy metrics

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Collapsed limits, special holonomy and physics

- Weak limits of special holonomy manifolds (more generally spaces with Ricci-curvature lower bounds) are a priori only *metric spaces*.
- The limit space can drop in dimension: then we say that collapse occurs and the limit metric space is a collapsed limit.
- In general the structure theory of collapsed Ricci-limit spaces is still poorly understood (noncollapsed limits now relatively well understood).

An ongoing research programme: Understand codimension 1 collapse for special holonomy metrics

- Construct families of special/exceptional holonomy metrics undergoing 'codimension 1' collapse, i.e. dimension of the limit space drops by 1.
- Conversely, understand the structure theory for codimension 1 collapsed limits of special and exceptional holonomy metrics.
- Codimension 1 collapse of G_2 holonomy metrics in 7 dimensions to Calabi–Yau metrics in 6 dimensions is important in physics:

  it underpins limit in which an 11-*dimensional* physical theory (M theory) reduces to a better-understood 10-*dimensional theory* (Type IIA String Theory).

Riemannian collapse with bounded curvature I

A family of Riemannian metrics g_ϵ on M is said to *collapse with bounded curvature* if i_p (the injectivity radius at p) converges uniformly to 0 at all points p of M, but the curvature K_ϵ stays bounded (independent of p and ϵ).

Collapse to a point by global rescaling.

If g is a complete Riemannian metric on a compact manifold M, then the rescaled family of metrics $(M, \epsilon^2 g)$ collapses, but curvature stays bounded only if (M, g) is *flat*.

Codimension one collapse by rescaling circle fibres in a circle bundle.

Let θ be any connection on a principal circle bundle M^{n+1} over a complete Riemannian manifold (B^n, g_B) (with bounded geometry) then the 1-parameter family of circle-invariant metrics g_ϵ on M

$$g_\epsilon = g_B + \epsilon^2 \theta^2$$

collapses with bounded curvature to (B, g_B).

(Berger 1962 considered this in the special case of Hopf fibration $S^3 \to S^2$).

Riemannian collapse with bounded curvature II

Higher codimension collapse: generalise to torus bundles and "localise". Whenever a Riemannian manifold admits (compatible local) isometric torus actions T, then can construct families of (T-invariant) metrics by shrinking the metric in directions tangent to T. This leads to notion of an *F-structure of positive rank* (Cheeger–Gromov 1986).

• M admits an F-structure $\Rightarrow$ there is a family of metrics on M that collapses with bounded curvature; a suitable converse also holds.

• Existence of an F-structure forces topological constraints on M, e.g. $\chi(M) = 0$ if M is compact.

Loosely speaking, a sufficiently highly collapsed metric must admit continuous families of (almost) isometries. This raises 2 apparent problems in considering *highly collapsed Ricci-flat* manifolds.

1. A simply connected compact Ricci-flat metric admits no Killing fields!
2. The K3 surface has $\chi = 24$, so it does not admit *any* F-structure. So Ricci-flat metrics on K3 cannot collapse with curvature bounded everywhere.

Collapsing hyperKähler metrics on K3

Foscolo gave the following general construction of collapsing families of hyperKähler metrics on the K3 surface utilising so-called *ALF gravitational instantons*, i.e. a complete hyperKähler 4-manifold with "finite energy", and specific type of asymptotic geometry. The limit space is 3-dimensional.

Theorem (**Foscolo** 2016, **JDG**) Every collection of 8 ALF gravitational instantons of dihedral type $M_1, \ldots, M_8$ and n ALF gravitational instantons of cyclic type $N_1, \ldots, N_n$ satisfying

$$\sum_{j=1}^{8} \chi(M_j) + \sum_{i=1}^{n} \chi(N_i) = 24$$

appears as the collection of "bubbles" forming in a sequence of Kähler Ricci-flat metrics on the K3 surface collapsing to the flat orbifold $T^3/\mathbb{Z}_2$ with bounded curvature away from $n + 8$ points.

I will discuss ALF gravitational instantons, cyclic and dihedral, shortly.

