Codimension one collapse and special holonomy metrics

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joint with Lorenzo Foscolo and Johannes Nordström

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1st-order PDE system for G_2 holonomy metrics

- A G_2-structure on an oriented 7-manifold M is a choice of smoothly varying pointwise isomorphism between T_pM and $(\mathbb{R}^7, \varphi_0)$ $\forall p \in M$.
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Theorem: Let (M, φ, g_φ) be a G_2-structure; the following are equivalent

1. $\mathrm{Hol}(g_\varphi) \subset G_2$ and φ is the induced 3-form
2. $d\varphi = d^*\varphi = 0$, where d^* is defined using Hodge star $*$ w.r.t. g_φ.

Call such a G_2-structure a torsion-free G_2 structure.

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By writing equation for 3-form φ (not metric g directly) and allowing $\mathrm{Hol}(g_\varphi) \subset G_2$ we obtain *differential* (not integro-differential) equations.

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2 is a 1st-order system of 49 equations on the 35 coefficients of φ!

Methods to construct torsion-free G_2 structures

1. Methods from theory of overdetermined systems of PDE
   - Bryant's original 1987 Annals paper used Cartan–Kähler theory (heavily uses real analyticity) to construct the first *local incomplete* examples of metrics with holonomy equal to G_2 (and $Spin_7$).
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3. Degeneration/Perturbation methods

- Use nonlinear PDE methods to deform a closed G_2–structure with *small torsion*, i.e. $d^*\varphi$ is small, to torsion-free. (Pioneered by Joyce 1996 JDG).
- Allows construction of *compact* G_2-holonomy manifolds.
- Begs question: how do we obtain closed G_2–structures with *small torsion*?

Degenerations of special holonomy spaces

Moduli spaces of special holonomy metrics and their compactifications.

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In this talk I will concentrate on the *collapsed case*, where the degenerate limit drops in dimension.

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- Codimension 1 collapse of G_2 holonomy metrics in 7 dimensions to Calabi–Yau metrics in 6 dimensions is very important in physics: it underpins an important limit in which an 11-*dimensional* physical theory (M theory) is supposed to reduce to another better-understood 10-*dimensional theory* (Type IIA String Theory).

Riemannian collapse with bounded curvature I

A family of Riemannian metrics g_ϵ on M is said to *collapse with bounded curvature* if i_p (the injectivity radius at p) converges uniformly to 0 at all points p of M, but the curvature K_ϵ stays bounded (independent of p and ϵ).

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Codimension one collapse by rescaling circle fibres in a circle bundle.

Let θ be any connection on a principal circle bundle M^{n+1} over a complete Riemannian manifold (B^n, g_B) (with bounded geometry) then the 1-parameter family of circle-invariant metrics g_ϵ on M

$$g_\epsilon = g_B + \epsilon^2 \theta^2.$$

collapses with bounded curvature to (B, g_B).

(Berger 1962 considered this in the special case of Hopf fibration $S^3 \to S^2$).

Riemannian collapse with bounded curvature II

Higher codimension collapse: generalise to torus bundles and "localise". Whenever a Riemannian manifold admits (compatible local) isometric torus actions T, then can construct families of (T-invariant) metrics by shrinking the metric in directions tangent to T. This leads to notion of an *F-structure of positive rank* (Cheeger–Gromov 1986).

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Loosely speaking, a sufficiently highly collapsed metric must admit continuous families of (almost) isometries. This raises 2 apparent problems in considering highly collapsed Ricci-flat manifolds.

1. A simply connected compact Ricci-flat metric admits no Killing fields!
2. The K3 surface has $\chi = 24$, so it does not admit *any* F-structure. So (Ricci-flat) metrics on K3 cannot collapse with curvature bounded everywhere.

Collapsing hyperKähler metrics on K3

Foscolo gave the following general construction of collapsing families of hyperKähler metrics on the K3 surface utilising so-called ALF gravitational instantons, i.e. a complete hyperKähler 4-manifold with “finite energy”, and a particular type of asymptotic geometry. The limit space is 3-dimensional.

