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Universal structures in enumerative invariant theories

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Joint work with Jacob Gross and Yuuji Tanaka,

available as J. Gross, D. Joyce and Y. Tanaka,
'Universal structures in enumerative invariant theories, I',
arXiv:2005.05637.

These slides available at
http://people.maths.ox.ac.uk/∼joyce/.

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1. Outline of the conjectural picture

An *enumerative invariant theory* in Algebraic or Differential Geometry is the study of invariants $I_\alpha(\tau)$ which 'count' τ-semistable objects E with fixed topological invariants $[\![E]\!] = \alpha$ in some geometric problem, usually by means of a virtual class $[\mathcal{M}_\alpha^{ss}(\tau)]_{virt}$ for the moduli space $\mathcal{M}_\alpha^{ss}(\tau)$ of τ-semistable objects in some homology theory, with $I_\alpha(\tau) = \int_{[\mathcal{M}_\alpha^{ss}(\tau)]_{virt}} \mu_\alpha$ for some natural cohomology class μ_α. We call the theory $\mathbb{C}$-*linear* if the objects E live in a $\mathbb{C}$-linear additive category $\mathcal{A}$. For example:

- Mochizuki-style invariants counting coherent sheaves on surfaces. (Think of as algebraic Donaldson invariants.)
- Donaldson–Thomas invariants of Calabi–Yau or Fano 3-folds.
- Donaldson–Thomas type invariants of Calabi–Yau 4-folds.
- $U(m)$ Donaldson invariants of 4-manifolds (with $b_+^2 = 1$).

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We conjecture that many such theories share a common universal structure. Here is an outline of this structure:

(a) We form two moduli stacks $\mathcal{M}, \mathcal{M}^{\mathrm{pl}}$ of all objects E in $\mathcal{A}$, where $\mathcal{M}$ is the usual moduli stack, and $\mathcal{M}^{\mathrm{pl}}$ the 'projective linear' moduli stack of objects E modulo 'projective isomorphisms', i.e. quotient by $\lambda \operatorname{id}_E$ for $\lambda \in \mathbb{G}_m$ or $\mathrm{U}(1)$.

(b) We are given a quotient $K_0(A) \twoheadrightarrow K(\mathcal{A})$, where $K(\mathcal{A})$ is the lattice of topological invariants $[\![E]\!]$ of E (e.g. fixed Chern classes). We split $\mathcal{M} = \bigoplus_{\alpha \in K(\mathcal{A})} \mathcal{M}_\alpha$, $\mathcal{M}^{\mathrm{pl}} = \bigoplus_{\alpha \in K(\mathcal{A})} \mathcal{M}_\alpha^{\mathrm{pl}}$.

(c) There is a symmetric biadditive *Euler form* $\chi : K(\mathcal{A}) \times K(\mathcal{A}) \to \mathbb{Z}$.

(d) We can form the homology $H_*(\mathcal{M}), H_*(\mathcal{M}^{\mathrm{pl}})$ over $\mathbb{Q}$, with $H_*(\mathcal{M}) = \bigoplus_{\alpha \in K(\mathcal{A})} H_*(\mathcal{M}_\alpha)$, $H_*(\mathcal{M}^{\mathrm{pl}}) = \bigoplus_{\alpha \in K(\mathcal{A})} H_*(\mathcal{M}_\alpha^{\mathrm{pl}})$. Define shifted versions $\hat{H}_*(\mathcal{M}), \check{H}_*(\mathcal{M}^{\mathrm{pl}})$ by $\hat{H}_n(\mathcal{M}_\alpha) = H_{n-\chi(\alpha,\alpha)}(\mathcal{M}_\alpha)$, $\check{H}_n(\mathcal{M}_\alpha^{\mathrm{pl}}) = H_{n+2-\chi(\alpha,\alpha)}(\mathcal{M}_\alpha^{\mathrm{pl}})$. Then previous work by me makes $\hat{H}_*(\mathcal{M})$ into a *graded vertex algebra*, and $\check{H}_*(\mathcal{M}^{\mathrm{pl}})$ into a *graded Lie algebra*.

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(e) There is a notion of *stability condition* τ on $\mathcal{A}$. When $\mathcal{A} = \mathrm{coh}(X)$, this can be Gieseker stability for a polarization on X. For Donaldson theory for a compact oriented 4-manifold X with $b_+^2(X) = 1$, the stability condition is the splitting $H_{\mathrm{dR}}^2(X, \mathbb{R}) = H_+^2(X) \oplus H_-^2(X)$ induced by a metric g. For each $\alpha \in K(\mathcal{A})$ we can form moduli spaces $\mathcal{M}_\alpha^{\mathrm{st}}(\tau) \subseteq \mathcal{M}_\alpha^{\mathrm{ss}}(\tau)$ of τ-(semi)stable objects in class α. Here $\mathcal{M}_\alpha^{\mathrm{st}}(\tau)$ is a substack of $\mathcal{M}_\alpha^{\mathrm{pl}}$, and has the structure of a 'virtual oriented manifold' (in Algebraic Geometry, it may be a $\mathbb{C}$-scheme with perfect obstruction theory; in Differential Geometry, under genericness it may be an oriented manifold). Also $\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)$ is compact (proper). Thus, if $\mathcal{M}_\alpha^{\mathrm{st}}(\tau) = \mathcal{M}_\alpha^{\mathrm{ss}}(\tau)$ we have a virtual class $[\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)]_{\mathrm{virt}}$, which we regard as an element of $H_*(\mathcal{M}_\alpha^{\mathrm{pl}})$. The virtual dimension is $\mathrm{vdim}_\mathbb{R}[\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)]_{\mathrm{virt}} = 2 - \chi(\alpha, \alpha)$, so $[\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)]_{\mathrm{virt}}$ lies in $\check{H}_0(\mathcal{M}_\alpha^{\mathrm{pl}}) \subset \check{H}_0(\mathcal{M}^{\mathrm{pl}})$, which is a Lie algebra by (b).

