Conjectures on counting associative 3-folds in G_2-manifolds

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These slides available at
`http://people.maths.ox.ac.uk/~joyce/`.

Plan of talk:

1. G_2-manifolds and associative 3-folds
2. Different kinds of invariants
3. Conjectures on counting associative 3-folds

Disclaimer

Everything may be false.

1. G_2-manifolds and associative 3-folds

Let (X, g) be a Riemannian manifold, and $x \in X$. The *holonomy group* $\mathrm{Hol}(g)$ is the group of isometries of $T_x X$ given by parallel transport using the Levi-Civita connection ∇ around loops in X based at x. They were classified by Berger:

Theorem (Berger, 1955)

Suppose X is simply-connected of dimension n and g is irreducible and nonsymmetric. Then either: (i) $\mathrm{Hol}(g) = \mathrm{SO}(n)$ *[generic];*

(ii) $n = 2m \geqslant 4$ *and* $\mathrm{Hol}(g) = \mathrm{U}(m)$, *[Kähler manifolds];*

(iii) $n = 2m \geqslant 4$ *and* $\mathrm{Hol}(g) = \mathrm{SU}(m)$, *[Calabi–Yau m-folds];*

(iv) $n = 4m \geqslant 8$ *and* $\mathrm{Hol}(g) = \mathrm{Sp}(m)$, *[hyperkähler];*

(v) $n = 4m \geqslant 8$ *and* $\mathrm{Hol}(g) = \mathrm{Sp}(m)\,\mathrm{Sp}(1)$, *[quaternionic Kähler];*

(vi) $n = 7$ *and* $\mathrm{Hol}(g) = G_2$, *[exceptional holonomy] or*

(vii) $n = 8$ *and* $\mathrm{Hol}(g) = \mathrm{Spin}(7)$ *[exceptional holonomy].*

We are interested in 7-manifolds (X, g) with holonomy G_2. Then g is Ricci-flat. Any such X has a natural closed 3-form φ and closed 4-form $*\varphi$, and I refer to $(X, \varphi, *\varphi)$ as a *G_2-manifold*. Many examples of compact 7-manifolds $(X, \varphi, *\varphi)$ with holonomy G_2 were constructed by Joyce (1996), Kovalev (2003), and Corti–Haskins–Nordström–Pacini (2015). They are important in String Theory and M-theory. The moduli space of holonomy G_2-metrics on a compact X is smooth of dimension $b^3(X)$.

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If $(X, \varphi, *\varphi)$ is a G_2-manifold then $\varphi, *\varphi$ are calibrations on (X, g), in the sense of Harvey–Lawson, so we have natural classes of calibrated submanifolds, called *associative 3-folds*, and *coassociative 4-folds*. They are minimal submanifolds in (X, g). The deformation theory of compact associatives and coassociatives was studied by McLean (1998). Both are controlled by an elliptic equation, and so are well behaved. Moduli spaces of associative 3-folds N may be obstructed, and have virtual dimension 0. Moduli spaces of coassociative 4-folds C are smooth of dimension $b^2_+(C)$.

An analogy between G_2-manifolds and Calabi–Yau 3-folds

There is a strong analogy:

G_2-manifolds $(X, \varphi, *\varphi)$	$\leftrightarrow$	Calabi–Yau 3-folds (Y, J, h)
G_2 3-form φ	$\leftrightarrow$	Kähler form ω of h
G_2 4-form $*\varphi$	$\leftrightarrow$	complex structure J
Associative 3-folds N	$\leftrightarrow$	J-holomorphic curves Σ in Y
Coassociative 4-folds C	$\leftrightarrow$	special Lagrangians L in Y.

For instance, if Y is a Calabi–Yau 3-fold then $X = Y \times \mathcal{S}^1$ is a G_2-manifold, and J-holomorphic curves Σ, special Lagrangians L map to associative 3-folds $\Sigma \times \mathcal{S}^1$ and coassociative 4-folds $L \times \mathcal{S}^1$ in $Y \times \mathcal{S}^1$. Now lots of exciting mathematics is known about Calabi–Yau 3-folds, so we can ask how much of this may extend to G_2-manifolds. Today we ask whether theories on 'counting' J-holomorphic curves in Calabi–Yau 3-folds extend to theories on 'counting' associative 3-folds in G_2-manifolds.