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Theorem (Foscolo 2016) Every collection of 8 ALF spaces of dihedral type $M_1, \ldots, M_8$ and n ALF spaces of cyclic type $N_1, \ldots, N_n$ satisfying

$$\sum_{j=1}^{8} \chi(M_j) + \sum_{i=1}^{n} \chi(N_i) = 24$$

appears as the collection of "bubbles" forming in a sequence of Kähler Ricci-flat metrics on the K3 surface collapsing to the flat orbifold $T^3/\mathbb{Z}_2$ with bounded curvature away from $n+8$ points.

ALF gravitational instantons

Defn: A gravitational instanton (M, g) is called **ALF** if there exists a compact set $K \subset M$, $R > 0$ and a finite group $\Gamma \subset O(3)$ acting freely on $\mathbb{S}^2$ such that $M \setminus K$ is the total space of a circle fibration over $(\mathbb{R}^3 \setminus \mathbb{B}_R)/\Gamma$ and the metric g is asymptotic to a Riemannian submersion

$$g = \pi^* g_{\mathbb{R}^3/\Gamma} + \theta^2 + O(r^{-\tau})$$

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if $\Gamma = \mathbb{Z}_2$ we say M is ALF of *dihedral* type.

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- The ALF spaces of **dihedral** type are more complicated to construct because they do not arise from the Gibbons–Hawking ansatz (except asymptotically). The *Atiyah–Hitchin manifold* D_0 and its double cover D_1 are vital to the gluing construction. They contribute effective negative 'charge'. Without them only have collapse via the Kummer construction.

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- Complete the resulting hyperkähler metrics by gluing ALF spaces at the $n+8$ punctures:

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- Deform the resulting approximately hyperKähler metric using the Implicit Function Theorem. The setting of *definite triples* seems the most convenient framework to use.

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We would like to develop analogues of these key features in the G_2 case.

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Q1: What's the correct G_2 analogue of an ALF gravitational instanton?

ALF in higher dims: ALC manifolds

- (Σ, g_Σ) closed (connected) Riemannian manifold of dim $n-2$.
- $\pi : N \to \Sigma$ a principal circle bundle (up to passing to a double cover).
- θ a connection 1–form on $\pi : N \to \Sigma$, and a constant $\ell > 0$

Data $(\Sigma, g, \pi, \theta, \ell) \rightsquigarrow$ model metric on $M_\infty = \mathbb{R}^+ \times N$

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Remark: ALC reduces to ALF when $n = 4$, $N = \mathbb{S}^3/\Gamma$ where Γ is a cyclic or binary dihedral group, and $\Sigma = \mathbb{S}^2$ or $\mathbb{RP}^2$ respectively.

ALC G_2 manifolds

G_2 holonomy: specified by a closed and coclosed 3–form φ.

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$$\varphi_\infty = \theta \wedge \omega_C + \mathrm{Re}\,\Omega_C - \tfrac{1}{2} r^2 \eta \wedge d\theta$$

where η is the contact 1-form on Σ.

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- Geometrically the two parameters control the size of the exceptional orbit (a round 3-sphere) and the length of the asymptotic circle.
- Geometric degenerations occur at special points in the parameter space.
  - An AC limit or a conically singular ALC space (Foscolo-H-Nordström).
  - A **collapsed limit** where the ALC G_2 manifold Gromov–Hausdorff converges to a noncompact Calabi–Yau 3-fold.

ALC G_2 metrics via AC CY metrics

Basic idea: Use analytic methods to understand ALC G_2 manifolds close to the collapsed Calabi–Yau limit (another adiabatic limit problem).

Theorem: Foscolo–H–Nordström arXiv:1709.04904
Let $(B, g_0, \omega_0, \Omega_0)$ be a simply connected AC Calabi–Yau 3–fold asymptotic with rate $\mu < 0$ to the Calabi–Yau cone $\mathrm{C}(\Sigma)$ over a smooth Sasaki–Einstein 5–manifold Σ. Let $M \to B$ be a principal $U(1)$–bundle such that $c_1(M) \neq 0$ but $c_1(M) \cup [\omega_0] = 0 \in H^4(B)$.
Then there exists $\epsilon_0 > 0$ such that for every $\epsilon \in (0, \epsilon_0)$ the 7–manifold M carries an $\mathbb{S}^1$–invariant torsion-free G_2–structure φ_ϵ such that the Riemannian metric g_ϵ on M induced by φ_ϵ is ALC with (restricted) holonomy equal to G_2 and collapses with bounded curvature as $\epsilon \to 0$ to (B, g_0).