Ingredients of proof

- Construct (incomplete) $\mathbb{S}^1$–invariant hyperkähler metrics on circle bundles over a punctured 3–torus using **Gibbons-Hawking ansatz**.
- Fix involution τ with 8 fixed points on T^3, choose a $\mathbb{Z}_2$–invariant configuration of $2n+8$ punctures and construct a monopole with Dirac-type singularities at these points. Pass to a $\mathbb{Z}_2$ quotient.
- Complete the resulting hyperkähler metrics by gluing ALF spaces at the $n+8$ punctures:

  an ALF space of *dihedral* type at each of the 8 fixed points of τ,
  an ALF space of *cyclic* type at each of the other punctures.
- Deform the resulting approximately hyperKähler metric using the Implicit Function Theorem. The setting of *definite triples* (due to **Donaldson**) seems the most convenient framework to use.

Key features: collapsing hK metrics on K3

The main ingredients of the construction are the following:

1. A good perturbation theory for closed definite triples that are almost hyperKähler.
2. The Gibbons–Hawking ansatz for constructing $\mathbb{S}^1$-invariant hK metrics was used in several ways
   - To construct the ALF gravitational instantons of cycle type (multi-centered Taub-NUT)
   - To construct the incomplete $\mathbb{S}^1$-invariant metric on the total space of a suitable circle bundle over a punctured 3-torus.
   - It provides a good approximation to the asymptotic geometry of the ALF gravitational instantons of dihedral type (up to a double cover).
3. The existence of ALF gravitational instantons of *dihedral type*, not just of cyclic type, including the exceptional ones D_0 and D_1, the Atiyah–Hitchin manifold and its double cover.

We would like to develop analogues of these key features in the G_2 case.

The Gibbons–Hawking ansatz

hyperKähler metrics in dimension 4 with a *triholomorphic* circle action

- U open subset of $\mathbb{R}^3$; h positive function on U
- $\pi : P \to U$ a principal $U(1)$–bundle with a connection θ

Gibbons–Hawking Ansatz (1978): the $U(1)$-invariant metric on P

$$g := h\,\pi^* g_{\mathbb{R}^3} + h^{-1}\theta^2$$

is hyperKähler iff (h, θ) is an abelian **monopole**: $*dh = d\theta$.

Monopole eqn $\Rightarrow$ h is **harmonic**. $h = \frac{1}{2|x|}$ gives Euclidean metric.

$$h = \sum_{i=1}^{n} \frac{1}{2|x - x_i|}, \quad \text{with } x_1, \ldots x_n \in \mathbb{R}^3,$$

yields the **multi-centre Eguchi–Hanson** metric. Decay of $h \Rightarrow g$ still has *Euclidean* volume growth but asymptotic cone is $\mathbb{C}^2/\mathbb{Z}_n$ with $\mathbb{Z}_n \subset SU(2)$. a.k.a. **ALE gravitational instantons** of **cyclic** type A_n

ALF gravitational instantons: cyclic and dihedral

If we add a constant term to the harmonic functions h then get complete $\mathbb{S}^1$-invariant hK metrics with **cubic** (ALF) not Euclidean volume growth (because h^{-1} and so length of the fibre now bounded at infinity).

$n = 1$ gives **Taub-NUT metric**: complete hK metric on $\mathbb{C}^2$ with cubic volume growth. $n > 1$ gives us the **multi-center Taub-NUT metrics**. Complete ALF hK metrics on minimal resolutions of $\mathbb{C}^2/\mathbb{Z}_n$.

These are **ALF gravitational instantons of cyclic type**.

One way to obtain **dihedral ALF spaces**:

- Take quotient of Taub-NUT metric on $\mathbb{C}^2$ by a dihedral group Γ,
- Resolve the resulting orbifold singularity by gluing in an ALE gravitational instanton of dihedral type (Biquard–Minerbe).
- Could also obtain the cyclic ALF spaces this way by a gluing construction.

NB: There are also 3 exceptional ALF dihedral gravitational instantons: the *Atiyah-Hitchin manifold* D_0 and the Dancer deformations D_1 on its simply-connected cover and D_2, the Page–Hitchin metrics.

G_2 analogues of ALF gravitational instantons

I explained in talk 1 that the correct G_2–analogue of an ALF gravitational instanton is an *ALC* G_2*–space*. Two ways we can construct ALC G_2 spaces.