We can prove all of (a)–(e) already in the cases we care about.

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Here is the conjectural part of the picture:

(f) For many theories, there is a problem defining the invariants
$[\mathcal{M}_\alpha^{ss}(\tau)]_{virt}$ when $\mathcal{M}_\alpha^{st}(\tau) \neq \mathcal{M}_\alpha^{ss}(\tau)$, i.e. when the moduli
spaces $\mathcal{M}_\alpha^{ss}(\tau)$ contain *strictly τ-semistable points* (in gauge
theory, these are *reducible connections*).
We conjecture there is a systematic way to define $[\mathcal{M}_\alpha^{ss}(\tau)]_{virt}$
in homology over $\mathbb{Q}$ (not $\mathbb{Z}$) in these cases. (In gauge theory,
this requires a condition analogous to $b_+^2 \geqslant 1$.)

(g) If $\tau, \tilde{\tau}$ are stability conditions and $\alpha \in K(\mathcal{A})$, we expect that

$$[\mathcal{M}_\alpha^{ss}(\tilde{\tau})]_{virt} = \sum_{\alpha_1 + \cdots + \alpha_n = \alpha} \tilde{U}(\alpha_1, \ldots, \alpha_n; \tau, \tilde{\tau}) \cdot \big[\big[\ldots \big[[\mathcal{M}_{\alpha_1}^{ss}(\tau)]_{virt}, \\ [\mathcal{M}_{\alpha_2}^{ss}(\tau)]_{virt} \big], \ldots \big], [\mathcal{M}_{\alpha_n}^{ss}(\tau)]_{virt} \big], \qquad (1)$$

where $\tilde{U}(-)$ are combinatorial coefficients defined in my
previous work on wall-crossing formulae for motivic invariants,
and $[,]$ is the Lie bracket on $\check{H}_0(\mathcal{M}^{pl})$ from (b).

(h) We can often give an explicit, inductive definition of the
$[\mathcal{M}_\alpha^{ss}(\tau)]_{virt}$ using (1) and the method of *pair invariants*.

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We prove our conjectures completely when $\mathcal{A} = \mathrm{mod}\text{-}\mathbb{C}Q$ is the category of representations of a quiver Q without oriented cycles, and stability conditions τ are slope stability. In this case, if $\mathcal{M}_\alpha^{\mathrm{st}}(\tau) = \mathcal{M}_\alpha^{\mathrm{ss}}(\tau)$ then $\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)$ is a smooth projective $\mathbb{C}$-scheme (a compact complex manifold), given by a GIT quotient $\mathbb{A}^N /\!/_\tau \mathrm{PGL}_\alpha$, so it has a fundamental class $[\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)]_{\mathrm{fund}}$, and we set $[\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)]_{\mathrm{virt}} = [\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)]_{\mathrm{fund}}$. But we also define $[\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)]_{\mathrm{virt}}$ if $\mathcal{M}_\alpha^{\mathrm{st}}(\tau) \neq \mathcal{M}_\alpha^{\mathrm{ss}}(\tau)$.

- I know how to extend the proof to quivers with relations $\mathrm{mod}\text{-}\mathbb{C}Q/I$ and slope stability μ with Behrend–Fantechi virtual cycles when $\mathcal{M}_\alpha^{\mathrm{st}}(\mu) = \mathcal{M}_\alpha^{\mathrm{ss}}(\mu)$, and to 4-Calabi–Yau dg-quivers, with Borisov–Joyce virtual cycles.
- I believe I am close to proving the conjectures for $\mathrm{coh}(X)$ for X a curve, surface or Fano 3-fold, with Gieseker stability and Behrend–Fantechi virtual cycles.
- An adapted proof should work for Calabi–Yau 4-folds, if B–J virtual cycles behave well under torus localization.
- Donaldson theory for $b_+^2 = 1$ looks difficult for now.

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Remarks on counting strictly τ-semistables

When $\mathcal{M}_\alpha^{\mathrm{st}}(\tau) = \mathcal{M}_\alpha^{\mathrm{ss}}(\tau)$, the virtual classes $[\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)]_{\mathrm{virt}}$ are defined using a geometric structure on $\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)$ (e.g. smooth $\mathbb{C}$-schemes, or $\mathbb{C}$-schemes with perfect obstruction theories, or -2-shifted symplectic derived schemes) by a known construction.

When $\mathcal{M}_\alpha^{\mathrm{st}}(\tau) \neq \mathcal{M}_\alpha^{\mathrm{ss}}(\tau)$, we currently have *no definition* of $[\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)]_{\mathrm{virt}}$ in terms of a geometric structure on $\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)$.

