Recent Developments in Special and Exceptional Holonomy Metrics

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Plan of talk

1. Special versus exceptional holonomy: general philosophy and outlook
2. Kähler Ricci-flat metrics on the K3 surface
3. Kähler Ricci-flat metrics for Calabi–Yau 3-folds
4. Recent progress on G_2–holonomy metrics on noncompact manifolds, compact manifolds and compact singular spaces.

Holonomy SU(n): Calabi–Yau metrics via Monge–Ampère

- We have an exceptionally powerful method to construct **Kähler Ricci-flat (KRF)** metrics on *compact smooth manifolds*—Yau's solution to the Calabi conjecture
  - We reduce to solving a complex Monge–Ampère (CMA) equation, a scalar nonlinear elliptic equation. Yau solved this CMA equation by the continuity method and establishing fundamental a priori estimates.
  - A compact Kähler manifolds M admit KRF metrics iff $c_1(M) = 0 \in H^2(M, \mathbb{R})$, and then we have existence and uniqueness of KRF metrics in every Kähler class on M.

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  - The original CMA method has more trouble with KRF metrics on *singular* spaces. Viscosity/pluripotential theoretic methods give existence and uniqueness results for KRF metrics on mildly singular spaces, but understanding the regularity of solutions and their detailed metric asymptotics at singularities remains a challenge.
- None of this survives for the exceptional holonomy groups G_2 or Spin(7).

Calabi–Yau metrics via gluing constructions

- The CMA method is so general that it is difficult to extract detailed geometric control of the resulting KRF metric. A related difficulty is how to understand how the variation of the KRF metrics as we change the complex structure and/or Kähler class. In particular what happens to the KRF metrics as we undergo a complex structure degeneration or approach the boundary of the Kähler cone?

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- Drastically different metric behaviour can occur at different scales: gluing methods require existence of suitable *model metrics* that well approximate the sought-for KRF metric at different scales. Often these are *noncompact complete* KRF metrics (but *incomplete metrics* are also used). Begs question: How to construct those?
- Gluing methods fundamentally rely on different scales not interacting too strongly.

Exceptional holonomy metrics via gluing

- In the compact setting *all* currently known constructions of **exceptional holonomy** spaces use *gluing methods* because we have no analogue of Yau's solution of the Calabi conjecture. So we currently only understand exceptional holonomy metrics that are close to degenerate in some way.
- We need to guess in advance some way in which we expect exceptional holonomy spaces might degenerate and then develop a gluing construction realising that possibility. Thinking about such metric degenerations for Calabi–Yau surfaces or 3-folds is often instructive: either as guiding examples, warm-up problems or more directly as factors of the model metrics.

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  More likely to find exceptional holonomy analogues of KRF model metrics exhibiting a high degree of symmetry.

Gluing constructions of KRF metrics on the K3 surface

- Immediate Goal: give a very brief description of the 5 currently known different gluing constructions of KRF metrics on the K3 surface, some of which occurred within the lifetime of the Collaboration with the involvement of its members.
- Some of these gluing constructions serve as motivation for either higher dimensional Calabi–Yau or exceptional holonomy analogues. Some of those will be described later in the talk. In at least one case the exceptional holonomy considerations directly motivated study of the K3 construction.

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- One of the K3 gluing constructions was motivated by desire to understand the metric behaviour of KRF metrics on K3 surfaces undergoing complex structure degenerations such as a quartic K3 degenerating to a (generic) pair of quadrics. (There is also recent progress on similar complex degenerations in higher dimensions).

Gluing constructions of KRF metrics on K3

In all cases the gluing constructions produce families of KRF metrics on K3 and a limiting metric space can be extracted from the family; it can be singular and can drop in dimension. It is convenient to organise the construction accordings to the dimension of the limit space, or equivalently in terms of the *codimension* of the collapse that occurs in the limit space as in the list below.

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3b. 1-dimensional limit space; fibres with nontrivial nilpotent collapsing structure Hein–Sun–Viaclovsky–Zhang (2018).

(The latter is expected to be related to complex structure degenerations of K3 such as a quartic K3 degenerating to a pair of quadrics).

Codimension 1 collapse of KRF metrics on K3

Foscolo recently gave the following general construction of collapsing families of KRF metrics on the K3 surface utilising so-called ALF gravitational instantons, i.e. a complete hyperKähler 4-manifold with “finite energy”, and a particular type of asymptotic geometry. The limit space is 3-dimensional: $T^3/\mathbb{Z}_2$.