- Method I: find highly collapsed ALC G_2 spaces (explained in talk 1).
- **Method II: find the highly symmetric ALC G_2 spaces.**

(Review of ALC versus ALF)

Key feature: the metric on an ALF space M is asymptotically a circle bundle over an exterior domain in $\mathbb{R}^3$ (or in $\mathbb{R}^3/\mathbb{Z}_2$) where the circle fibre has asymptotically constant length.

In higher dimensional generalisations (called ALC spaces by physicists) we:

- replace the base $\mathbb{R}^3$ with a Riemannian cone $C = C(\Sigma)$
- $C(\Sigma)$ should be a Calabi–Yau cone if we want to consider ALC G_2 spaces
- $C(\Sigma)$ is Calabi–Yau iff Σ is Sasaki–Einstein.
- Many Sasaki–Einstein metrics are now known to exist.

Uncollapsing highly collapsed limits

Main Theorem from talk 1 provides a very general construction of highly collapsed ALC G_2–metrics.

What happens to families of highly collapsed G_2*–metrics as we try to "open up" the size of the circle fibre?*

- In general currently intractable, but if Calabi–Yau base is *cohomogeneity one* we can answer this using cohomogeneity one methods.
- The most interesting case is $B = K_{\mathbb{P}^1 \times \mathbb{P}^1}$: it has a 2-dimensional Kähler cone, so we have many circle bundles that lead to simply connected 7-manifolds. We were therefore led to conjecture the following:
- There exists an infinite family of new complete *asymptotically conical* G_2*–metrics* $M_{m,n}$. The cross-section of the asymptotic cone of $M_{m,n}$ is the quotient of the standard nK structure on $S^3 \times S^3$ by a freely acting cyclic subgroup $\mathbb{Z}_{2(m+n)}$ of the group of isometries of $S^3 \times S^3$.

Previously only three asymptotically conical G_2–holonomy metrics were known (from Bryant-Salamon's original work).

Input from physics I: late 1990s, early 2000s

Development of M-theory, string dualities and understanding limits of M-theory led to expectation that there should be lots of G_2 holonomy spaces (possibly singular ones), but actual constructions were mainly lacking.

Some concrete developments in the noncompact setting assuming symmetry.

- In 2001 Brandhuber–Gomis–Gubser–Gukov constructed a new explicit complete G_2–metric on $\mathcal{S}S^3$ and suggested that their example should be a member of a 1-parameter family of such complete G_2–metrics.
- Numerical studies by Brandhuber & Cvetic–Gibbons–Lu–Pope supported that belief, but explicit solutions or existence proofs were lacking.
- The BGGG example has cohomogeneity one, but smaller symmetry group than Bryant–Salamon metric on $\mathcal{S}S^3$: $\mathrm{SU}(2)^2 \times \mathrm{U}(1)$ versus $\mathrm{SU}(2)^3$.
- The asymptotic geometry of the BGGG example is also different from the Bryant–Salamon metric. Its volume growth is r^6 not r^7. CGLP coined the term *ALC (asymptotically locally conical)* to describe its asymptotic geometry (generalising ALF spaces like the Taub–NUT metric.)

Input from physics II

- Further work by CGLP, and Hori–Hosomichi–Page–Rabadan–Walcher (2005) suggested there should be four 1-parameter families of complete cohomogeneity one ALC G_2–metrics: $\mathbb{A}_7$, $\mathbb{B}_7$, $\mathbb{C}_7$, $\mathbb{D}_7$.
- BGGG example belongs to the $\mathbb{B}_7$ family. In 2013 Bogoyavlenskaya proved existence of the whole 1-parameter family by qualitative ODE methods.
- No existence proof was available for any members of the $\mathbb{A}_7$, $\mathbb{C}_7$ and $\mathbb{D}_7$ families. Challenge is to control which solutions of ODE system give rise to complete solutions, even though explicit general solutions are lacking.
- In physics the $\mathbb{A}_7$ and $\mathbb{B}_7$ families were viewed as deriving from certain 4-dimensional ALF manifolds:
  the Atiyah–Hitchin metrics and Taub–NUT metrics respectively.
  The meaning of this was not clear (to mathematicians!)
- It is natural/important to understand the behaviour of these metrics at the two extremes of the possible range of parameters. Here a simpler hyperKähler analogy is illustrative.