Physically our Theorem gives a very general existence result for Type IIA supergravity solutions with only Ramond–Ramond flux (but no $D6$-branes).

ALC G_2 metrics via AC CY metrics

Remarks on the Theorem:

- If B is the Candelas–de la Ossa metric on the small resolution of the conifold or the Calabi metric on $K_{\mathbb{P}^1 \times \mathbb{P}^1}$ then we obtain (parts) of the families called $\mathbb{D}_7$ and $\mathbb{C}_7$ respectively by physicists. In this case the metrics are actually cohomogeneity 1.

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- If D is any smooth del Pezzo surface then we can take $B = K_D$ with the Calabi metric for any Kähler–Einstein metric on D. For appropriate Kähler classes on D we can satisfy the Chern class condition and get corresponding ALC G_2 manifolds collapsing to K_D. (Can also consider KE degenerations of D.)

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- Many other examples arise from taking B to be a crepant resolution of a toric Calabi–Yau cone (existence of toric CY cone metrics by Futaki–Ono–Wang). These all have AC CY metrics by work of van Coevering and Goto.

ALC G_2 metrics via AC CY metrics

Further remarks on the Theorem:

- If B is any *small resolution* of a Calabi–Yau cone C then the condition of $c_1(M)$ is vacuous.
- Infinitely many Calabi–Yau cones admitting small resolutions were recently constructed by Collins–Székelyhidi using results about K-stability in the Sasaki–Einstein context. Any such small resolution B admits an AC CY metric by Conlon–Hein.

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- In the $\mathbb{B}_7$ family expect to see collapse to the AC CY metric on the smoothing of the conifold. This is not covered by the current theorem because in that case the isometric circle action has fixed set $\mathbb{S}^3$ and collapsed occurs with unbounded curvature close to this $\mathbb{S}^3$.

General expectation from theory of collapse: a highly collapsed space should be composed of pieces that are approximately invariant under an isometric circle action.

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Recall **Gibbons–Hawking Ansatz** for circle–invariant hyperKähler 4-manifolds : the $U(1)$-invariant metric on P^4

$$g := h\,\pi^* g_{\mathbb{R}^3} + h^{-1}\theta^2$$

is hyperKähler iff (h, θ) is an abelian **monopole**: $*dh = d\theta$.

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Reduces us to solving a **linear** PDE $\Delta h = 0$ (plus some topological constraints).

G_2 analogue of Gibbons-Hawking

- **Apostolov–Salamon** (2004): Any circle-invariant G_2 structure φ on $M^7 \to B^6$ a principal circle bundle can be written as

$$\varphi = \theta \wedge \omega + h^{\frac{3}{4}} \mathrm{Re}\Omega,$$

where (ω, Ω) is an SU(3)–structure on B, h is a positive function on B and θ is a connection 1-form. The induced metric is

$$g = \sqrt{h}\, g_M + h^{-1}\theta^2$$

and the 4-form is given by

$$*_\varphi \varphi = -h^{-1/4}\theta \wedge \mathrm{Im}\Omega + \frac{1}{2} h\omega^2.$$

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φ is an $\mathbb{S}^1$-invariant torsion-free G_2-structure iff

$$d\omega = 0, \qquad d\left(h^{\frac{3}{4}}\mathrm{Re}\Omega\right) = -d\theta \wedge \omega,$$
$$d\left(h^{\frac{1}{4}}\mathrm{Im}\Omega\right) = 0, \qquad \tfrac{1}{2} dh \wedge \omega^2 = h^{\frac{1}{4}} d\theta \wedge \mathrm{Im}\Omega.$$