For quivers, our proof works by showing that there are unique $[\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)]_{\mathrm{virt}}$ when $\mathcal{M}_\alpha^{\mathrm{st}}(\tau) \neq \mathcal{M}_\alpha^{\mathrm{ss}}(\tau)$, extending the given ones when $\mathcal{M}_\alpha^{\mathrm{st}}(\tau) = \mathcal{M}_\alpha^{\mathrm{ss}}(\tau)$, which also satisfy the wall-crossing formula (1). So the definition involves *all stability conditions*, not just one.

For Joyce–Song Donaldson–Thomas invariants, counting strictly τ-semistables is a complicated mess, and uses rational weights.

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Motivic invariants versus homology

An invariant $I(X)$ of algebraic $\mathbb{K}$-varieties X in a commutative ring R is *motivic* if $I(X) = I(Y) + I(X \setminus Y)$ if $Y \subset X$ is closed subvariety, and $I(X \times Y) = I(X)I(Y)$. Examples are the Euler characteristic $\chi(X)$, with $R = \mathbb{Z}$, and virtual Poincaré polynomials. Over 2003-8 I worked on invariants $I_\alpha^{ss}(\tau)$ which 'counted' Algebro-Geometric moduli stacks $\mathcal{M}_\alpha^{st}(\tau) \subseteq \mathcal{M}_\alpha^{ss}(\tau)$ using motivic invariants, including wall-crossing formulae under change of stability condition. An important tool was Ringel–Hall algebras and Lie algebras of stack functions $SF(\mathcal{M})$ on moduli spaces $\mathcal{M}$. (See 'Configurations in abelian categories I–IV', and Joyce–Song.) Homology $H_*(\mathcal{M})$ and virtual classes $[\mathcal{M}_\alpha^{ss}(\tau)]_{virt}$ are *not* motivic, so this old work does not apply. But the new theory works by taking the old results on invariants and wall-crossing formulae in a Lie algebra of stack functions $SF_{al}^{ind}(\mathcal{M})$, and replacing this by the Lie algebra $\check{H}_0(\mathcal{M}^{pl})$ that comes out of my vertex algebra work.

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Which invariant theories fit into our framework?

I expect the following to satisfy versions of our conjecture:

- Counting vector bundles or sheaves on projective curves X.
- Counting sheaves plus extra data (Higgs fields, . . .) on curves.
- Counting sheaves on surfaces with $h^{2,0}(X) = 0$, à la Mochizuki.
- Donaldson–Thomas invariants of Fano 3-folds.
- Donaldson–Thomas type invariants of Calabi–Yau 4-folds.
- $\mathrm{U}(m)$ Donaldson invariants of 4-manifolds with $b_+^2 = 1$.
- Quivers, quivers with relations, CY4 dg-quivers.

Donaldson–Thomas invariants of Calabi–Yau 3-folds are related, but don't fit the structure above exactly (the virtual dimension is not $2 - \chi(\alpha, \alpha)$). Similarly for Donaldson invariants with $b_+^2 > 1$, and surfaces with $h^{2,0}(X) > 0$. (Actually our theory works here (?), but the invariants are zero; fix determinants to make them nonzero.)

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Interesting questions and future projects

- Do the invariants $[\mathcal{M}^{\mathrm{ss}}_\alpha(\tau)]_{\mathrm{virt}}$ in these theories have a common universal structure determined by a small amount of data? (Something like Seiberg–Witten $\Rightarrow$ Donaldson invariants, MNOP Conjecture, etc.)

- Does the vertex algebra structure relate to deep properties of enumerative invariants? (Modularity of generating functions, etc.)

- How should the picture be modified for theories like Donaldson theory for $b^2_+ > 1$, surfaces with $h^{2,0}(X) > 0$? (Now no wall-crossing, but counting strictly τ-semistables and pair invariants make sense, so we may have something to say.)

- Replace $H_*(-)$ by a complex oriented generalized homology theory $E_*(-)$? K-theory enumerative invariants already studied.

- Extension to triangulated categories, Bridgeland stability?

- Does our picture have an interpretation in String Theory?

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2. More details
2.1. Vertex and Lie algebras on homology of moduli stacks

We will explain the Algebraic Geometry version of our theory. Let $\mathcal{A}$ be a $\mathbb{C}$-linear abelian or triangulated category from Algebraic Geometry or Representation Theory, e.g. $\mathcal{A} = \mathrm{coh}(X)$ or $D^b \mathrm{coh}(X)$ for X a smooth projective $\mathbb{C}$-scheme, or $\mathcal{A} = \mathrm{mod}\text{-}\mathbb{C}Q$ or $D^b \mathrm{mod}\text{-}\mathbb{C}Q$. Write $\mathcal{M}$ for the moduli stack of objects in $\mathcal{A}$, which is an Artin $\mathbb{C}$-stack in the abelian case, and a higher $\mathbb{C}$-stack in the triangulated case. There is a morphism $\Phi : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ acting by $([E], [F]) \to [E \oplus F]$ on $\mathbb{C}$-points.

Now $\mathbb{G}_m$ acts on objects E in $\mathcal{A}$ with $\lambda \in \mathbb{G}_m$ acting as $\lambda \, \mathrm{id}_E : E \to E$. This induces an action $\Psi : [*/\mathbb{G}_m] \times \mathcal{M} \to \mathcal{M}$ of the group stack $[*/\mathbb{G}_m]$ on $\mathcal{M}$. We write $\mathcal{M}^{\mathrm{pl}} = \mathcal{M}/[*/\mathbb{G}_m]$ for the quotient, called the 'projective linear' moduli stack. There is a morphism $\mathcal{M} \to \mathcal{M}^{\mathrm{pl}}$ which is a $[*/\mathbb{G}_m]$-fibration on $\mathcal{M} \setminus \{[0]\}$.

