An overview of special Lagrangian construction methods

Mark Haskins

University of Bath

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Basics of calibrated geometry

A *calibrated geometry* is a distinguished class of minimal submanifolds, i.e. mean curvature $H = 0$, associated with a differential form.

- A *calibrated form* is a **closed** differential p-form ϕ on a Riemannian manifold (M, g) satisfying $\phi \leq vol_g$.

  i.e. $$\phi(e_1, \ldots, e_p) \leq 1$$

  for any orthonormal set of p vectors $e_1, \ldots, e_p$.
- For $m \in M$ associate with ϕ the subset $G_m(\phi)$ of oriented p-planes for which equality holds in $(*)$ – the *calibrated* planes.
- An oriented p-dim submanifold S is *calibrated* by ϕ if its oriented tangent plane is a ϕ-calibrated p-plane at each point $s \in S$.

Lemma: (Harvey–Lawson Acta 1982)
Calibrated submanifolds minimize volume in their homology class.

- How to find interesting calibrated forms?
- For interesting calibrated forms construct calibrated submanifolds.

Calibrated forms and special holonomy

Interesting calibrated forms on $\mathbb{R}^n$ arise from constant coefficient differential forms invariant under (large) subgroups $G \subset \mathsf{SO}(n)$, for instance

- $\mathsf{U}(n) \subset \mathsf{SO}(2n)$, $\hat{\omega}_p = \frac{1}{p!}\omega^p$, ω standard Kähler form on $\mathbb{C}^n$. Calibrated submanifolds are *p-diml complex submanifolds*.
- $\mathsf{SU}(n) \subset \mathsf{U}(n)$, $\phi = \mathrm{Re}\,\Omega$, Ω is standard complex volume form on $\mathbb{C}^n$. Calibrated submanifolds are called *special Lagrangian submanifolds*.
- $\mathsf{G}_2 \subset \mathsf{SO}(7)$, ϕ is the fundamental G_2–invariant 3-form or its Hodge dual 4-form ψ. Calibrated submanifolds called *associative* & *coassociative*.
- $\mathsf{Spin}_7 \subset \mathsf{SO}(8)$, ϕ is the fundamental Spin_7-invariant 4-form on $\mathbb{R}^8$. Calibrated submanifolds called *Cayley* 4-manifolds.

By the **holonomy/parallel tensor correspondence** any G-invariant tensor on $\mathbb{R}^n$ with $G \subset \mathsf{SO}(n)$ gives rise to a corresponding *parallel tensor* on any Riemannian manifold (M, g) with holonomy group contained in G.

In particular *parallel calibrated forms* exist for Kahler, *Calabi–Yau*, G_2-manifolds and Spin_7-manifolds (modelled on the constant coefficient calibrations described above).

Some fundamental questions

1. What homology classes of a compact special holonomy manifold can be represented by a calibrated submanifold (more general calibrated object)?
2. Can we "count" calibrated submanifolds? If so are these counts invariant under some deformations of the underlying structure, e.g. Gromov–Witten theory for pseudo-holomorphic curves in a symplectic manifold with a tame almost complex structure.
3. Existence of *(singular) calibrated fibrations of special holonomy manifolds*, e.g. special Lagrangian T^3 fibrations of CY 3-folds, coassociative $K3$ fibrations of G_2–manifolds, Cayley calibrations of $Spin_7$–manifolds.
4. Regularity theory for weak solutions to calibration equations, e.g. calibrated rectifiable currents or varifolds. **2 talks by C. Bellettini talks on Weds**. For the Kähler calibrations there are very strong classical regularity theorems due to King, Shiffman and others. In particular the regularity theory is much better than for general mass-minimising currents and the proofs much easier.

   Open question: what about for the other special holonomy calibrations?

Geometry of Lagrangian planes in $\mathbb{C}^n$

- Oriented Lagn n-planes in $\mathbb{C}^n \leftrightsquigarrow \mathrm{U}(n)/\mathrm{SO}(n)$.
- Map $\det_{\mathbb{C}} : \mathrm{U}(n)/\mathrm{SO}(n) \longrightarrow \mathbb{S}^1$ gives us the *phase* of the Lagrangian n-plane.
- Write phase of oriented Lagrangian submfd L of $\mathbb{C}^n$ as $e^{i\theta} : L \to \mathbb{S}^1$. Locally can lift phase $e^{i\theta}$ to a function $\theta : L \to \mathbb{R}$, the *Lagrangian angle*.
- θ not necessarily globally well-defined, but $d\theta$ is and has geometric meaning:

  $$d\theta = \iota_H \omega \quad \text{or} \quad H = -J\nabla\theta$$

  where H is mean curvature vector of L.
- Hence $H = 0 \longleftrightarrow d\theta_L = 0 \longleftrightarrow \theta_L$ locally constant
  $\theta_L = 0 \longleftrightarrow L$ is special Lagrangian.
- We also say that L is *Hamiltonian stationary* if $d\theta$ is harmonic.
- If $[d\theta] = 0$ we say that L is a *Maslov-zero* Lagrangian. Then L admits a global Lagrangian angle θ and (L, θ) is called a *graded Lagrangian*.

