Donaldson–Thomas theory of Calabi–Yau 3-folds

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Pre-recorded introductory talk for
Simons Collaboration online meeting
"Donaldson–Thomas invariants and Resurgence"
January 11-15, 2021.

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1.1. Calabi–Yau manifolds

A *Calabi–Yau m-fold* is a compact $2m$-dimensional manifold X equipped with four geometric structures:

- a Riemannian metric g;
- a complex structure J;
- a symplectic form (Kähler form) ω; and
- a complex volume form Ω.

These satisfy pointwise compatibility conditions: $\omega(u,v)=g(Ju,v)$, $|\Omega|_g\equiv 2^{m/2}$, Ω is of type $(m,0)$ w.r.t. J, and p.d.e.s: J is integrable, and $\mathrm{d}\omega\equiv\mathrm{d}\Omega\equiv 0$. Usually we also require $H^1(X;\mathbb{R})=0$. This is a rich geometric structure, and very interesting from several points of view.

Complex algebraic geometry: (X,J) is a projective complex manifold. That is, we can embed X as a complex submanifold of $\mathbb{CP}^N$ for some $N\gg 0$, and then X is the zero set of finitely many homogeneous polynomials on $\mathbb{C}^{N+1}$. Also Ω is a holomorphic section of the canonical bundle K_X, so K_X is trivial, and $c_1(X)=0$.

Analysis: For fixed (X, J), Yau's solution of the Calabi Conjecture by solving a nonlinear elliptic p.d.e. shows that there exists a family of Kähler metrics g on X making X Calabi–Yau. Combining complex algebraic geometry and analysis proves the existence of huge numbers of examples of Calabi–Yau m-folds.

Riemannian geometry: (X, g) is a Ricci-flat Riemannian manifold with holonomy group $\mathrm{Hol}(g) \subseteq \mathrm{SU}(m)$.

Symplectic geometry: (X, ω) is a symplectic manifold with $c_1(X) = 0$.

Calibrated geometry: there is a distinguished class of minimal submanifolds in (X, g) called *special Lagrangian m-folds*.

String Theory: a branch of theoretical physics aiming to combine Quantum Theory and General Relativity. String Theorists believe that space-time is not 4 dimensional, but 10-dimensional, and is locally modelled on $\mathbb{R}^{3,1} \times X$, where $\mathbb{R}^{3,1}$ is Minkowski space, our observed universe, and X is a Calabi–Yau 3-fold with radius of order 10^{-33}cm, the Planck length.

1.2. String Theory and Mirror Symmetry

String Theorists believe that each Calabi–Yau 3-fold X has a quantization, a *Super Conformal Field Theory* (SCFT), not yet rigorously defined. Invariants of X such as the Dolbeault groups $H^{p,q}(X)$ and the Gromov–Witten invariants of X translate to properties of the SCFT. Using physical reasoning they made amazing predictions about Calabi–Yau 3-folds, an area known as *Mirror Symmetry*, conjectures which are slowly turning into theorems.

Part of the picture is that Calabi–Yau 3-folds should occur in pairs $X, \hat{X}$, such that $H^{p,q}(X) \cong H^{3-p,q}(\hat{X})$, and the complex geometry of X is somehow equivalent to the symplectic geometry of $\hat{X}$, and vice versa. This is very strange. It is an exciting area in which to work.

1.3. Invariants in Geometry

When geometers talk about *invariants*, they tend to have a particular, quite complex set-up in mind:

- Let X be a manifold (usually compact).
- Let $\mathcal{G}$ be a geometric structure on X that we are interested in.
- Let $\mathcal{A}$ be some auxiliary geometric structure on X.
- Let α be some topological invariant, e.g. a homology class on X.

We define a *moduli space* $\mathcal{M}(\mathcal{G},\mathcal{A},\alpha)$ which parametrizes isomorphism classes of some kind of geometric object on X (e.g. submanifolds, or bundles with connection) which satisfy a p.d.e. depending on $\mathcal{G}$ and $\mathcal{A}$, and have topological invariant α.
Then we define $I(\mathcal{G},\alpha)$ in $\mathbb{Z}$ or $\mathbb{Q}$ or $H_*(X;\mathbb{Q})$ which 'counts' the number of points in $\mathcal{M}(\mathcal{G},\mathcal{A},\alpha)$. The 'counting' often has to be done in a complicated way. Usually we need $\mathcal{M}(\mathcal{G},\mathcal{A},\alpha)$ compact.