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Theorem (Foscolo JDG 2019) Every collection of 8 ALF spaces of dihedral type $M_1, \ldots, M_8$ and n ALF spaces of cyclic type $N_1, \ldots, N_n$ satisfying

$$\sum_{j=1}^{8} \chi(M_j) + \sum_{i=1}^{n} \chi(N_i) = 24$$

appears as the collection of "bubbles" forming in a sequence of Kähler Ricci-flat metrics on the K3 surface collapsing to the flat orbifold $T^3/\mathbb{Z}_2$ with bounded curvature away from $n + 8$ points.

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- Complete the resulting hyperkähler metrics by gluing ALF spaces at the $n+8$ punctures:

  an ALF space of *dihedral* type at each of the 8 fixed points of τ,

  an ALF space of *cyclic* type at each of the other punctures.

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Foscolo, Nordström and I have been developing a **$\mathbf{G}_2$ analogue** of this construction starting with the construction of appropriate model metrics: 3 fundamentally different types of model are needed. This is described later in the talk.

KRF metrics on K3-fibred Calabi–Yau 3-folds

Gross–Wilson (JDG 2000) studied Kähler degenerations of *elliptic K3 surfaces* with a Kähler class approaching the wall of the Kähler cone corresponding to the fibration. Using a semi-flat metric away from the singular fibres and the Ooguri–Vafa metric close to the singular fibres they gave a gluing construction of such degenerating KRF metrics.

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Away from the singular fibres Tosatti (JDG 2010) proved (at appropriate scales) that

- the KRF metric on any fibre itself becomes KRF w.r.t. the induced Kahler class
- the base of the fibration inherits a so-called *generalised Kähler–Einstein metric*: its Ricci form is equal to a Weil–Petersson type metric ω_{WP} determined by the variation of complex structure of the fibres.

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This leads to the so-called *semi-Ricci-flat* metric approximation of the true KRF metric, but it only applies *away from singular fibres*.

A major breakthrough was achieved by Yang Li in 2017 who constructed an exotic non-flat complete KRF metric on $\mathbb{C}^3$ with maximal volume growth.

Theorem 1 (Y Li Inventiones 2019) There is a complete Calabi–Yau metric $\omega_{\mathbb{C}^3}$ of maximal volume growth on $\mathbb{C}^3$ with the standard Euclidean holomorphic volume form, whose tangent cone at infinity is the singular cone $\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$.

Theorem 1 ran contrary to expectations, falsifying a conjecture of Tian. Significant analytic difficulties arise because the initial metric ansatz has errors that decay only slowly at infinity. Therefore it needs to be corrected by developing suitable linear analysis wrt to this asymptotic geometry with its singular tangent cone at infinity, before the noncompact nonlinear CMA techniques developed by Hein can be applied. Conlon–Rochon and Székelyhidi generalised Li's construction soon after.

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Theorem 2 (Y Li GAFA 2019) There is a gluing construction of the KRF metric on a sufficiently collapsed K3 Lefschetz fibred CY 3-fold. The metric ansatz used in the construction is obtained by gluing the model metric $\omega_{\mathbb{C}^3}$ from Theorem 1 to a regularisation of the semi-Ricci-flat metric. It follows from the construction that $\omega_{\mathbb{C}^3}$ appears as a blow-up limit of the collapsing KRF metrics near a node.

KRF metrics on nodal CY 3-folds

A nodal (ODP) singularity is (locally analytically) isomorphic to the quadric

$$Q = \{z \in \mathbb{C}^{n+1} | \sum_{i=1}^{n+1} z_i^2 = 0\}.$$

Q admits a Kähler Ricci–flat cone metric ω_Q, which is not flat for $n \geq 3$. For $n \geq 3$ these give natural isolated KRF singularities not of orbifold type.

Compact Calabi–Yau varieties with only nodes are very natural mild degenerations of smooth Calabi–Yaus (Friedman 1986). Nodal Calabi–Yau 3-folds have been much studied in physics in the context of *conifold transitions*, e.g. Candelas–de la Ossa (1990).

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Conjecture (C-dlO 1990): There exists a singular KRF metric on any nodal quintic hypersurface in $\mathbb{P}^4$ that is asymptotic to ω_Q at each node.

This conjecture (and more) was proven only recently by Hein–Song (Pub Math IHES 2017) and provided the first known examples of compact Ricci–flat spaces with non-orbifold isolated conical singularities.