Generalises to Calabi–Yau n-folds using the parallel holomorphic n-form Ω.

Existence theory for special Lagrangians

1. Existence results in special holonomy spaces with *additional structure*, e.g. *noncompact Calabi–Yau manifolds with large symmetry groups* like $\mathbb{C}^3$ or T^*S^3 with the Candelas–de la Ossa–Stenzel metric.
   Here we can study special Lagrangians with additional structure and derive easier PDEs or ODEs for these. Since the Calabi–Yau metric is explicit we can be explicit about the PDEs we need to solve.
2. Existence theory in *general compact special holonomy manifolds*.
   In the Calabi–Yau case we rely on Yau to provide the Kahler Ricci-flat metric g. Since we do not have explicit information about g, the PDE for SL n-folds is not so explicit.
3. Theory in special holonomy manifolds close to an *adiabatic limit*, e.g.
   - CY 3-folds fibred by *almost holomorphic $K3$ surfaces* or *almost flat 3-tori*
   - Smooth CY 3-folds close to singular ones, e.g. with nodal singularities.
   - Smooth CY 3-folds obtained by neck-stretching constructions.

In **3** parts of our compact manifold have some approximate special structure, e.g. **see Yang Li's talk on Thursday**.

Existence theory in a general Calabi–Yau

Warning! So far very limited success in actual constructions of special Lagrangians by general theory.

1. Parabolic methods: (Lagrangian) mean curvature flow.
   We will have 3 lectures related to LMCF this week (Lee, Lotay, Joyce), so I won't say more here.
2. Direct minimisation methods.

Since SLG n-folds are homological volume(mass) minimisers we could try to minimise volume(mass) in a given homology class.

- Classical compactness theory in Geometric Measure Theory gives mass-minimising rectifiable currents representing every homology class. GMT regularity theory tells us how far these are from submanifolds.
- However we don't know which homology classes should have a special Lagrangian representative. (Does follow from calibration argument that if a class it has any SL representative then any other mass-minimiser in the same class is also SL.)

So need another approach (still won't end up producing SL manifolds in general..)

Schoen–Wolfson approach to existence in 4 dim

Basic idea: Find canonical representatives of (certain) homology classes by seeking an object that minimises volume among Lagrangian competitors.

Obvious possible problem: no guarantee that an object that minimises volume along Lagrangian competitors must minimise among *all submanifolds* in the same homology class. So it is not obvious this approach yields a special Lagrangian. However if minimiser is *smooth* then in CY setting there is an argument to show this must be true.

Main ingredients of SW approach

- Restrict to case of 2-dim Lagrangians in 4-dim Kähler manifold M.
- Work only in Lagrangian homology classes $\alpha \in H_2(M, \mathbb{Z})$, i.e. $[\omega](\alpha) = 0$. (this imposes "rationality" conditions on $[\omega]$)
- Find a suitable weak formulation of Lagrangian constraint with good compactness properties: use weakly Lagrangian maps $\ell \in W^{1,2}(\Sigma, M)$, i.e. $\ell^*\omega = 0$ a.e. on Σ.
- Study the *regularity theory of Lagrangian minimisers*. Here Hölder continuity is relatively easy, but *higher regularity* fails in general.

Structure of Lagrangian constrained minimisers

Basic problem: Even in the case of K3 surfaces the weak minimising Lagrangian surfaces produced by the SW approach need not be smooth and need not be special Lagrangian.

- A minimiser is a branched immersion except at a finite number of singular points.
- Each singular point is represented by a Hamiltonian stationary Lagrangian cone with an associated integer local Maslov index.
- Hamiltonian stationary cones in $\mathbb{C}^2$ are classified: there are infinitely many cones $C_{p,q}$ exist in $\mathbb{C}^2$; they are cones over explicit torus knots. ıOnly cones of local Maslov index ± 1 can be area minimisers. This rules out most HS cones but there are still infinitely many HS cones of local Maslov index ± 1. ($C_{p,p\pm 1}$ for any p).
- Wolfson (2005) exhibited a large class of Kähler structures on K3 for which there exists a homology class that has an area minimiser among Lagrangian 2-spheres that is not regular, i.e. admits singular points with non-flat tangent cones.