Sometimes one can prove $I(\mathcal{G},\alpha)$ is *independent of the choice of auxiliary geometric structure* $\mathcal{A}$, even though $\mathcal{M}(\mathcal{G},\mathcal{A},\alpha)$ depends very strongly on $\mathcal{A}$, and even though we usually have no way to define $I(\mathcal{G},\alpha)$ without choosing $\mathcal{A}$. Then we call $I(\mathcal{G},\alpha)$ an *invariant*. Invariants are interesting as they may be part of some deep underlying structure, perhaps some kind of Quantum Geometry coming from String Theory. Some examples:

- *Donaldson invariants* and *Seiberg–Witten invariants* of 4-manifolds 'count' self-dual connections. They are independent of the Riemannian metric used to define them. They can distinguish homeomorphic, non-diffeomorphic 4-manifolds.
- *Gromov–Witten invariants* of a compact symplectic manifold (X,ω) 'count' J-holomorphic curves in X for an almost complex structure J compatible with ω, but are independent of J.
- *Donaldson–Thomas invariants* of a Calabi–Yau 3-fold (X,J,g,Ω) 'count' coherent sheaves on X, and are independent of the complex structure J up to deformation.

2. Donaldson–Thomas invariants

Let X be a Calabi–Yau 3-fold. A *holomorphic vector bundle* $\pi : E \to X$ of *rank* r is a complex manifold E with a holomorphic map $\pi : E \to X$ whose fibres are complex vector spaces $\mathbb{C}^r$. A *morphism* $\phi : E \to F$ of holomorphic vector bundles $\pi : E \to X$, $\pi' : F \to X$ is a holomorphic map $\phi : E \to F$ with $\pi' \circ \phi \equiv \pi$, that is linear on the vector space fibres. Then $\mathrm{Hom}(E, F)$ is a finite-dimensional vector space. Holomorphic vector bundles form an exact category $\mathrm{Vect}(X)$.

A holomorphic vector bundle E has topological invariants, the *Chern character* $\mathrm{ch}_*(E)$ in $H^{\mathrm{even}}(X, \mathbb{Q})$, with $\mathrm{ch}_0(E) = r$, the rank of E. Holomorphic vector bundles are very natural objects to study.

Roughly speaking, D–T invariants are integers which 'count' (semi)stable holomorphic vector bundles. But we actually consider a larger category, the *coherent sheaves* $\mathrm{coh}(X)$ on X.

A coherent sheaf is a (possibly singular) vector bundle $E \to Y$ on a complex submanifold (subscheme) Y in X. We need coherent sheaves for two reasons:
Firstly, moduli spaces of semistable holomorphic vector bundles are generally noncompact; to get compact moduli spaces, we have to allow singular vector bundles, that is, coherent sheaves.
Secondly, if $\phi : E \to F$ is a morphism of vector bundles then $\mathrm{Ker}\,\phi$ and Cokerϕ are generally coherent sheaves, not vector bundles.
The category $\mathrm{coh}(X)$ is better behaved than Vect(X) (it is an *abelian category*, has kernels and cokernels).

One cannot define invariants 'counting' all coherent sheaves with a fixed Chern character α, as the number would be infinite (the moduli spaces are not of finite type). Instead, one restricts to *(semi)stable* coherent sheaves. A coherent sheaf E is Gieseker (semi)stable if all subsheaves $F \subset E$ satisfy some numerical conditions. These conditions depend on an ample line bundle on X; essentially, on the cohomology class $[\omega] \in H^2(X;\mathbb{R})$ of the Kähler form ω of X. We will write τ for Gieseker stability. Every coherent sheaf can be decomposed into τ-semistable sheaves in a unique way, the *Harder–Narasimhan filtration*. So counting τ-semistable sheaves is related to counting all sheaves.