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We need some extra data:

- A quotient $K_0(A) \twoheadrightarrow K(\mathcal{A})$ giving splittings
  $\mathcal{M} = \bigoplus_{\alpha \in K(\mathcal{A})} \mathcal{M}_\alpha$, $\mathcal{M}^{\mathrm{pl}} = \bigoplus_{\alpha \in K(\mathcal{A})} \mathcal{M}_\alpha^{\mathrm{pl}}$.
- A symmetric biadditive *Euler form* $\chi : K(\mathcal{A}) \times K(\mathcal{A}) \to \mathbb{Z}$.
- A perfect complex $\Theta^\bullet$ on $\mathcal{M} \times \mathcal{M}$ satisfying some
  assumptions, including $\mathrm{rank}\, \Theta|_{\mathcal{M}_\alpha \times \mathcal{M}_\beta} = \chi(\alpha, \beta)$.
  If $\mathcal{A}$ is a 4-Calabi–Yau category, and we will use Borisov–Joyce
  virtual classes, we take $\Theta^\bullet = (\mathcal{E}xt^\bullet)^\vee$, where $\mathcal{E}xt^\bullet \to \mathcal{M} \times \mathcal{M}$
  is the *Ext complex*. Otherwise we take $\Theta^\bullet = (\mathcal{E}xt^\bullet)^\vee +$
  $\sigma^*(\mathcal{E}xt^\bullet)$, where $\sigma : \mathcal{M} \times \mathcal{M} \to \mathcal{M} \times \mathcal{M}$ swaps the factors.
- Signs $\epsilon_{\alpha,\beta} \in \{\pm 1\}$ for $\alpha, \beta \in K(\mathcal{A})$ with $\epsilon_{\alpha,\beta} \cdot \epsilon_{\alpha+\beta,\gamma} =$
  $\epsilon_{\alpha,\beta+\gamma} \cdot \epsilon_{\beta,\gamma}$ and $\epsilon_{\alpha,\beta} \cdot \epsilon_{\beta,\alpha} = (-1)^{\chi(\alpha,\beta)+\chi(\alpha,\alpha)\chi(\beta,\beta)}$.
  (These compare orientations on $\mathcal{M}_\alpha, \mathcal{M}_\beta, \mathcal{M}_{\alpha+\beta}$.)

Then we can make the homology $H_*(\mathcal{M})$, with grading shifted to
$\hat{H}_*(\mathcal{M})$ as above, into a *graded vertex algebra*.

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Writing $H_*([*/\mathbb{G}_m]) = \mathbb{Q}[t]$ with $\deg t = 2$, the state-field correspondence $Y(z)$ is given by, for $u \in H_a(\mathcal{M}_\alpha)$, $v \in H_b(\mathcal{M}_\beta)$

$$Y(u,z)v = \epsilon_{\alpha,\beta}(-1)^{a\chi(\beta,\beta)} z^{\chi(\alpha,\beta)} \cdot H_*(\Phi \circ (\Psi \times \mathrm{id})) \qquad (2)$$
$$\left\{ \left(\sum_{i \geqslant 0} z^i t^i \right) \boxtimes \left[(u \boxtimes v) \cap \exp \left(\sum_{j \geqslant 1} (-1)^{j-1}(j-1)! z^{-j} \mathrm{ch}_j([\Theta^\bullet]) \right) \right] \right\}.$$

The identity $\mathbb{1}$ is $1 \in H_0(\mathcal{M}_0)$. Define $e^{zD} : \check{H}_*(\mathcal{M}) \to \check{H}_*(\mathcal{M})[[z]]$ by $Y(v,z)\mathbb{1} = e^{zD}v$. Then $(\check{H}_*(\mathcal{M}), \mathbb{1}, e^{zD}, Y)$ is a graded vertex algebra. By a standard construction in vertex algebra theory, $\check{H}_{*+2}(\mathcal{M})/D(\check{H}_*(\mathcal{M}))$ is a graded Lie algebra. In the abelian category case at least, there is a canonical isomorphism $\check{H}_*(\mathcal{M}^{\mathrm{pl}}) \cong \check{H}_{*+2}(\mathcal{M})/D(\check{H}_*(\mathcal{M}))$. This makes $\check{H}_*(\mathcal{M}^{\mathrm{pl}})$ into a graded Lie algebra, and $\check{H}_0(\mathcal{M}^{\mathrm{pl}})$ into a Lie algebra.