Invariant theories in Symplectic Geometry

Let (Y, ω) be a compact symplectic manifold (e.g. a Calabi–Yau 3-fold) and J an almost complex structure on Y compatible with ω. Then symplectic geometers can define:

- The *Gromov–Witten invariants* $GW_{g,k}(\alpha)$ of (Y, ω) 'counting' J-holomorphic curves Σ with genus g and k marked points in homology class $\alpha \in H_2(Y; \mathbb{Z})$.
- The *Lagrangian Floer cohomology groups* $HF^*(L_1, L_2)$ of compact Lagrangians L_1, L_2 in Y.
- The *Fukaya category* $D^b\mathcal{F}(Y)$.

All of these are defined by 'counting' J-holomorphic curves Σ in Y satisfying some conditions, but have the magic property that they are independent of the choice of J. That is, we can deform J through a smooth family $J_t : t \in [0, 1]$, but the structures defined using J_t do not change (up to canonical isomorphism).

Invariant theories in G_2 geometry?

So by analogy with Calabi–Yau 3-folds, we can ask:

Question 1

Given a compact G_2-manifold $(X, \varphi, *\varphi)$, can we define interesting theories analogous to Gromov–Witten invariants, etc., by 'counting' associative 3-folds in X, so that the answer is unchanged under continuous deformations of $*\varphi$?

Actually, this is not yet a good question. In the symplectic case we fix ω and vary J. We have an analogy $\varphi \leftrightarrow \omega$, $*\varphi \leftrightarrow J$, so it would seem natural to consider varying the G_2-structure so that φ is fixed and $*\varphi$ varies. But $*\varphi$ is determined by φ, so this makes no sense. Our solution is to enlarge the class of G_2-manifolds we consider. Following Donaldson and Segal, we define *Tamed Almost G_2-manifolds*, or *TA G_2-manifolds*, (X, φ, ψ) to be a 7-manifold X with a closed G_2 3-form φ and a closed G_2 4-form ψ which satisfy a pointwise compatibility condition that is weaker than $\psi = *\varphi$.

We can add an extra line to our analogy:

TA G_2-manifolds $(X, \varphi, \psi) \leftrightarrow$ symplectic manifold (Y, ω)
with compatible almost structure J.

Then associative 3-folds N in a TA G_2-manifold (X, φ, ψ) depend only on ψ, but the natural notion of 'volume' is $\mathrm{vol}(N) = \int_N \varphi$, just as J-holomorphic curves Σ depend only on J but have volume $\int_\Sigma \omega$. We refine Question 1 to:

Question 2

Given a compact TA G_2-manifold (X, φ, ψ), can we define interesting theories analogous to Gromov–Witten invariants, etc., by 'counting' associative 3-folds in X, so that the answer is unchanged under continuous deformations of (φ, ψ) fixing φ?

This is related to understanding 'topological M Theory'. Answering this depends on understanding the singular behaviour of associative 3-folds, as singularities can break deformation-invariance.

2. Different kinds of invariants

Many important areas of geometry concern *invariants*. These have the following general structure:

- We begin with a primary geometric object X we want to study, e.g. a compact oriented 4-manifold, or a symplectic manifold.
- We choose some secondary geometric data $\mathcal{G}$ on X, e.g. a Riemannian metric g on the 4-manifold, or an almost complex structure J on the symplectic manifold (X,ω). Usually $\mathcal{G}$ lives in a large, connected, infinite-dimensional family.
- Using $X,\mathcal{G}$ we define a nonlinear elliptic equation for objects $E\rightarrow X$, e.g. instantons on oriented Riemannian 4-manifold (X,J), or J-holomorphic curves in (X,ω). These objects E form (compactified) moduli spaces $\bar{\mathcal{M}}_E$.
- We 'count' the moduli spaces $\bar{\mathcal{M}}_E$ to get 'invariants' $[\bar{\mathcal{M}}_E]_{\rm virt}$. We prove the resulting numbers (or homology groups, etc.) are independent of the secondary data $\mathcal{G}$, and so depend only on X.

We can divide 'invariant' problems into four types, in decreasing order of niceness:

(A) **Absolute invariants:** the numbers (or homology classes, etc.) $[\bar{\mathcal{M}}_E]_{\rm virt}$ are completely independent of secondary data $\mathcal{G}$.