2.1. Thomas' definition of Donaldson–Thomas invariants

Let X be a Calabi–Yau 3-fold. The *Donaldson–Thomas invariants* $DT^\alpha(\tau)$ of X were defined by Richard Thomas in 1998. Fix a Chern character α in $H^{\rm even}(X;\mathbb{Q})$. Then one can define *coarse moduli schemes* $\mathcal{M}_{\rm st}^\alpha(\tau)$, $\mathcal{M}_{\rm ss}^\alpha(\tau)$ parametrizing equivalence classes of τ-(semi)stable sheaves with Chern character α. They are not manifolds, but schemes which may have bad singularities. Two good properties:

- $\mathcal{M}_{\rm ss}^\alpha(\tau)$ is a projective $\mathbb{C}$-scheme, so in particular it is compact and Hausdorff.
- $\mathcal{M}_{\rm st}^\alpha(\tau)$ is an open subset in $\mathcal{M}_{\rm ss}^\alpha(\tau)$, and has an extra structure, a *symmetric obstruction theory*, which does not extend to $\mathcal{M}_{\rm ss}^\alpha(\tau)$ in general.

If $\mathcal{M}_{\rm ss}^\alpha(\tau)=\mathcal{M}_{\rm st}^\alpha(\tau)$, that is, there are no strictly τ-semistable sheaves in class α, then $\mathcal{M}_{\rm st}^\alpha(\tau)$ is compact with a symmetric obstruction theory. Thomas used the *virtual class* of Behrend and Fantechi to define the 'number' $DT^\alpha(\tau)\in\mathbb{Z}$ of points in $\mathcal{M}_{\rm st}^\alpha(\tau)$, and showed $DT^\alpha(\tau)$ is unchanged under deformations of the complex structure of X.

Virtual classes are *non-local*. But Behrend (2005) showed that $DT^\alpha(\tau)$ can be written as a *weighted Euler characteristic*

$$DT^\alpha(\tau)=\int_{\mathcal{M}_{\rm st}^\alpha(\tau)}\nu\,{\rm d}\chi, \qquad (1)$$

where ν is the 'Behrend function', a $\mathbb{Z}$-valued constructible function on $\mathcal{M}_{\rm st}^\alpha(\tau)$ depending only on $\mathcal{M}_{\rm st}^\alpha(\tau)$ as a $\mathbb{C}$-scheme. We think of ν as a *multiplicity function*, so (1) counts points with multiplicity.

Donaldson–Thomas invariants are of interest in String Theory. The *MNOP Conjecture*, an important problem, relates the rank 1 Donaldson–Thomas invariants to the Gromov–Witten invariants counting holomorphic curves in X.

Thomas' definition of $DT^\alpha(\tau)$ has two disadvantages:

- $DT^\alpha(\tau)$ is undefined if $\mathcal{M}_{\rm ss}^\alpha(\tau) \neq \mathcal{M}_{\rm st}^\alpha(\tau)$.
- It was not understood how $DT^\alpha(\tau)$ depends on the choice of stability condition τ (effectively, on the Kähler class $[\omega]$ of X).

I will explain a theory which solves these two problems (joint work with Yinan Song).

2.2. Joyce–Song's generalized D–T invariants

We will define *generalized Donaldson–Thomas invariants* $\bar{DT}^{\alpha}(\tau) \in \mathbb{Q}$ for all Chern characters α, such that:

- $\bar{DT}^{\alpha}(\tau)$ is unchanged by deformations of the underlying CY3.
- If $\mathcal{M}_{\rm st}^{\alpha}(\tau) = \mathcal{M}_{\rm ss}^{\alpha}(\tau)$ then $\bar{DT}^{\alpha}(\tau) = DT^{\alpha}(\tau)$ in $\mathbb{Z} \subset \mathbb{Q}$.
- The $\bar{DT}^{\alpha}(\tau)$ transform according to a known transformation law under change of stability condition.
- For 'generic' τ, we have a conjecture rewriting the $\bar{DT}^{\alpha}(\tau)$ in terms of $\mathbb{Z}$-valued 'BPS invariants' $\hat{DT}^{\alpha}(\tau)$. (Cf. Gromov– Witten and Gopakumar–Vafa invariants).
- The theory generalizes to invariants counting representations of a quiver with relations coming from a superpotential. (Cf. 'noncommutative D–T invariants').