The Gibbons–Hawking ansatz again

Example: ALF and ALE metrics of cyclic type

$$g_m = \left(m + \sum_{i=1}^{n} \frac{1}{2|x - a_i|} \right) dx \cdot dx + \left(m + \sum_{i=1}^{n} \frac{1}{2|x - a_i|} \right)^{-1} \theta^2$$

- $a_1, \ldots, a_n$ distinct $\Longrightarrow$ complete metric
  $a_1 = \cdots = a_{k+1} \Longrightarrow$ orbifold singularity $\mathbb{C}^2/\mathbb{Z}_k$
- m is called the mass
  - $m > 0 \Longrightarrow$ **ALF** (= ALC with flat asymptotic cone)
  - $m = 0 \Longrightarrow$ **ALE** (= AC with flat asymptotic cone)

Limits of ALF geometries

We can see three different limits

$$g_m = \left(m + \sum_{i=1}^{n} \frac{1}{2|x - a_i|} \right) dx \cdot dx + \left(m + \sum_{i=1}^{n} \frac{1}{2|x - a_i|} \right)^{-1} \theta^2$$

- $m \to \infty$: **collapse** to $\mathbb{R}^3$ (with curvature blow-up at finitely many points)
- $m \to 0$: smooth convergence to **ALE limit**

- By scaling get different picture of limit $m \to 0$:

$$m\, g_m = m \left(m + \sum_{i=1}^{n} \frac{1}{2|x - a_i|} \right) dx \cdot dx + m \left(m + \sum_{i=1}^{n} \frac{1}{2|x - a_i|} \right)^{-1} \theta^2$$

$$\underset{y = m\,x}{=} \left(1 + \sum_{i=1}^{n} \frac{1}{2|y - m\, a_i|} \right) dy \cdot dy + \left(1 + \sum_{i=1}^{n} \frac{1}{2|y - m\, a_i|} \right)^{-1} \theta^2$$

- $m \to 0$: convergence to **orbifold ALF**
- **orbifold ALF + ALE $\rightsquigarrow$ smooth ALF**

Cohomogeneity 1 AC CY 3-folds

- The simplest Calabi–Yau cone (Candelas–de la Ossa): the **conifold** $\{z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0\} \subset \mathbb{C}^4$
- AC Calabi–Yau 3-folds modelled on the conifold:
  - the **smoothing** of the conifold: T^*S^3 (tip of the cone replaced by a round totally geodesic special Lagrangian S^3)
  - the **small resolution** of the conifold: total space of $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$ (tip of the cone replaced by a round totally geodesic holomorphic S^2)
  - $K_{\mathbb{P}^1 \times \mathbb{P}^1}$ with **Calabi's metric** and its deformations: asymptotic to conifold$/\mathbb{Z}_2$ (tip of the cone replaced by an exceptional divisor $\mathbb{P}^1 \times \mathbb{P}^1$)
  - We can also consider the quotient of T^*S^3 by the standard antiholomorphic involution, i.e. we form an **orientifold** of the CY metric on T^*S^3.
- The conifold itself and its asymptotically conical CY desingularisations are **cohomogeneity one**: $\mathrm{SU}(2) \times \mathrm{SU}(2)$ acts isometrically with generic orbit of codimension one.
- In the *highly collapsed limit* the four 1-parameter families $\mathbb{B}_7$, $\mathbb{D}_7$, $\mathbb{C}_7$, and $\mathbb{A}_7$ correspond to these four AC Calabi-Yau spaces respectively.
- What happens in the uncollapsed limit? (A conical singularity develops).