(B) **Invariants with cohomological wall crossing.** The $[\bar{\mathcal{M}}_E]_{\rm virt}$ do depend on $\mathcal{G}$, but in a nice way: from $\mathcal{G}$ we can define some cohomology classes $c(\mathcal{G})$, and the $[\bar{\mathcal{M}}_E]_{\rm virt}$ change according to a rigid wall-crossing formula when $c(\mathcal{G})$ crosses certain real hypersurfaces in its cohomology group.

(C) **No invariant numbers, but subtle homological information conserved.** The $[\bar{\mathcal{M}}_E]_{\rm virt}$ depend on $\mathcal{G}$ in a nasty way, the individual numbers change chaotically and unpredictably with $\mathcal{G}$, but from the family of all $[\bar{\mathcal{M}}_E]_{\rm virt}$ we can extract some nontrivial information – such as a cohomology group – which is independent of $\mathcal{G}$.

(D) **No conserved information.** Nothing works.

It's not always easy to tell the difference between (C) and (D).

Examples of invariant problems of each type

(A) **Absolute invariants. 'Closed String.'** Donaldson or Seiberg–Witten invariants counting instantons on closed oriented 4-manifolds X with $b^2_+(X) > 1$. Gromov–Witten invariants counting J-holomorphic curves in symplectic manifolds/smooth schemes.

(B) **Invariants with wall-crossing. 'Counting branes/BPS states.'** Donaldson/SW invariants counting instantons on closed oriented 4-manifolds X with $b^2_+(X) = 1$. Donaldson–Thomas invariants.

(C) **Invariant structures from homological algebra. 'Open string'.** Morse flow lines and Morse homology. Instanton/SW Floer homology. J-holomorphic curves with boundary in Lagrangians; Lagrangian Floer cohomology; Fukaya categories.

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More about problems of type (C)
A simple example: Morse homology and Morse flow lines

Let X be a compact manifold, and $f : X \to \mathbb{R}$ a fixed Morse function. Pick some generic Riemannian metric g on X (the secondary data $\mathcal{G}$), and for critical points p, q of f, consider the moduli spaces $\bar{\mathcal{M}}(p, q)$ of gradient flow-lines $\gamma : \mathbb{R} \to X$ of f (i.e. $\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t) = \nabla f|_{\gamma(t)}$) with $\lim_{t\to-\infty}\gamma(t) = p$, $\lim_{t\to\infty}\gamma(t) = q$. Then (simplifying a bit) $\bar{\mathcal{M}}(p, q)$ is an oriented manifold with corners, of dimension $\mu(p) - \mu(q) - 1$, where $\mu(p)$ is the Morse index (number of negative e-values of $\mathrm{Hess}_p(f)$), with boundary

$$\partial\bar{\mathcal{M}}(p, r) = \coprod_{q\in\mathrm{Crit}(f)} \bar{\mathcal{M}}(p, q) \times \bar{\mathcal{M}}(q, r). \tag{1}$$

This boundary behaviour comes from 'broken flow-lines' – a real codimension 1 singular behaviour of flow-lines.

We define *Morse homology* $MH_*(X)$ as the homology of the complex $(MC_*(X), \partial)$, where $MC_k(X)$ has basis $p \in \mathrm{Crit}(f)$ with $\mu(p) = k$, and $\partial p = \sum_{q \in \mathrm{Crit}(f):\mu(q)=k-1} \#\bar{\mathcal{M}}(p,q) \cdot q$.
Here $\bar{\mathcal{M}}(p,q)$ is a compact oriented 0-manifold, i.e. a finite set with signs, and we count with signs to get $\#\bar{\mathcal{M}}(p,q)$.
Note that these numbers $\#\bar{\mathcal{M}}(p,q)$ are *not invariant under deformations of* g.
However, the homology $MH_*(X)$ of the complex $(MC_*(X), \partial)$ is canonically isomorphic to $H_*(X;\mathbb{R})$, and so is independent of g.
The deformation-invariant information is contained in the collection of all numbers $\#\bar{\mathcal{M}}(p,q)$, not in individual numbers.
As we deform g through a family g_t: $t \in [0,1]$, in real codimension 1 in t, a flow line $\gamma : q \to q'$ can appear with $\mu(q) = \mu(q')$, so that $\mathrm{vdim}(\bar{\mathcal{M}}(q,q')) = -1$. Then the numbers change by:

$$\begin{aligned} \#\bar{\mathcal{M}}(p,q') &\longmapsto \#\bar{\mathcal{M}}(p,q') \pm \#\bar{\mathcal{M}}(p,q), \quad \mu(p) = \mu(q)+1, \\ \#\bar{\mathcal{M}}(q,r) &\longmapsto \#\bar{\mathcal{M}}(q,r) \mp \#\bar{\mathcal{M}}(q',r), \quad \mu(r) = \mu(q)-1. \end{aligned}$$

3. Conjectures on counting associative 3-folds

Let (X, φ, ψ) be a TA G_2-manifold, and $\alpha \in H_3(X;\mathbb{Z})$ a homology class. Write $\mathcal{M}^\alpha(\psi)$ for the moduli space of compact associative 3-folds N in X with $[N] = \alpha$ in $H_3(X;\mathbb{Z})$. McLean proved the deformation theory of $\mathcal{M}^\alpha(\psi)$ is elliptic, possibly obstructed, with virtual dimension 0. We expect that if ψ is generic, then there are no obstructions:

Conjecture 1

Let (X, φ, ψ) be a compact TA-G_2-manifold with ψ generic in the infinite-dimensional family of closed 4-forms on X. Then moduli spaces $\mathcal{M}^\alpha(\psi)$ of associative 3-folds N with $[N] = \alpha \in H_3(X;\mathbb{Z})$ are smooth compact 0-manifolds (i.e. finite sets).

Counting invariants: elementary considerations

Assume Conjecture 1, and consider how we might define invariants counting associative 3-folds. Two big issues are:

• **Orientations of moduli spaces.** We want a way to define orientations on the moduli spaces $\mathcal{M}^\alpha(\psi)$. For ψ generic, the moduli spaces are compact 0-manifolds, and orientations are maps $\mathrm{Or} : \mathcal{M}^\alpha(\psi) \to \{\pm 1\}$. Thus we have counts

$$\#\mathcal{M}^\alpha(\psi) = \sum_{[N]\in\mathcal{M}^\alpha(\psi)} \mathrm{Or}([N]).$$

• **Dependence on ψ.** Let $(\varphi_0, \psi_0), (\varphi_1, \psi_1)$ be given with ψ_0, ψ_1 generic, and $(\varphi_t, \psi_t) : t \in [0, 1]$ be a generic smooth path of TA-G_2-structures between them. Then we have families $\mathcal{M}^\alpha(\psi_t)$ for $t \in [0, 1]$. We expect that at finite sets of values $0 < t_1 < \cdots < t_k < 1$ of t, the moduli spaces can change, becoming singular/obstructed. Then $\#\mathcal{M}^\alpha(\psi_t)$ may change discontinuously at $t = t_i$, and so not be invariants.

Real codimension 1 singular behaviour

So, the really important issue to understand is: how can moduli spaces $\mathcal{M}^\alpha(\psi_t)$ change in generic 1-parameter families $(\varphi_t, \psi_t) : t \in [0, 1]$? That is, what is the possible singular behaviour of associative 3-folds which happens in codimension 1 of possible G_2 4-forms ψ?

Although general singularities of associatives may be too horrible to feasibly understand, it seems plausible that there may be *only finitely many kinds of singularity* that occur generically in codimension 1 – the most common kinds of singularity. We could hope to understand all these types of codimension 1 singular behaviour, at least conjecturally. Then we would understand the only ways moduli spaces $\mathcal{M}^\alpha(\psi_t)$ can change in generic families $(\varphi_t, \psi_t) : t \in [0, 1]$, and we could try to build invariants, cohomology groups, etc., unaffected by these changes.

This is the approach of my paper (also compare Donaldson–Segal).