Collapsing the circle fibres

Let φ_ϵ be a family of S^1–invariant torsion-free G_2–structures on $M \to B$ with circle fibres shrinking to zero length as $\epsilon \to 0$. By rescaling along the fibres we write

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The previous AS system is equivalent to

$$d\omega_\epsilon = 0, \qquad \tfrac{1}{2} dh_\epsilon \wedge \omega_\epsilon^2 = \epsilon\,(h_\epsilon)^{\frac{1}{4}} d\theta_\epsilon \wedge \mathrm{Im}\Omega_\epsilon, \qquad d\theta_\epsilon \wedge \omega_\epsilon^2 = 0,$$
$$d\mathrm{Re}\Omega_\epsilon = -\tfrac{3}{4} h_\epsilon^{-1} dh_\epsilon \wedge \mathrm{Re}\Omega_\epsilon - \epsilon (h_\epsilon)^{-\frac{3}{4}} d\theta_\epsilon \wedge \omega_\epsilon,$$
$$d\mathrm{Im}\Omega_\epsilon = -\tfrac{1}{4} h_\epsilon^{-1} dh_\epsilon \wedge \mathrm{Im}\Omega_\epsilon.$$

In the formal limit where $\epsilon \to 0$ the second equation implies $h_0 = \lim h_\epsilon$ is constant. Wlog $h_0 = 1$ and then (ω_0, Ω_0) is Calabi–Yau, i.e. $d\omega_0$ and $d\Omega_0$ both vanish.

The adiabatic limit of Apostolov–Salamon eqns

- *Linearised* equations over the collapsed limit: a **CY monopole** on B

$$*dh = d\theta \wedge \mathrm{Re}\,\Omega_0, \qquad d\theta \wedge \omega_0^2 = 0$$

  together with a stable 3–form ρ satisfying a coupled linear equation

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A solution of these equations yields a 1-parameter family of highly collapsed **closed** ALC G_2 structures

$$\varphi_\epsilon^{(1)} = \epsilon\,\theta \wedge \omega_0 + \mathrm{Re}\Omega_0 + \epsilon\rho$$

with torsion of order $O(\epsilon^2)$.

Proof strategy

- Understand how to solve the linearised Apostolov–Salamon equations on AC manifold B in weighted Holder spaces for appropriate choice of weights.
- Understand appropriate gauge-fixing conditions to apply.
- Construct successive higher–order approximations $\varphi_\epsilon^{(k)}$ to torsion-free structure with torsion of order $O(\epsilon^{k+1})$. This requires a full understanding of the mapping properties of the linearisation of the Apostolov–Salamon equations.
- Construct a formal power series solution to the Apostolov–Salamon equations.
- Prove convergence of this formal power series solution for ϵ sufficiently small.

END OF TALK

Collapsing G_2-metrics on compact spaces

Want to use our highly collapsed ALC G_2-spaces $\mathbb{B}_7$ as bubbles in a gluing construction. $\mathbb{B}^7$ has a global isometric circle action that fixes the exceptional orbit, which is a round $\mathbb{S}^3$. In the limit $\mathbb{B}^7$ converges to the Stenzel metric on the smoothing $T^*\mathbb{S}^3$ of the conifold.

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Collapsing G_2-metrics on compact spaces II

We have to require that the nodal CY 3-fold X admits an anti-holomorphic involution whose fixed set is a subset of the nodes. This appears to be very restrictive.

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We observe that Hitchin's work on gerbes and SL 3-folds can be used to solve this system on the complement of a homologically trivial collection of smooth compact disjoint SL 3-folds. Argument uses Hodge theory for currents plus clever arguments using type decomposition repeatedly.

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Instead we proof existence of the $\mathbb{A}_7$ family in the highly collapsed limit by perturbation methods. This is also more involved than in the previous cases. We needed to construct a better approximation to the geometry in the neighbourhood of the singular orbit. We do this by rescaling in the normal directions and adapting ideas from Simon Donaldson's adiabatic limits of coassociative fibrations. Gives a rigorous way to interpret the physics statement that these manifolds are families of "AH metrics fibred over $\mathbb{S}^3$".