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Remarks

• One can often write down $\check{H}_*(\mathcal{M})$ and $\check{H}_*(\mathcal{M}^{\mathrm{pl}})$ with their algebraic structures explicitly. The answer is usually simpler in the derived category case. For example, Jacob Gross showed that if a smooth projective $\mathbb{C}$-scheme X is a curve, surface, or toric variety, and $\mathcal{M}$ is the moduli stack of $D^b\operatorname{coh}(X)$, then

$$\hat{H}_*(\mathcal{M}, \mathbb{Q}) \cong \mathbb{Q}[K^0_{\mathrm{sst}}(X)] \otimes_R \operatorname{Sym}^*(K^0(X^{\mathrm{an}}) \otimes_{\mathbb{Z}} t^2\mathbb{Q}[t^2])$$
$$\otimes_R \bigwedge\nolimits^*(K^1(X^{\mathrm{an}}) \otimes_{\mathbb{Z}} t\mathbb{Q}[t^2]), \qquad (3)$$

with a super-lattice vertex algebra structure. Thus we can use this for explicit computations in examples, as well as for abstract theory.
• It helps to study $[\mathcal{M}^{\mathrm{ss}}_\alpha(\tau)]_{\mathrm{virt}}$ in $\operatorname{coh}(X)$ using $H_*(\mathcal{M})$, $H_*(\mathcal{M}^{\mathrm{pl}})$ for $D^b\operatorname{coh}(X)$, so we can use the presentation (3).
• Although Lie algebras are much simpler than vertex algebras, it is difficult to write down the Lie bracket on $\check{H}_*(\mathcal{M}^{\mathrm{pl}})$ explicitly: the best way seems to be via the vertex algebra structure on $\hat{H}_*(\mathcal{M})$.

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2.2. Virtual classes of moduli spaces

The vertex and Lie algebras $\hat{H}_*(\mathcal{M})$, $\check{H}_*(\mathcal{M}^{\mathrm{pl}})$ above work for $\mathcal{M}$ the moduli stack of objects in $\mathrm{coh}(X)$ or $D^b\mathrm{coh}(X)$ for X a smooth projective $\mathbb{C}$-scheme of any dimension. However, defining virtual classes $[\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)]_{\mathrm{virt}}$ when $\mathcal{M}_\alpha^{\mathrm{st}}(\tau){=}\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)$ is much more restrictive:

- If $\dim \mathcal{A} = 1$, say if $\mathcal{A} = \mathrm{mod}\text{-}\mathbb{C}Q$ or $\mathcal{A} = \mathrm{coh}(X)$ for X a curve, then $\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)$ is a smooth projective $\mathbb{C}$-scheme, and has a fundamental class $[\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)]_{\mathrm{fund}}$.
- If $\dim \mathcal{A} = 2$, say if $\mathcal{A} = \mathrm{mod}\text{-}\mathbb{C}Q/I$ or $\mathcal{A} = \mathrm{coh}(X)$ for X a surface, then $\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)$ is a projective $\mathbb{C}$-scheme with obstruction theory, and has a Behrend–Fantechi virtual class $[\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)]_{\mathrm{virt}}$.
- If $\mathcal{A} = \mathrm{coh}(X)$ for X a Calabi–Yau or Fano 3-fold, one can also define Behrend–Fantechi virtual classes $[\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)]_{\mathrm{virt}}$.
- If $\mathcal{A} = \mathrm{coh}(X)$ for X a Calabi–Yau 4-fold, Borisov–Joyce define a different kind of virtual class $[\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)]_{\mathrm{virt}}$, with *half the expected dimension* of the Behrend–Fantechi class.

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2.3 On moduli stacks and moduli schemes

There are two main ways of forming moduli spaces in Algebraic Geometry: as *schemes* or *stacks*. An important difference is that if $\mathcal{M}$ is a moduli stack of objects E, then automorphism groups are remembered in the isotropy groups of $\mathcal{M}$ by $\mathrm{Iso}_{\mathcal{M}}([E]) = \mathrm{Aut}(E)$, but moduli schemes forget automorphism groups.

Our moduli stacks $\mathcal{M}, \mathcal{M}^{\mathrm{pl}}$ differ in that their isotropy groups are $\mathrm{Iso}_{\mathcal{M}}([E]) = \mathrm{Aut}(E)$, but $\mathrm{Iso}_{\mathcal{M}^{\mathrm{pl}}}([E]) = \mathrm{Aut}(E)/(\mathbb{G}_m \cdot \mathrm{id}_E)$.

If E is τ-stable then $\mathrm{Aut}(E) = \mathbb{G}_m \cdot \mathrm{id}_E$, so $\mathrm{Iso}_{\mathcal{M}^{\mathrm{pl}}}([E]) = \{1\}$.

Because of this, the τ-stable moduli scheme $\mathcal{M}_\alpha^{\mathrm{st}}(\tau)$ is actually an *open substack* in $\mathcal{M}^{\mathrm{pl}}$ (but not $\mathcal{M}$). This makes $\mathcal{M}^{\mathrm{pl}}$ useful for us.

The τ-semistable moduli scheme $\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)$ has the *good property* that it is usually compact (proper). But it has the *bad properties* that it does not map to $\mathcal{M}^{\mathrm{pl}}$ or $\mathcal{M}$, and the obstruction theory (or other nice structure) on $\mathcal{M}_\alpha^{\mathrm{st}}(\tau)$ does not extend to $\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)$, so we cannot define a virtual class $[\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)]_{\mathrm{virt}}$ unless $\mathcal{M}_\alpha^{\mathrm{st}}(\tau) = \mathcal{M}_\alpha^{\mathrm{ss}}(\tau)$.