On the face of it, the problem is just to decide how to 'count' strictly τ-semistable sheaves with the correct multiplicity, which sounds simple. But the solution turns out to be very long and very complex, and involves a lot of interesting mathematics. I will just explain a few of the key ideas involved.

Key idea 1: work with Artin stacks

Kinds of space used in complex algebraic geometry, in decreasing order of 'niceness':

- complex manifolds (very nice)
- varieties (nice)
- schemes (not bad): Thomas' $DT^\alpha(\tau)$.
- algebraic spaces (getting worse)
- Deligne–Mumford stacks (not nice)
- Artin stacks (horrible): our $\bar{DT}^\alpha(\tau)$.
- derived stacks (deeply horrible).

For classical D–T theory we work with moduli spaces which are *Artin stacks*, rather than coarse moduli schemes as Thomas does. One reason is that strictly τ-semistable sheaves can have nontrivial automorphism groups, and Artin stacks keep track of automorphism groups, but schemes do not.

For generalizations of D–T theory, we will need to work with derived stacks, and the theory of Pantev–Toën–Vaquié–Vezzosi.

Key idea 2: Ringel–Hall algebras

Write $\mathfrak{M}$ for the moduli stack of coherent sheaves on X. The 'stack functions' $\mathrm{SF}(\mathfrak{M})$ is the $\mathbb{Q}$-vector space generated by isomorphism classes $[(\mathfrak{R},\rho)]$ of morphisms $\rho:\mathfrak{R}\to\mathfrak{M}$ for $\mathfrak{R}$ a finite type Artin $\mathbb{C}$-stack, with the relation

$$[(\mathfrak{R},\rho)]=[(\mathfrak{S},\rho)]+[(\mathfrak{R}\setminus\mathfrak{S},\rho)]$$

for $\mathfrak{S}$ a closed substack of $\mathfrak{R}$.
There is an interesting associative, noncommutative product $*$ on $\mathrm{SF}(\mathfrak{M})$ defined using short exact sequences in $\mathrm{coh}(X)$; for $f,g\in\mathrm{SF}(\mathfrak{M})$, think of $(f*g)(F)$ as the 'integral' of $f(E)g(G)$ over all exact sequences $0\to E\to F\to G\to 0$ in $\mathrm{coh}(X)$.

The substack $\mathfrak{M}_{\rm ss}^\alpha(\tau)$ of $\mathfrak{M}$ of τ-semistable sheaves with Chern character α has finite type, so $\bar\delta_{\rm ss}^\alpha(\tau)=[(\mathfrak{M}_{\rm ss}^\alpha(\tau),{\rm inc})]\in{\rm SF}(\mathfrak{M})$. There is a Lie subalgebra ${\rm SF}^{\rm ind}(\mathfrak{M})$ of ${\rm SF}(\mathfrak{M})$ of stack functions 'supported on virtual indecomposables'. Define elements

$$\bar\epsilon^\alpha(\tau)=\sum_{n\geqslant 1,\;\alpha_1+\cdots+\alpha_n=\alpha,\;\tau(\alpha_i)=\tau(\alpha),\text{ all } i}(-1)^{n-1}/n\cdot\bar\delta_{\rm ss}^{\alpha_1}(\tau)*\bar\delta_{\rm ss}^{\alpha_2}(\tau)*\cdots*\bar\delta_{\rm ss}^{\alpha_n}(\tau).$$

Then $\bar\epsilon^\alpha(\tau)\in{\rm SF}^{\rm ind}(\mathfrak{M})$.