$\mathrm{U}(1)$-invariant associative 3-folds in $\mathbb{R}^7$

Consider associative 3-folds N in $\mathbb{R}^7$ invariant under the $\mathrm{U}(1)$-action

$$e^{i\theta} : (x_1, \ldots, x_7) \longmapsto (x_1, x_2, x_3, \cos\theta\, x_4 - \sin\theta\, x_5, \sin\theta\, x_4 + \cos\theta\, x_5, \\ \cos\theta\, x_6 + \sin\theta\, x_7, -\sin\theta\, x_6 + \cos\theta\, x_7).$$

Define $\mathrm{U}(1)$-invariant quadratic polynomials y_1, y_2, y_3 on $\mathbb{R}^7$ by

$$\begin{aligned} y_1(x_1, \ldots, x_7) &= x_4^2 + x_5^2 - x_6^2 - x_7^2, \\ y_2(x_1, \ldots, x_7) &= 2(x_4x_7 + x_5x_6), \\ y_3(x_1, \ldots, x_7) &= 2(x_4x_6 - x_5x_7). \end{aligned}$$

Then $y_1^2 + y_2^2 + y_3^2 = (x_4^2 + x_5^2 + x_6^2 + x_7^2)^2$. Consider the map

$$\Pi = (x_1, x_2, x_3, y_1, y_2, y_3) : \mathbb{R}^7 \longrightarrow \mathbb{R}^6 = \mathbb{C}^3.$$

This induces a homeomorphism $\bar{\Pi} : \mathbb{R}^7/\mathrm{U}(1) \to \mathbb{R}^6$. The $\mathrm{U}(1)$-fixed set $\mathbb{R}^3 \subset \mathbb{R}^7$ maps to $L = \mathbb{R}^3 = \{(x_1, x_2, x_3, 0, 0, 0) : x_j \in \mathbb{R}\}$ in $\mathbb{R}^6 = \mathbb{C}^3$.

Proposition

There is a 1-1 correspondence between $\mathrm{U}(1)$*-invariant associative 3-folds* N *in* $\mathbb{R}^7$ *and* J*-holomorphic curves* Σ *in* $\mathbb{R}^6$ *with boundary* $\partial\Sigma \subset L = \mathbb{R}^3 \subset \mathbb{R}^6$ *by* $\Sigma = \bar{\Pi}(N/\mathrm{U}(1))$*, where* J *is an almost complex structure on* $\mathbb{R}^6$ *with singularities on* $L = \mathbb{R}^3 \subset \mathbb{R}^6$*.*

Explicitly, writing $u : \mathbb{R}^6 \to [0, \infty)$, $u(x_1, x_2, x_3, y_1, y_2, y_3) = (y_1^2 + y_2^2 + y_3^2)^{1/2}$, then in block diagonal form on $\mathbb{R}^6 = \mathbb{R}^3 \oplus \mathbb{R}^3$, we have

$$J = \begin{pmatrix} 0 & -\frac{1}{2}u^{-1/2}\,\mathrm{id}_{\mathbb{R}^3} \\ 2u^{1/2}\,\mathrm{id}_{\mathbb{R}^3} & 0 \end{pmatrix},$$

so that J is singular when $u = 0$. But I do not expect the singularities of J to affect the heuristic behaviour of curves Σ.

Conclusion

Counting closed associative 3-folds should be similar to counting J*-holomorphic curves* **with boundary**. *So it may be of type (C), and we should* **not** *look for invariants, even with wall-crossing, but for Floer-type groups and subtle homological algebra information.*

Invariant information from counting associative 3-folds

In my paper I make a proposal for extracting invariant information from numbers of associatives $\#\mathcal{M}^\alpha(\psi)$. There are three parts to it:
(a) 'Canonical flags' of associative 3-folds, and defining natural orientations for associative moduli spaces $\mathcal{M}^\alpha(\psi)$.
(b) Conjectural description of codimension 1 singular behaviour of associative 3-folds – six different kinds of singularity of moduli spaces.
(c) Define a superpotential $\Phi_\psi : \mathcal{U} \to \Lambda$, where Λ is a Novikov ring of formal power series, and $\mathcal{U}$ is an open set in $H^3(X;\Lambda)$. This Φ_ψ is very roughly a generating function for $\#\mathcal{M}^\alpha(\psi)$ for associative $\mathbb{Q}$-homology spheres N, plus higher contributions involving 'linking numbers' and 'self-linking numbers' of such N. **Conjecture:** Φ_ψ is not invariant, but changes under deformations by $\Phi_\psi \mapsto \Phi_\psi \circ \Upsilon$ for $\Upsilon : \mathcal{U} \to \mathcal{U}$ a diffeomorphism of $\mathcal{U}$ as a rigid analytic space.
Given a choice of critical point of Φ_ψ, we define *G_2-quantum cohomology*, an associative, supercommutative Λ-algebra. I expect it to be deformation-invariant up to isomorphism.