Special Lagrangian 3-folds in $\mathbb{C}^3$

Various types of additional special structure can be imposed in this case due to the combination of rotational and translational symmetry available.

Possible special structures

1. Large continuous symmetry groups: Harvey–Lawson 1982, Bryant, Haskins, Joyce 2000, and others.
2. Bundle constructions and ruled submanifolds: HL 1982, Borisenko 1993, Bryant and Joyce 2000.
3. Foliations by special hypersurfaces, e.g. quadrics: Lawlor, Harvey, Joyce.
4. Graphs of a potential function (HL 1982). For a function $u : \Omega \subset \mathbb{R}^3 \to \mathbb{R}$, the graph of ∇u is a SL 3-fold in $\mathbb{C}^3$ iff

   $$\Delta u = \det \mathrm{Hess} u.$$

   Caffarelli–Nirenberg-Spruck solved the Dirichlet problem on bounded strictly convex domains for this equation (and more general ones).
5. Conical and asymptotically conical structure (Haskins, Joyce, McIntosh, Haskins–Kapouleas, Carberry–McIntosh)
6. Nongeneric second fundamental form: 2nd order SL 3-folds (Bryant).

Special Lagrangians with T^2-symmetry

SL level sets (Harvey – Lawson 1982)

Define $F : \mathbb{C}^3 \to \mathbb{R}^3$ by

$$F = \left(|z_1|^2 - |z_2|^2,\ |z_1|^2 - |z_3|^2,\ \operatorname{Im} z_1 z_2 z_3\right)$$

$F^{-1}(c)$ is a (possibly singular) SL 3-fold invariant under a T^2 action.

- For a generic c, $F^{-1}(c)$ is nonsingular and diffeomorphic to $T^2 \times \mathbb{R}$.
- For 1-dimensional set of critical values of c, $F^{-1}(c)$ is a smoothly immersed 3-fold, with two components each diffeomorphic to $S^1 \times \mathbb{R}^2$ and intersecting in a circle.
- $F^{-1}(\mathbf{0})$ is a singular cone C. The cross-section of C has 2 (antipodal) components. Each component is flat 2-torus (with equilateral conformal structure). I call this the *Legendrian Clifford torus in* $\mathbb{S}^5$.
- $F^{-1}(c)$ is *asymptotically conical* for every $c \neq 0 \in \mathbb{R}^3$.
- We understand all SL 3-folds asymptotic (in a multiplicity one sense) to the cone over the Legendrian Clifford torus: Joyce, Haskins, Imagi.

Singular special Lagrangians

Singular special Lagrangians play an important role.

- A family of smooth compact SL 3-folds (in a family of CY 3-folds) can degenerate to a singular SL 3-fold.

  Understanding what singular SL 3-folds can arise as a limit of smooth SL 3-folds is an important problem. It has applications to understanding invariance properties of "counts" of SL homology 3-spheres.
- Singular fibres must appear in SYZ-type special Lagrangian torus fibrations of CY 3-folds (for Euler characteristic reasons).
- The starting point of any singularity analysis in geometric PDE is to pass to "tangent cones". Tangent cones to even very singular special Lagrangians exist by general GMT compactness results, but are not currently known to be unique.
- Uniqueness is known if any of the tangent cones is *regular*, i.e. its cross-section Σ is a smooth submanifold of the unit sphere. (L Simon)
- There exists a reasonably well-developed theory (Joyce) of compact singular SL 3-folds with *"isolated conical singularities"*; these have only finitely many singular points modelled on *regular SL cones in* $\mathbb{C}^3$.

Regular SL cones in $\mathbb{C}^3$

Link of a *regular SL cone* is a cpt oriented Riemannian surface Σ of genus g.
Σ is a minimal Legendrian surface in $\mathbb{S}^5$.

The trichotomy:

1. $g = 0$. Only example is standard round $\mathbb{S}^2$
2. $g = 1$
   - $\exists$ infinitely many explicit examples with cts symmetries
   - Integrable systems/ loop group / algebraic geometry methods produce all examples
3. $g > 1$
   - Continuous symmetry not possible
   - Integrable systems methods not currently effective
   - Use geometric PDE 'gluing' methods to construct many examples (Haskins-Kapouleas)

SL cones via gluing constructions

Th$^{\text{m}}$ 1: (H-Kapouleas Inventiones 2007)
For any $d \in \mathbb{N}$ there exist infinitely many SL cones C with a link a surface of genus $g = 2d + 1$. Also true for genus 4.