There are many important *universal identities* in the Ringel–Hall algebra ${\rm SF}(\mathfrak{M})$. For instance, if $\tau,\tilde\tau$ are different stability conditions, we have

$$\bar\delta_{\rm ss}^\alpha(\tilde\tau)=\sum_{n\geqslant 1,\;\alpha_1+\cdots+\alpha_n=\alpha}S(\alpha_1,\ldots,\alpha_n;\tau,\tilde\tau)\cdot\bar\delta_{\rm ss}^{\alpha_1}(\tau)*\cdots*\bar\delta_{\rm ss}^{\alpha_n}(\tau),\qquad(2)$$

$$\bar\epsilon^\alpha(\tilde\tau)=\sum_{n\geqslant 1,\;\alpha_1+\cdots+\alpha_n=\alpha}U(\alpha_1,\ldots,\alpha_n;\tau,\tilde\tau)\cdot\bar\epsilon^{\alpha_1}(\tau)*\cdots*\bar\epsilon^{\alpha_n}(\tau),\qquad(3)$$

for combinatorial coefficients $S,U(\cdots;\tau,\tilde\tau)$.

Key idea 3: local structure of the moduli stack

We prove that the moduli stack of coherent sheaves $\mathfrak{M}$ can be written locally in the complex analytic topology as $[\mathrm{Crit}(f)/G]$, where G is a complex Lie group, U a complex manifold acted on by G, and $f : U \to \mathbb{C}$ a G-invariant holomorphic function.
This is a complex analytic analogue for $\mathfrak{M}$ of the fact that $\mathcal{M}^\alpha_{\mathrm{st}}(\tau)$ has a symmetric obstruction theory.
It requires X to be a Calabi–Yau 3-fold. The proof in Joyce–Song from 2008 is non-algebraic, using gauge theory on complex vector bundles over X, and works only over the field $\mathbb{C}$. However, more recently Ben-Bassat–Bussi–Brav–Joyce used PTVV's shifted symplectic derived algebraic geometry to give an algebraic proof, in the Zariski/smooth topologies, which works over fields $\mathbb{K}$ of characteristic zero.

Key idea 4: Behrend function identities

For each Artin $\mathbb{C}$-stack $\mathfrak{M}$ we can define a *Behrend function* $\nu_{\mathfrak{M}}$, a $\mathbb{Z}$-valued constructible function we interpret as a *multiplicity function*. If we can write $\mathfrak{M}$ locally as $[\mathrm{Crit}(f)/G]$ for $f : U \to \mathbb{C}$ holomorphic and U a complex manifold then $\nu_{\mathfrak{M}}(uG) = (-1)^{\dim U - \dim G}(1 - \chi(MF_f(u)))$ for $u \in \mathrm{Crit}(f)$, where $MF_f(u)$ is the *Milnor fibre* of f at u.
Using Key idea 3 we prove two identities on the Behrend function of the moduli stack $\mathfrak{M}$:

$$\nu_{\mathfrak{M}}(E_1 \oplus E_2) = (-1)^{\bar{\chi}([E_1],[E_2])}\nu_{\mathfrak{M}}(E_1)\nu_{\mathfrak{M}}(E_2), \tag{4}$$

$$\begin{aligned}\int_{\substack{[\lambda]\in\mathbb{P}(\mathrm{Ext}^1(E_2,E_1)):\\ \lambda \Leftrightarrow 0\to E_1\to F\to E_2\to 0}} \nu_{\mathfrak{M}}(F)\mathrm{d}\chi - \int_{\substack{[\lambda']\in\mathbb{P}(\mathrm{Ext}^1(E_1,E_2)):\\ \lambda' \Leftrightarrow 0\to E_2\to F'\to E_1\to 0}} \nu_{\mathfrak{M}}(F')\mathrm{d}\chi \\ = \big(\dim \mathrm{Ext}^1(E_2,E_1) - \dim \mathrm{Ext}^1(E_1,E_2)\big)\nu_{\mathfrak{M}}(E_1 \oplus E_2).\end{aligned} \tag{5}$$