(a) Canonical flags and orientations

Let X be an oriented 7-manifold, and $N \subset X$ a compact, connected, oriented 3-submanifold. A *flag* $[s]$ on N is (roughly) an isotopy class of nonvanishing sections s of the normal bundle $\nu \to N$ of N in X. It is like a framing of a knot in $\mathbb{R}^3$. The set $\mathrm{Flag}(N)$ of flags on N is a $\mathbb{Z}$-torsor.

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I define a *flag structure* F on 7-manifolds X, a new algebro-topological structure, giving a sign $F(N,[s]) = \pm 1$ to each immersed flagged 3-submanifold $(N,[s])$, satisfying some rules.

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Let (X, φ, ψ) be a TA-G_2-manifold, and $N \subset X$ be a compact associative 3-fold, with unobstructed deformation theory.

McLean's deformation theory for N gives a twisted Dirac operator $\mathbb{D}_N : \Gamma^\infty(\nu) \to \Gamma^\infty(\nu)$, self-adjoint as $\mathrm{d}\psi = 0$. By comparing $\mathbb{D}_N$ with $\mathrm{d} + *\mathrm{d} : \Gamma^\infty(\Lambda^0 T^*N \oplus \Lambda^2 T^*N) \to \Gamma^\infty(\Lambda^0 T^*N \oplus \Lambda^2 T^*N)$ and using spectral flow, I define a *canonical flag* $[s_N]$ for N.

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Given a choice of flag structure F on X, we define orientations $\mathrm{Or} : \mathcal{M}^\alpha(\psi) \to \{\pm 1\}$ on associative moduli spaces by $\mathrm{Or}(N) = F(N, [s_N])$. I claim this is a natural thing to do.

(b) Conjectural codim 1 singular behaviour of associatives

In my paper I describe 6 kinds of codim 1 singularities of associatives. Here I will explain one. Suppose we have a generic 1-parameter family of TA-G_2-manifolds (X, φ_t, ψ_t), $t \in [0,1]$, and compact, unobstructed associatives N_t^1, N_t^2 in (X, φ_t, ψ_t) with $[N_t^i] = \alpha^i$ in $H_3(X;\mathbb{Z})$. For generic $t \in [0,1]$ we expect $N_t^1 \cap N_t^2 = \emptyset$, but at $t_0 \in (0,1)$ we may have $N_t^1 \cap N_t^2 = \{p\}$. Then (Nordström) we can create a new family of associatives N_t^3 for $t \in (t_0, 1]$ with topology the connect sum $N_t^3 \cong N_t^1 \# N_t^2$ and homology class $[N_t^3] = \alpha^1 + \alpha^2$, such that for t close to t_0, N_t^3 near p resembles a small 'Lawlor neck' $\mathcal{S}^2 \times \mathbb{R}$ in $T_pX \cong \mathbb{C}^3 \oplus \mathbb{R}$.

Thus $\#\mathcal{M}^{\alpha^1+\alpha^2}(\psi_t)$ changes by ± 1 as t crosses t_0, so it is not invariant.

(b) Conjectural codim 1 singular behaviour of associatives

In my paper I describe 6 kinds of codim 1 singularities of associatives. Here I will explain one. Suppose we have a generic 1-parameter family of TA-G_2-manifolds (X, φ_t, ψ_t), $t \in [0,1]$, and compact, unobstructed associatives N_t^1, N_t^2 in (X, φ_t, ψ_t) with $[N_t^i] = \alpha^i$ in $H_3(X;\mathbb{Z})$. For generic $t \in [0,1]$ we expect $N_t^1 \cap N_t^2 = \emptyset$, but at $t_0 \in (0,1)$ we may have $N_t^1 \cap N_t^2 = \{p\}$. Then (Nordström) we can create a new family of associatives N_t^3 for $t \in (t_0, 1]$ with topology the connect sum $N_t^3 \cong N_t^1 \# N_t^2$ and homology class $[N_t^3] = \alpha^1 + \alpha^2$, such that for t close to t_0, N_t^3 near p resembles a small 'Lawlor neck' $\mathcal{S}^2 \times \mathbb{R}$ in $T_pX \cong \mathbb{C}^3 \oplus \mathbb{R}$.