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3. Proof in the case of quivers

Let $Q = (Q_0, Q_1, h, t)$ be a quiver, with finite sets Q_0 of vertices and Q_1 of edges, and head and tail maps $h, t : Q_1 \to Q_0$. Then we have a $\mathbb{C}$-linear abelian category $\mathrm{mod}\text{-}\mathbb{C}Q$ of *representations* (V_v, ρ_e) of Q, comprising a finite-dimensional $\mathbb{C}$-vector space V_v for each $v \in Q_0$ and a linear map $\rho_e : V_{t(e)} \to V_{h(e)}$ for each $e \in Q_1$. The *dimension vector* of (V_v, ρ_e) is $\boldsymbol{d} \in \mathbb{N}^{Q_0}$, where $\boldsymbol{d}(v) = \dim V_v$. We can work out our theory very explicitly for $\mathcal{A} = \mathrm{mod}\text{-}\mathbb{C}Q$. We take $K(\mathcal{A}) = \mathbb{Z}^{Q_0}$. Then $\mathcal{M} = \coprod_{\boldsymbol{d} \in \mathbb{N}^{Q_0}} \mathcal{M}_{\boldsymbol{d}}$, $\mathcal{M}^{\mathrm{pl}} = \coprod_{\boldsymbol{d} \in \mathbb{N}^{Q_0}} \mathcal{M}_{\boldsymbol{d}}^{\mathrm{pl}}$, where $\mathcal{M}_{\boldsymbol{d}} = [R_{\boldsymbol{d}} / \mathrm{GL}_{\boldsymbol{d}}]$, $\mathcal{M}_{\boldsymbol{d}}^{\mathrm{pl}} = [R_{\boldsymbol{d}} / \mathrm{PGL}_{\boldsymbol{d}}]$ with

$$R_{\boldsymbol{d}} = \prod_{e \in Q_1} \mathrm{Hom}(\mathbb{C}^{t(\boldsymbol{d}(e))}, \mathbb{C}^{h(\boldsymbol{d}(e))}), \ \mathrm{GL}_{\boldsymbol{d}} = \prod_{v \in Q_0} \mathrm{GL}(\boldsymbol{d}(v)),$$

and $\mathrm{PGL}_{\boldsymbol{d}} = \mathrm{GL}_{\boldsymbol{d}} / \mathbb{G}_m$. Hence $H_*(\mathcal{M}_{\boldsymbol{d}}) \cong H_*(B\,\mathrm{GL}_{\boldsymbol{d}})$ and $H_*(\mathcal{M}_{\boldsymbol{d}}^{\mathrm{pl}}) \cong H_*(B\,\mathrm{PGL}_{\boldsymbol{d}})$, which we can write explicitly.

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Slope stability conditions

Fix $\mu_v \in \mathbb{R}$ for all $v \in Q_0$. Define $\mu : \mathbb{N}^{Q_0} \setminus \{0\} \to \mathbb{R}$ by

$$\mu(\boldsymbol{d}) = \left(\textstyle\sum_{v \in Q_0} \mu_v \boldsymbol{d}(v)\right) / \left(\textstyle\sum_{v \in Q_0} \boldsymbol{d}(v)\right).$$

We call μ a *slope function*. An object $0 \neq E \in \mathrm{mod}\text{-}\mathbb{C}Q$ is called
μ-*semistable* (or μ-*stable*) if whenever $0 \neq E' \subsetneq E$ is a subobject
we have $\mu(\mathbf{dim}E') \geqslant \mu(\mathbf{dim}E)$ (or $\mu(\mathbf{dim}E') > \mu(\mathbf{dim}E)$).
Recall that $\mathcal{M}_{\boldsymbol{d}}^{\mathrm{pl}} = [R_{\boldsymbol{d}} / \mathrm{PGL}_{\boldsymbol{d}}]$ as a quotient stack. King (1994)
showed that there is a linearization θ of the action of $\mathrm{PGL}_{\boldsymbol{d}}$ on
$R_{\boldsymbol{d}}$, such that a $\mathbb{C}$-point $[E] \in [R_{\boldsymbol{d}} / \mathrm{PGL}_{\boldsymbol{d}}]$ is μ-(semi)stable in
$\mathrm{mod}\text{-}\mathbb{C}Q$ iff the corresponding point in $R_{\boldsymbol{d}}$ is GIT (semi)stable.
Hence there are moduli schemes $\mathcal{M}_{\boldsymbol{d}}^{\mathrm{st}}(\mu) \subseteq \mathcal{M}_{\boldsymbol{d}}^{\mathrm{ss}}(\mu)$ which are the
GIT (semi)stable quotients $R_{\boldsymbol{d}} /\!\!/_\theta^{\mathrm{st}} \mathrm{PGL}_{\boldsymbol{d}} \subseteq R_{\boldsymbol{d}} /\!\!/_\theta^{\mathrm{ss}} \mathrm{PGL}_{\boldsymbol{d}}$.
If Q has *no oriented cycles* then a $\mathbb{G}_m$ subgroup of $\mathrm{PGL}_{\boldsymbol{d}}$ acts on
$R_{\boldsymbol{d}}$ with positive weights, so $\mathcal{M}_{\boldsymbol{d}}^{\mathrm{ss}}(\mu) = R_{\boldsymbol{d}} /\!\!/_\theta^{\mathrm{ss}} \mathrm{PGL}_{\boldsymbol{d}}$ is a
projective $\mathbb{C}$-scheme. Also $\mathcal{M}_{\boldsymbol{d}}^{\mathrm{st}}(\mu) = R_{\boldsymbol{d}} /\!\!/_\theta^{\mathrm{st}} \mathrm{PGL}_{\boldsymbol{d}}$ is a smooth
quasi-projective $\mathbb{C}$-scheme, an open substack of $\mathcal{M}_{\boldsymbol{d}}^{\mathrm{pl}} = [R_{\boldsymbol{d}} / \mathrm{PGL}_{\boldsymbol{d}}]$.