Key idea 5: A morphism from a Ringel–Hall Lie algebra

Let $K(X) \subset H^{\mathrm{even}}(X;\mathbb{Q})$ be the lattice of Chern characters of coherent sheaves. Then $K(X) \cong \mathbb{Z}^l$, and there is an antisymmetric *Euler form* $\bar\chi : K(X) \times K(X) \to \mathbb{Z}$.
Define a Lie algebra $L(X)$ to have basis, as a $\mathbb{Q}$-vector space, symbols λ^α for $\alpha \in K(X)$, and Lie bracket

$$[\lambda^\alpha, \lambda^\beta] = (-1)^{\bar\chi(\alpha,\beta)} \bar\chi(\alpha,\beta)\, \lambda^{\alpha+\beta}.$$

We define a *Lie algebra morphism* $\Psi : \mathrm{SF}^{\mathrm{ind}}(\mathfrak{M}) \to L(X)$. Roughly speaking this is given by

$$\Psi\bigl([\mathfrak{R},\rho]\bigr) = \sum_{\alpha \in K(X)} \chi^{\mathrm{stk}}(\mathfrak{R} \times_{\mathfrak{M}} \mathfrak{M}^\alpha, \rho^*(\nu_{\mathfrak{M}}))\, \lambda^\alpha,$$

where χ^{stk} is a kind of stack-theoretic weighted Euler characteristic.

However, Euler characteristics of stacks are not well-defined: we want $\chi([X/G]) = \chi(X)/\chi(G)$ for X a scheme and G a Lie group, but $\chi(G) = 0$ whenever $\mathrm{rank}\, G > 0$.
The point of using $\mathrm{SF}^{\mathrm{ind}}(\mathfrak{M})$ is that it is generated by elements $[(U \times [\mathrm{Spec}\,\mathbb{C}/\mathbb{C}^*], \rho)]$ for U a $\mathbb{C}$-variety, and we set

$$\Psi\bigl([(U \times [\mathrm{Spec}\,\mathbb{C}/\mathbb{C}^*], \rho)]\bigr) \\ = \textstyle\sum_{\alpha \in K(X)} \chi\bigl(U \times_{\mathfrak{M}} \mathfrak{M}^\alpha, \rho^*(\nu_{\mathfrak{M}})\bigr)\, \lambda^\alpha,$$

which is well-defined as $U \times_{\mathfrak{M}} \mathfrak{M}^\alpha$ is a variety. We do not yet know how to extend Ψ from $\mathrm{SF}^{\mathrm{ind}}(\mathfrak{M})$ to $\mathrm{SF}(\mathfrak{M})$. To prove Ψ is a Lie algebra morphism we use the Behrend function identities (4)–(5).

We can now define *generalized Donaldson–Thomas invariants* $\bar{DT}^\alpha(\tau) \in \mathbb{Q}$: we set $\Psi(\bar\epsilon^\alpha(\tau)) = \bar{DT}^\alpha(\tau)\lambda^\alpha$ for all $\alpha \in K(\mathcal{A})$. The transformation law (3) for the $\bar\epsilon^\alpha(\tau)$ under change of stability condition can be written as a Lie algebra identity in $\mathrm{SF}^{\mathrm{ind}}(\mathfrak{M})$. So applying the Lie algebra morphism Ψ yields a transformation law for the $\bar{DT}^\alpha(\tau)$:

$$\bar{DT}^\alpha(\tilde\tau) = \sum_{\substack{\text{iso. classes}\\ \text{of } \Gamma, I, \kappa}} \pm\, U(\Gamma, I, \kappa; \tau, \tilde\tau) \cdot \prod_{i \in I} \bar{DT}^{\kappa(i)}(\tau) \cdot \prod_{\substack{\text{edges}\\ i - j \text{ in } \Gamma}} \bar\chi(\kappa(i), \kappa(j)). \tag{6}$$

Here Γ is a connected, simply-connected undirected graph with vertices I, $\kappa : I \to K(\mathcal{A})$ has $\sum_{i \in I} \kappa(i) = \alpha$, and $U(\Gamma, I, \kappa; \tau, \tilde\tau)$ in $\mathbb{Q}$ are explicit combinatorial coefficients.