AC metrics and conically singular ALC manifolds

Theorem A (Foscolo–H–Nordström, arixv:1805.02612)

- For every pair of coprime positive integers m, n there exists a **complete AC** G_2–metric (unique up to scale) on the (simply connected) total space $M_{m,n}$ of the circle bundle over $K_{\mathbb{P}^1 \times \mathbb{P}^1}$ with first Chern class $(m, -n)$.
- $M_{m,n}$ is asymptotic to the cone over $S^3 \times S^3/\mathbb{Z}_{2(m+n)}$.
- There is a 1-parameter family of ALC G_2–metrics on $M_{m,n}$ that collapses to a Calabi–Yau metric on $K_{CP^1 \times CP^1}$ at one extreme and "opens up" at the other extreme to the unique AC G_2–metric on $M_{m,n}$.

Theorem B (Foscolo–H–Nordström, arixv:1805.02612)

- There exists a (unique up to scale) G_2–metric g_0 on $M_0 = (0, \infty) \times S^3 \times S^3$ such that
- (M, g_0) has an **isolated conical singularity** modelled on the G_2–cone over the homogeneous nearly Kähler structure over $S^3 \times S^3$;
- (M, g_0) has a **complete ALC end**.

(M_0, g_0) and some of its quotients can be also be desingularised by analytic methods using the AC metrics and gives an analytic construction of smooth ALC metrics degenerating to (M_0, g_0).

The $\mathbb{A}_7$ family: G_2 analogue of Atiyah–Hitchin

We are still missing one type of bubble to glue in, the analogues of ALF *dihedal* gravitational instantons. (We need a family of ALC G_2 manifolds that collapses to Stenzel metric on $T^*\mathbb{S}^3$ quotiented by its standard antiholomorphic involution to glue at fixed points of τ.)

Physicists suggested that there should be an "M-theory lift" of the Atiyah–Hitchin manifold: the $\mathbb{A}_7$ family of cohomogeneity one metrics (studied by Hori et al numerically). Still have $SU(2) \times SU(2)$ symmetry but the absence of $U(1)$ symmetry makes the ODE system much less tractable.

We prove existence of the $\mathbb{A}_7$ family in two limits: the *highly collapsed* limit and close to *a conically singular limit* by perturbation theory.

- In the highly collapsed regime we need a better approximation to the geometry in the neighbourhood of the singular orbit.
  - We do this by rescaling in the normal directions and adapting ideas from Donaldson's work on adiabatic limits of coassociative fibrations.
  - Gives a rigorous way to interpret the physics statement that these manifolds are families of "Atiyah–Hitchin metrics fibred over $\mathbb{S}^3$".

Collapsing G_2-metrics on compact spaces

Want to use our highly collapsed ALC G_2-spaces $\mathbb{B}_7$ as bubbles in a gluing construction. $\mathbb{B}^7$ has a global isometric circle action that fixes the exceptional orbit, which is a round $\mathbb{S}^3$. In the limit $\mathbb{B}^7$ converges to the Stenzel metric on the smoothing $T^*\mathbb{S}^3$ of the conifold.

Q: What should replace the singular limit space $\mathbb{T}^3/\mathbb{Z}_2$ in the hK case?

A: Quotient of a Calabi-Yau 3-fold by an anti-holomorphic involution τ. Fixed point set of τ is a totally geodesic special Lagrangian 3-fold L

Obvious problem: In Stenzel have a round $\mathbb{S}^3$. How do we get round SL 3-spheres in CY 3-folds?

Answer: Start with a CY 3-fold X_0 that has only ordinary double point (conifold) singularities and assume that X is smoothable. By recent work of Hein and Sun we know that X admits an incomplete CY metric that is asymptotic to the standard KRF cone metric on the conifold at each ODP. Can now construct nearby smooth CY 3-folds Y_t by gluing in Stenzel metrics at each ODP. This way we get SL 3-spheres as close as required to round.

Collapsing G_2-metrics on compact spaces II

We have to require that the nodal CY 3-fold X admits an anti-holomorphic involution whose fixed set is a subset of the nodes. This appears to be very restrictive.

Because we need to find a circle bundle over our 6-dimensional space (with the correct topological properties) and we need to lift the involution to the circle bundle we also need the existence of a divisor passing through the nodes of X in a specified way,

We do know at least one such nodal CY 3-fold satisfying our wish list!

On the total space of the circle bundle over Y' we also need to construct a highly collapsed circle-invariant approximate G_2-metric. Here again we use the linearised version of Apostolov-Salamon as in Talk 1.