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Thus, if Q has no oriented cycles, and μ is a slope function on mod-$\mathbb{C}Q$, and $\boldsymbol{d} \in \mathbb{N}^{Q_0} \setminus \{0\}$ with $\mathcal{M}^{\mathrm{st}}_{\boldsymbol{d}}(\mu) = \mathcal{M}^{\mathrm{ss}}_{\boldsymbol{d}}(\mu)$, then $\mathcal{M}^{\mathrm{ss}}_{\boldsymbol{d}}(\mu)$ is a smooth projective $\mathbb{C}$-scheme and an open substack of $\mathcal{M}^{\mathrm{pl}}_{\boldsymbol{d}}$, and has a fundamental class $[\mathcal{M}^{\mathrm{ss}}_{\boldsymbol{d}}(\mu)]_{\mathrm{fund}}$ in $H_*(\mathcal{M}^{\mathrm{pl}}_{\boldsymbol{d}})$. It has dimension $2 - \chi(\boldsymbol{d}, \boldsymbol{d})$, where $\chi : \mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0} \to \mathbb{Z}$ is

$$\chi(\boldsymbol{d}, \boldsymbol{e}) = 2 \sum_{v \in Q_0} \boldsymbol{d}(v)\boldsymbol{e}(v) - \sum_{e \in Q_1} (\boldsymbol{d}(h(e))\boldsymbol{e}(t(e)) + \boldsymbol{d}(t(e))\boldsymbol{e}(h(e))).$$

Theorem 1

Let Q be a quiver with no oriented cycles. Then for all slope functions μ on mod-$\mathbb{C}Q$ and $\boldsymbol{d} \in \mathbb{N}^{Q_0} \setminus \{0\}$, there exist unique classes $[\mathcal{M}^{\mathrm{ss}}_{\boldsymbol{d}}(\mu)]_{\mathrm{virt}} \in H_{2-\chi(\boldsymbol{d},\boldsymbol{d})}(\mathcal{M}^{\mathrm{pl}}_{\boldsymbol{d}}) = \check{H}_0(\mathcal{M}^{\mathrm{pl}}_{\boldsymbol{d}})$ such that:

(i) *If $\mathcal{M}^{\mathrm{st}}_{\boldsymbol{d}}(\mu) = \mathcal{M}^{\mathrm{ss}}_{\boldsymbol{d}}(\mu)$ then $[\mathcal{M}^{\mathrm{ss}}_{\boldsymbol{d}}(\mu)]_{\mathrm{virt}} = [\mathcal{M}^{\mathrm{ss}}_{\boldsymbol{d}}(\mu)]_{\mathrm{fund}}$.*

(ii) *The $[\mathcal{M}^{\mathrm{ss}}_{\boldsymbol{d}}(\mu)]_{\mathrm{virt}}$ transform according to the wall-crossing formula (1) above in the Lie algebra $\check{H}_0(\mathcal{M}^{\mathrm{pl}})$ under change of stability condition.*

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We also prove:

Theorem 2

*There is a notion of **morphism of quivers** $\lambda : Q \to Q'$, which induces a functor $\lambda_* : \text{mod-}\mathbb{C}Q \to \text{mod-}\mathbb{C}Q'$, and morphisms of vertex algebras $\Omega : \hat{H}_*(\mathcal{M}) \to \hat{H}_*(\mathcal{M}')$ and of Lie algebras $\Omega^{\text{pl}} : \check{H}_*(\mathcal{M}^{\text{pl}}) \to \check{H}_*(\mathcal{M}'^{\text{pl}})$. If μ' is a slope function on $\text{mod-}\mathbb{C}Q'$ then $\mu = \mu \circ \lambda_*$ is a slope function on $\text{mod-}\mathbb{C}Q$. Then for each $\boldsymbol{d} \in \mathbb{N}^{Q_0} \setminus \{0\}$ with $\lambda_*(\boldsymbol{d}) = \boldsymbol{d}' \in \mathbb{N}^{Q'_0} \setminus \{0\}$, the virtual classes $[\mathcal{M}^{\text{ss}}_{\boldsymbol{d}}(\mu)]_{\text{virt}}$ of Theorem 1 satisfy*

$$\prod_{v \in Q_0} \boldsymbol{d}(v)! \cdot \Omega^{\text{pl}}\big([\mathcal{M}^{\text{ss}}_{\boldsymbol{d}}(\mu)]_{\text{virt}}\big) = \prod_{v' \in Q'_0} \boldsymbol{d}'(v')! \cdot [\mathcal{M}'^{\text{ss}}_{\boldsymbol{d}'}(\mu')]_{\text{virt}}.$$

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Sketch proof of Theorems 1 and 2

We call a slope function μ *decreasing* if for all edges $\overset{e}{\bullet} \longrightarrow \overset{w}{\bullet}$ in Q we have $\mu_v > \mu_w$. Such μ exist if and only if Q has no oriented cycles. If μ is decreasing, for each $\boldsymbol{d} \in \mathbb{N}^{Q_0} \setminus \{0\}$, either:

(a) $\boldsymbol{d} = \delta_v$ for some $v \in Q_0$, that is, $\boldsymbol{d}(v) = 1$ and $\boldsymbol{d}(w) = 0$ for $w \neq v$. Then $\mathcal{M}_{\boldsymbol{d}}^{\mathrm{st}}(\mu) = \mathcal{M}_{\boldsymbol{d}}^{\mathrm{ss}}(\mu)$ is a single point $*$.