Th$^{\text{m}}$ 2: (H-Kapouleas)
For every $n \geq 3$ there are infinitely many topological types of special Lagrangian cone in $\mathbb{C}^n$ each of which admits infinitely many geometrically distinct representatives.

- Proofs use delicate *gluing* techniques – singular perturbation theory for geometric PDE à la Schoen (singular Yamabe problem), Kapouleas (CMC surfaces) and others ...
- Links composed of *large* number of *almost spherical regions* connected to each other via small highly curved regions.
- Reminiscent of use of Delaunay surfaces to construct new CMC surfaces in $\mathbb{R}^3$ by Kapouleas.

Gluing constructions also of use in constructing compact SL 3-folds in compact CY 3-folds, e.g. Y-I Lee, Butscher, Joyce, Y-M Chan, Hein-Song.

Outline of proof of Theorems 1 and 2

1. Use ODE methods to construct SL analogues of Delaunay cylinders:
   • highly symmetric SL *necklaces* w/ many almost spherical regions connected by highly curved neck regions.
   • $\exists$ finitely many topological types of necklace in each dimension.
2. Fuse many SL necklaces to form topologically more complicated Legendrian submanifolds that are approximately minimal.
3. Linear theory.
   (a) Understand linearization $\mathcal{L}$ of condition that graph(f) over Σ is minimal.
   (b) Understand (approx) kernel of $\mathcal{L}$.
   (c) Understand how to compensate for the (approx) kernel of $\mathcal{L}$.
4. Deal with nonlinear terms.
   (a) Set up iteration/ fixed point scheme.
   (b) Prove nonlinear estimates to guarantee convergence.

SL analogues of Delaunay cylinders in $\mathbb{S}^5$

$SO(2)$-invariant SL cylinders in $\mathbb{S}^5$.
SO(3): std $\mathbb{C}$-linear SO(3) action on $\mathbb{C}^3$; SO(2) $\subset$ SO(3): stabilizer of $(1,0,0) \in \mathbb{S}^5$.

Thm: (H AJM 2004) There exists a 1-parameter family of SO(2)-invariant SL cylinders $X_\tau : \mathbb{R} \times \mathbb{S}^1 \to \mathbb{S}^5$ interpolating between:

- a flat cylinder when $\tau = \tau_{\text{max}}$, and
- the round sphere $\mathbb{S}^2 \setminus (\pm 1, 0, 0)$ when $\tau = 0$.

For $\tau \sim 0$, X_τ is periodic and consists of infinitely many almost spherical regions (ASRs) connected by small highly curved necks.

For any $m \gg 0$, $\exists\,!\ \tau_m \sim 0$ s.t. X_{τ_m} factors through an embedded SL **torus** with exactly m ASRs.

$\Rightarrow$ $\exists$ **embedded SL toroidal necklaces in $\mathbb{S}^5$ with any (sufficiently large) number of almost spherical beads.**

In higher dimensions consider SO(p) $\times$ SO(q)-invariant SL cones in $\mathbb{C}^{p+q}$. (Haskins–Kapouleas CAG 2010). So get necklaces that depend on (p, q).

Construction of initial submanifolds

Basic idea: fuse a finite number of SL necklaces at one common central sphere.

Initial configuration determined by

1. type (p, q) of SL necklaces used
2. position of attachment sets on central sphere
3. number of beads in SL necklace

To simplify treatment of approximate kernel look only at "maximally symmetric" configurations. $\Rightarrow$

A. only use one necklace type per construction and same number m of beads for all fused necklaces

B. attachment sets must be highly symmetrically arranged

The linearized operator

Can use the *Legendrian neighbourhood theorem* to express nearby Legendrians in terms of a scalar function f on Σ.

Write θ_f for the Legendrian angle of X_f, and θ_0 for the Legn angle of initial immersion of Σ (angle exists globally if initial immersion sufficiently close to special Legn).
Basic equation is:

$$\theta_f = \theta_0 + \mathcal{L}_\Sigma f + Q_{f,\Sigma}$$

where

$$\mathcal{L}_\Sigma = \Delta_\Sigma - 2n,$$

Δ_Σ is (geometer's) Laplacian and Q represents the quadratic and higher order terms.

Want to solve $\theta_f = 0$ when θ_0 is small (in some well-chosen norm) $\iff$ must solve

$$\mathcal{L}_\Sigma f = -(\theta_0 + Q_{\Sigma,f}).$$

The main difficulty

- The gluing error is small when the number m of beads is sufficiently large.
- Each almost spherical region (bead) contributes $d(n)$ small eigenvalues to the linearisation, where $d(n) = \dim \mathrm{SU}(n) - \dim \mathrm{SO}(n)$.
- Necessarily have to deal with a linearised operator that has a very large approximate kernel.