Key idea 6: pair invariants $PI^{\alpha,N}(\tau')$

We define an auxiliary invariant $PI^{\alpha,N}(\tau') \in \mathbb{Z}$ counting 'stable pairs' (E, s) with E a semistable sheaf in class α and $s \in H^0(E(N))$, for $N \gg 0$. The moduli space of stable pairs is a projective $\mathbb{C}$-scheme with a symmetric obstruction theory, so $PI^{\alpha,N}(\tau')$ is unchanged by deformations of X.

By a similar proof to (6) we show that $PI^{\alpha,N}(\tau')$ can be written in terms of the $\bar{DT}^{\beta}(\tau)$ by

$$PI^{\alpha,N}(\tau') = \sum_{\substack{\alpha_1,\ldots,\alpha_n \in K(\mathcal{A}): \\ \alpha_1+\cdots+\alpha_n=\alpha, \\ \tau(\alpha_i)=\tau(\alpha)\ \forall i}} \frac{(-1)^n}{n!} \cdot \prod_{i=1}^{n} (-1)^{\bar{\chi}([\mathcal{O}_X(-N)]-\alpha_1-\cdots-\alpha_{i-1},\alpha_i)} \cdot \bar{\chi}([\mathcal{O}_X(-N)] - \alpha_1 - \cdots - \alpha_{i-1}, \alpha_i) \bar{DT}^{\alpha_i}(\tau). \tag{7}$$

Since the $PI^{\alpha,N}(\tau')$ are deformation- invariant, we use (7) and induction on $\operatorname{rank}\alpha$ to prove that $\bar{DT}^\alpha(\tau)$ is *unchanged under deformations of* X for all $\alpha \in K(X)$.
The $PI^{\alpha,N}(\tau')$ are similar to Pandharipande–Thomas invariants.
Note that $\bar{DT}^\alpha(\tau)$ counts strictly semistables E in a complicated way: there are $\mathbb{Q}$-valued contributions from every filtration $0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$ with E_i τ-semistable and $\tau(E_i) = \tau(E)$, weighted by $\nu_M(E)$. One can show by example that more obvious, simpler definitions of $\bar{DT}^\alpha(\tau)$ do not give deformation-invariant answers.

3. Final remarks

3.1. Integrality properties of the invariants

Suppose E is stable and rigid in class α. Then $kE = E \oplus \cdots \oplus E$ is strictly semistable in class $k\alpha$, for $k \geqslant 2$. Calculations show that E contributes 1 to $\bar{DT}^{\alpha}(\tau)$, and kE contributes $1/k^2$ to $\bar{DT}^{k\alpha}(\tau)$.

So we do not expect the $\bar{DT}^{\alpha}(\tau)$ to be integers, in general.

Define new invariants $\hat{DT}^{\alpha}(\tau) \in \mathbb{Q}$ by

$$\bar{DT}^{\alpha}(\tau) = \sum_{k \geqslant 1 : k \text{ divides } \alpha} \frac{1}{k^2} \hat{DT}^{\alpha/k}(\tau).$$

Then kE above contributes 1 to $\hat{DT}^{\alpha}(\tau)$ and 0 to $\hat{DT}^{k\alpha}(\tau)$ for $k > 1$.

Conjecture (Joyce–Song, now proved Davison–Meinhardt)

Suppose τ is generic, in the sense that $\tau(\alpha) = \tau(\beta)$ implies $\bar{\chi}(\alpha, \beta) = 0$. Then $\hat{DT}^{\alpha}(\tau) \in \mathbb{Z}$ for all $\alpha \in K(X)$.

These $\hat{DT}^{\alpha}(\tau)$ may coincide with invariants conjectured by Kontsevich–Soibelman, and in String Theory should perhaps be interpreted as 'numbers of BPS states'.