(b) $\boldsymbol{d} = n\delta_v$ for some $v \in Q_0$ and $n > 1$. Then $\mathcal{M}_{\boldsymbol{d}}^{\mathrm{st}}(\mu) = \emptyset$ and $\mathcal{M}_{\boldsymbol{d}}^{\mathrm{ss}}(\mu) \cong [*/\operatorname{PGL}(n, \mathbb{C})]$. Note that $2 - \chi(\boldsymbol{d}, \boldsymbol{d}) = 2 - 2n^2 < 0$.

(c) $\boldsymbol{d} \neq n\delta_v$ for any $v \in Q_0$, $n \geqslant 1$. Then $\mathcal{M}_{\boldsymbol{d}}^{\mathrm{st}}(\mu) = \mathcal{M}_{\boldsymbol{d}}^{\mathrm{ss}}(\mu) = \emptyset$.

Hence the classes $[\mathcal{M}_{\boldsymbol{d}}^{\mathrm{ss}}(\mu)]_{\mathrm{virt}}$ in Theorem 1 must be

$$[\mathcal{M}_{\boldsymbol{d}}^{\mathrm{ss}}(\mu)]_{\mathrm{virt}} = \begin{cases} 1 \in H_0(\mathcal{M}_{\boldsymbol{d}}^{\mathrm{pl}}) \cong R, & \boldsymbol{d} = \delta_v, \ v \in Q_0, \\ 0, & \text{otherwise,} \end{cases} \quad (4)$$

as in case (b) $[\mathcal{M}_{\boldsymbol{d}}^{\mathrm{ss}}(\mu)]_{\mathrm{virt}} \in H_{<0}(\mathcal{M}_{\boldsymbol{d}}^{\mathrm{pl}}) = 0$.

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Equation (4) for some fixed decreasing μ, and the wall-crossing formula in Theorem 1(ii) from μ to $\dot{\mu}$, then determine unique classes $[\mathcal{M}_{\boldsymbol{d}}^{\mathrm{ss}}(\dot{\mu})]_{\mathrm{virt}}$ for all slope functions $\dot{\mu}$ on $\mathrm{mod}\text{-}\mathbb{C}Q$. We prove these satisfy Theorem 1(ii) for wall-crossing from $\dot{\mu}$ to $\ddot{\mu}$, for any two slope functions $\dot{\mu}, \ddot{\mu}$, by an associativity property of the wall-crossing formula proved in my 2003 work on motivic invariants. So far we have constructed classes $[\mathcal{M}_{\boldsymbol{d}}^{\mathrm{ss}}(\mu)]_{\mathrm{virt}}$ as in Theorem 1, satisfying Theorem 1(ii), but we do not yet know they satisfy (i). Next we prove these classes $[\mathcal{M}_{\boldsymbol{d}}^{\mathrm{ss}}(\mu)]_{\mathrm{virt}}$ satisfy Theorem 2, using the fact that since $\Omega^{\mathrm{pl}} : \check{H}_*(\mathcal{M}^{\mathrm{pl}}) \to \check{H}_*(\mathcal{M}'^{\mathrm{pl}})$ is a Lie algebra morphism, it takes the wall-crossing formula (1) in $\check{H}_*(\mathcal{M}^{\mathrm{pl}})$ used to define $[\mathcal{M}_{\boldsymbol{d}}^{\mathrm{ss}}(\mu)]_{\mathrm{virt}}$ to an identity in $\check{H}_*(\mathcal{M}'^{\mathrm{pl}})$. The factors $\prod_v \boldsymbol{d}(v)!$, $\prod_{v'} \boldsymbol{d}'(v')!$ arise because of a combinatorial identity relating the number of different ways of splitting $\boldsymbol{d} = \boldsymbol{d}_1 + \cdots + \boldsymbol{d}_n$ in $\mathbb{N}^{Q_0} \setminus \{0\}$, and $\boldsymbol{d}' = \boldsymbol{d}'_1 + \cdots + \boldsymbol{d}'_n$ in $\mathbb{N}^{Q'_0} \setminus \{0\}$.

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Finally we show the $[\mathcal{M}^{\mathrm{ss}}_{\boldsymbol{d}}(\mu)]_{\mathrm{virt}}$ satisfy Theorem 1(i). This is the most difficult part. If $\boldsymbol{d}(v) \in \{0,1\}$ and Q is a tree, we deduce the result using results of Joyce–Song on Donaldson–Thomas type invariants for quivers. Then we build up to progressively more general $Q, \boldsymbol{d}$ using Theorem 2 in different ways.

The methods we currently use to prove Theorem 1 are very special to quivers. I have three methods to extend to other categories $\mathcal{A}$:

- Compare $\mathcal{A}$ with $\mathrm{mod}\text{-}\mathbb{C}Q$ and deduce conjecture from theorem for quivers. Can use Alvarez-Consul–King 2007 to map $\mathrm{coh}(X)$ to $\mathrm{mod}\text{-}\mathbb{C}Q$ for a Kronecker quiver $\bullet \Rrightarrow \bullet$.

- Mochizuki-style proof involving Graber–Pandharipande $\mathbb{G}_m$-localization and 'enhanced master spaces'.

- Studying wall-crossing in GIT quotients using torus localization and Jeffrey–Kirwan residues, without needing master spaces.