Thus $\#\mathcal{M}^{\alpha^1+\alpha^2}(\psi_t)$ changes by ± 1 as t crosses t_0, so it is not invariant. We might hope to compensate for this by also counting pairs (N_t^1, N_t^2) weighted by a 'linking number' $\ell(N_t^1, N_t^2)$, such that $\ell(N_t^1, N_t^2)$ changes by ∓ 1 when N_t^1, N_t^2 cross, to cancel the change from N_t^3. However, no such $\ell(N_t^1, N_t^2)$ exists. On the face of it, this looks fatal for invariants counting associatives!

(c) A superpotential counting associatives

Consider TA-G_2-structures (φ,ψ) on X with $[\varphi]=\gamma\in H^3(X;\mathbb{R})$ fixed. Let $\mathbb{F}$ be the field $\mathbb{Q},\mathbb{R}$ or $\mathbb{C}$. Write Λ for the Novikov ring

$$\Lambda=\bigl\{\textstyle\sum_{i=1}^\infty c_i q^{\alpha_i} : c_i\in\mathbb{F},\ \alpha_i\in\mathbb{R},\ \alpha_i\to\infty \text{ as } i\to\infty\bigr\},$$

with q a formal variable. Write $\Lambda_{>0}\subset\Lambda$ for the ideal with all $\alpha_i>0$. Write $\mathcal{U}$ for the set of group homomorphisms

$$\mathcal{U}=\mathrm{Hom}\bigl(H_3(X;\mathbb{Z}),1+\Lambda_{>0}\bigr).$$

Then $\mathcal{U}$ is a smooth rigid analytic space, roughly an open set in $H^3(X;\Lambda)$ in the analytic topology. It has $\mathcal{U}\cong\Lambda_{>0}^{b_3(X)}$.

We define a superpotential $\Phi_\psi:\mathcal{U}\to\Lambda_{>0}$, of the form

$$\Phi_\psi=\sum_{\substack{\text{associatives } N:\\ H_1(N;\mathbb{Z})\text{ finite}}}\mathrm{Or}(N)\,|H_1(N;\mathbb{Z})|\,q^{[\varphi]\cdot[N]}+\text{higher order terms}.$$

These 'higher order terms' do involve a kind of 'linking number' $\ell(N^1,N^2)$, but it needs arbitrary choices to define. They also involve canonical flags $[s_N]$, interpreted as a 'self-linking number'.

I conjecture that different arbitrary choices, and deformations, change Φ_ψ by reparametrizations $\Phi_\psi\mapsto\Phi_\psi\circ\Upsilon$, Υ an isomorphism.

G_2 quantum cohomology

Motivated by an analogy with 'bounding cochains' in Fukaya–Oh–Ohta–Ono's Lagrangian Floer theory, I believe a natural thing to do is to choose a critical point θ of the superpotential Φ_ψ. This is, in a sense, a deformation-invariant thing to do, as if we change $\Phi_\psi \mapsto \Phi_{\psi'} = \Phi_\psi \circ \Upsilon$, then we also change $\theta \mapsto \theta' = \Upsilon^{-1}(\theta)$. Such critical points θ need not exist, and we call (X, φ, ψ) *unobstructed* if $\mathrm{Crit}(\Phi_\psi) \neq \emptyset$. We can then define the G_2 *quantum cohomology* $QH^*_\theta(X; \Lambda)$, which is a supercommutative algebra over the Novikov ring Λ, a natural deformation of $H^*(X; \Lambda)$, which is similar to Quantum Cohomology in Symplectic Geometry, and should be unchanged under deformations of (X, φ, ψ) fixing $[\varphi]$ in $H^3(X; \mathbb{R})$.

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In my paper I also point out some possible problems in Donaldson–Segal's programme for defining invariants counting G_2-instantons, and propose to fix these problems by including a choice of critical point θ of Φ_ψ in the picture.