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- The existence of such singular KRF metrics with controlled asymptotics on a nodal CY 3-fold X has immediate implications for the existence of *special Lagrangian 3-spheres* in smoothings of X via a gluing construction of the KRF metric on the smoothing: the controlled asymptotics of the singularity are crucial here.

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- M theorists would like to construct compact singular G_2–holonomy spaces with certain sorts of singularities, including conical singularities in codimension 7, to obtain realistic 4-dimensional physics. We have a programme to construct *compact singular* G_2*-spaces with isolated conical singularities* by considering spaces collapsing to suitable nodal CY 3-folds. It builds on the result of Hein–Song.

Codimension 1 collapse in the G_2 setting

Recall the basic ingredients in Foscolo's codimension 1 collapsing KRF metrics on K3.

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3 gives us incomplete $\mathbb{S}^1$-invariant hK metrics into which we glue spaces from 1 and 2 to give smooth but highly collapsed almost hK metrics on K3. We therefore think of 3 as giving us a suitable singular background hK metric into which we glue two different types of hK bubbles, according to the behaviour of points under the involution τ.

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3. The incomplete $\mathbb{S}^1$-invariant hK metric on the circle bundle over a punctured 3-torus.
    - Here we used the fact that $\mathbb{T}^3$ had a Ricci-flat metric (actually a flat metric);
    - the isometric involution τ also plays a fundamental role.

3 gives us incomplete $\mathbb{S}^1$-invariant hK metrics into which we glue spaces from 1 and 2 to give smooth but highly collapsed almost hK metrics on K3. We therefore think of 3 as giving us a suitable singular background hK metric into which we glue two different types of hK bubbles, according to the behaviour of points under the involution τ.

We want G_2–analogues of all these ingredients.

- Motivated by the programme above Foscolo–H–Nordström have developed a very general analytic method exploiting codimension 1 collapse to construct many *complete noncompact* G_2–manifolds as circle bundles over asymptotically conical (AC) Calabi–Yau 3-folds. The construction gives rise to infinitely many topological types of complete noncompact G_2–holonomy manifolds.

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- The construction is very powerful because there is only a mild restriction on the circle bundle and many AC Calabi–Yau 3-folds have been constructed. The most delicate part is the existence of Calabi–Yau cone metrics; this is now proven to be equivalent to K-stability of its cross-section—a Sasakian extension of the KE Fano result (Chen–Donaldson–Sun) proven by Collins–Székelyhidi (G&T 2019).

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Theorem A: Foscolo–H–Nordström arXiv:1709.04904

Let $(B, g_0, \omega_0, \Omega_0)$ be a simply connected AC Calabi–Yau 3–fold asymptotic to the Calabi–Yau cone $C(\Sigma)$ over a smooth Sasaki–Einstein 5–manifold Σ. Let $M \to B$ be a principal $U(1)$ bundle such that $c_1(M) \neq 0$ but $c_1(M) \cup [\omega_0] = 0 \in H^4(B)$.

Then there exists $\epsilon_0 > 0$ such that for $\epsilon \in (0, \epsilon_0)$ the 7–manifold M carries an $\mathbb{S}^1$ invariant torsion-free G_2–structure φ_ϵ such that the metric g_ϵ on M induced by φ_ϵ has holonomy equal to G_2 and collapses with bounded curvature as $\epsilon \to 0$ to (B, g_0).

Cohomogeneity 1 AC CY 3-folds and G_2–manifolds

- The simplest Calabi–Yau cone: the **conifold** $\{z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0\} \subset \mathbb{C}^4$

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- Asymptotically conical (AC) Calabi–Yau 3-folds modelled on the conifold:
  - the **smoothing** of the conifold: T^*S^3 (tip of the cone replaced by a round totally geodesic special Lagrangian S^3)
  - the **small resolution** of the conifold: total space of $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$ (tip of the cone replaced by a round totally geodesic holomorphic S^2)
  - $K_{\mathbb{P}^1 \times \mathbb{P}^1}$ with **Calabi's metric** and its deformations: asymptotic to conifold$/\mathbb{Z}_2$ (tip of the cone replaced by an exceptional divisor $\mathbb{P}^1 \times \mathbb{P}^1$)
  - We can also consider the quotient of T^*S^3 by the standard antiholomorphic involution, i.e. we form an **orientifold** of the CY metric on T^*S^3.

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- The conifold itself and its AC CY desingularisations are **cohomogeneity one**: $\mathrm{SU}(2) \times \mathrm{SU}(2)$ acts isometrically with generic orbit of codimension one.
- In the *highly collapsed limit* there are four 1-parameter families $\mathbb{B}_7$, $\mathbb{D}_7$, $\mathbb{C}_7$, and $\mathbb{A}_7$ of noncompact G_2–manifolds corresponding to these four AC Calabi-Yau spaces.