3.2. Relationship to Kontsevich and Soibelman's theory

Almost simultaneously with Joyce–Song, Kontsevich–Soibelman arXiv:0811.2435 proposed a much more ambitious, largely conjectural version of Donaldson–Thomas theory of Calabi–Yau 3-folds X. Rather than Euler characteristics χ, they worked over general motivic invariants Υ such as virtual Poincaré polynomials (now called 'refined D–T invariants'). Rather than using the abelian category $\mathrm{coh}(X)$ and Gieseker stability, as in Joyce–Song, they used the derived category $D^b\,\mathrm{coh}(X)$ and Bridgeland stability conditions $(Z,\mathcal{P})$ on $D^b\,\mathrm{coh}(X)$. (At the time no such $(Z,\mathcal{P})$ were known to exist.) By supposing that $\Upsilon(\mathbb{A}^1)-1$ is invertible in their base ring (impossible for Euler characteristics as $\chi(\mathbb{A}^1)=1$), they could work in an associative Ringel–Hall algebra, not a Lie algebra. Kontsevich–Soibelman had their own wall crossing formula for their motivic D–T invariants. This is in fact equivalent to a special case of my WCF for motivic invariants in math.AG/0410268, the prequel to Joyce–Song, but is written in a different way.

As in (6), my wall-crossing formula looks like

$$\bar{DT}^\alpha(\tilde\tau)=\sum_{\substack{\text{iso. classes}\\ \text{of }\Gamma,I,\kappa}}\pm U(\Gamma,I,\kappa;\tau,\tilde\tau)\cdot \prod_{i\in I}\bar{DT}^{\kappa(i)}(\tau)\cdot\prod_{\substack{\text{edges}\\ i-j\text{ in }\Gamma}}\bar\chi(\kappa(i),\kappa(j)), \tag{8}$$

where the $U(\Gamma,I,\kappa;\tau,\tilde\tau)$ are complicated combinatorial coefficients. The Kontsevich–Soibelman version roughly looks something like

$$\overrightarrow{\prod_{\substack{\alpha\in K(D^b\operatorname{coh}(X)):\tilde Z(\alpha)\in S\\ \text{in order of }\arg(\tilde Z(\alpha))\in(\theta_1,\theta_2)}}}\exp(\bar{DT}^\alpha(\tilde Z,\tilde{\mathcal P})\lambda_\alpha)=\overrightarrow{\prod_{\substack{\alpha\in K(D^b\operatorname{coh}(X)):Z(\alpha)\in S\\ \text{in order of }\arg(Z(\alpha))\in(\theta_1,\theta_2)}}}\exp(\bar{DT}^\alpha(Z,\mathcal P)\lambda_\alpha), \tag{9}$$

which happens in an explicit noncommutative algebra $\mathcal{A}$ (the 'quantum torus') generated by elements λ_α, and $S=\{re^{i\theta}:r>0,\ \theta\in(\theta_1,\theta_2)\}$ is a convex sector in $\mathbb{C}$, and we make the strong assumption that $Z(\alpha)\in S\Leftrightarrow\tilde Z(\alpha)\in S$. You can solve (9) inductively in $\mathcal{A}$ to write $\bar{DT}^\alpha(\tilde Z,\tilde{\mathcal P})$ in terms of $\bar{DT}^\beta(Z,\mathcal P)$ for many β, and the result will look something like (8). Equation (9) is the starting point for Bridgeland's work on 'Joyce structures'.

3.3. Generalization to non 3-Calabi–Yau invariants

In Gross–Joyce–Tanaka arXiv:2005.05637 we explain a conjectural universal framework for enumerative invariants counting semistable objects in additive categories $\mathcal{C}$, and their wall-crossing formulae, and prove the conjectures when $\mathcal{C} = \text{mod-}\mathbb{C}Q$ is the category of representations of a quiver Q with no oriented cycles. In work in progress, I am proving the conjectures in general for algebro-geometric invariants defined using Behrend–Fantechi virtual cycles. For example, this includes Mochizuki-style algebraic Donaldson invariants counting Gieseker semistable coherent sheaves on a projective surfaces X with $b^2_+(X) = 1$.

The picture is complicated, but roughly, we can define invariants like $\bar{DT}^\alpha(\tau)$ (though generally taking values in the homology of a moduli stack, not in $\mathbb{Q}$), which satisfy wall crossing formulae similar to (8)–(9) under change of stability condition. It seems an interesting question whether parts of the Bridgeland 'Joyce structures' picture generalize to these non CY3 invariants.