Theorem B (Foscolo–H–Nordström, arixv:1805.02612)

- There exists a (unique up to scale) G_2–metric g_0 on $M_0 = (0,\infty) \times S^3 \times S^3$ such that
- (M, g_0) has an **isolated conical singularity** modelled on the G_2–cone over the homogeneous nearly Kähler structure over $S^3 \times S^3$;
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Theorem C (Foscolo–H–Nordström, arixv:1805.02612)

- For every pair of coprime positive integers m, n there exists a **complete AC** G_2–metric (unique up to scale) on the (simply connected) total space $M_{m,n}$ of the circle bundle over $K_{\mathbb{P}^1\times\mathbb{P}^1}$ with first Chern class $(m, -n)$.
- $M_{m,n}$ is asymptotic to the cone over $S^3 \times S^3/\mathbb{Z}_{2(m+n)}$.
- There is a 1-parameter family of ALC G_2–metrics on $M_{m,n}$ that collapses to a Calabi–Yau metric on $K_{\mathbb{P}^1\times\mathbb{P}^1}$ at one extreme and "opens up" at the other extreme to the unique AC G_2–metric on $M_{m,n}$.

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Physicists suggested that there should be an "M-theory lift" of the Atiyah–Hitchin manifold: the $\mathbb{A}_7$ family of cohomogeneity one metrics (studied by Hori et al numerically). $SU(2) \times SU(2)$ symmetry remains but the absence of $U(1)$ symmetry makes the ODE system less tractable.

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In the highly collapsed regime we need a better approximation to the geometry in the neighbourhood of the singular orbit. We do this by rescaling in the normal directions and adapting ideas from Donaldson's work on adiabatic limits of coassociative fibrations. This gives a precise mathematical interpretation of the physics statement that these manifolds are families of "Atiyah–Hitchin metrics fibred over $\mathbb{S}^3$".

Collapsing G_2-metrics on compact manifolds

Want to use our highly collapsed ALC G_2-spaces $\mathbb{B}_7$ and $\mathbb{A}_7$ as bubbles in a gluing construction. $\mathbb{B}^7$ has a global isometric circle action that fixes the exceptional orbit, which is a round $\mathbb{S}^3$. In the limit $\mathbb{B}^7$ converges to the Stenzel metric on the smoothing $T^*\mathbb{S}^3$ of the conifold.

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Answer: Start with a nodal CY 3-fold X_0 that is smoothable. By Hein–Sun we know that X admits an incomplete CY metric that is asymptotic to the standard KRF cone metric on the conifold at each ODP. Can now construct nearby smooth CY 3-folds Y_t by gluing in Stenzel metrics at each ODP and get SL 3-spheres as close as required to round.

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On the total space of the circle bundle we also need to construct a highly collapsed circle-invariant approximate G_2-metric. This involves the same analytic method we developed to prove Theorem A: we can solve the Apostolov–Salamon equations by linearisation—the G_2–analogue of the Gibbons–Hawking equations—because we are in a highly collapsed setting. A homological condition enters here as a condition to solve these linearised equations. This is where we see that both $\mathbb{B}_7$ (branes of positive charge) and $\mathbb{A}_7$ (orientifolds of negative charge) are needed in the construction.

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We do know at least two such nodal CY 3-folds satisfying our wish list! One can compute the topology of the resulting 7-manifolds: they do not have the same Betti numbers as any currently known compact G_2–manifold...

Compact G_2–holonomy spaces with conical singularities

Recall that Theorem B gives us a highly symmetric **CS ALC $\mathbf{G_2}$–space**, i.e. a noncompact G_2–holonomy space with topology $(0,\infty) \times S^3 \times S^3$ which has an isolated conical singularity as $t \to 0$ and a complete ALC end as $t \to \infty$.

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Instead we glue in the CS ALC G_2–space at each of the nodes not fixed by τ, and a $\mathbb{Z}_2$ quotient of the same space at each node fixed by τ to a highly collapsed incomplete $\mathbb{S}^1$-invariant G_2-metric on an appropriate circle bundle.

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Now we correct this approximate solution using weighted analysis on this compact conically singular space to a torsion-free G_2–structure, maintaining the conical asymptotics at each remaining node. Some homological relations still appear in order to solve the linearised problem in appropriate spaces.