Uncollapsing highly collapsed G_2–holonomy metrics

Mark Haskins

Bath University

joint with Lorenzo Foscolo and Johannes Nordström

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1st-order PDE system for G_2 holonomy metrics

How to get a G_2-holonomy metric from a G_2-structure?

Theorem: Let (M, φ, g_φ) be a G_2-structure; the following are equivalent

1. $\mathrm{Hol}(g_\varphi) \subset G_2$ and φ is the induced 3-form
2. $d\varphi = d^*\varphi = 0$, where d^* is defined using Hodge star $*$ w.r.t. g_φ.

Call such a G_2-structure a torsion-free G_2 structure. **2** is *nonlinear* in φ because metric g_φ depends nonlinearly on φ and $*$, d^* depends on g_φ.

By writing equation for 3-form φ (not metric g directly) and allowing $\mathrm{Hol}(g_\varphi) \subset G_2$ we obtain *differential* (not integro-differential) equations.

2 is a 1st-order system of 49 equations on the 35 coefficients of φ!

Any torsion-free G_2 structure gives rise to a Ricci-flat metric.

Methods to construct torsion-free G_2 structures

1. Methods from theory of overdetermined systems of PDE (e.g. Bryant's original 1987 Annals paper)
2. Symmetry reduction methods
   - Assume a Lie group G acts isometrically on the manifold M with general orbit of codimension k, so-called *cohomogeneity k action.* In this case we get either PDEs in fewer variables, or if $k = 1$ a nonlinear system of ODEs.
   - Bryant–Salamon 1989: constructed the first complete G_2 holonomy metrics using cohomogeneity 1 methods for suitable groups.
3. Degeneration/Perturbation methods
   - Use nonlinear PDE methods to deform a closed G_2–structure with *small torsion*, i.e. $d^*\varphi$ is small, to torsion-free. (Pioneered by Joyce 1996 JDG).
   - Allows construction of *compact* G_2-holonomy manifolds provided we can obtain closed G_2–structures with *small torsion*.
   - Begs question: how can we obtain closed G_2–structures with *small torsion*?

NB: we have no analogue of Yau's solution of the Calabi conjecture that gives general existence results for G_2 holonomy metrics, and no analogue even of the Calabi conjecture is in sight.

G_2 analogues of ALF gravitational instantons

Key feature: the metric on an ALF hyperKähler space M is asymptotically a circle bundle over an exterior domain in $\mathbb{R}^3$ (or in $\mathbb{R}^3/\mathbb{Z}_2$) where the circle fibre has asymptotically constant length $\ell > 0$, $g_\infty = g_{\mathbb{R}^3} + \ell^2\theta^2$ for θ some fixed connection on the circle bundle.

In higher dimensional (ALC) generalisations we:

- replace the base $\mathbb{R}^3$ with a Riemannian cone $C = C(\Sigma)$, $g_\infty = g_C + \ell^2\theta^2$
- take C to be Ricci-flat if we want to consider Ricci-flat ALC spaces
- C should be Calabi–Yau if we want to consider G_2–holonomy spaces
- $C(\Sigma)$ is Calabi–Yau iff Σ is Sasaki–Einstein.
- Existence of Sasaki–Einstein metrics has close connections to existence of KE metrics with positive scalar curvature and hence to stability.
- Many Sasaki–Einstein metrics are now known to exist thanks to work on both physics and maths sides.

ALC G_2 metrics via AC CY metrics

Basic idea: Use analytic methods to understand ALC G_2 manifolds close to the collapsed Calabi–Yau limit—a so-called adiabatic limit problem.

Theorem: Foscolo–H–Nordström arXiv:1709.04904

Let $(B, g_0, \omega_0, \Omega_0)$ be an **asymptotically conical Calabi–Yau** 3**–fold** asymptotic to a Calabi–Yau cone (C, g_C) and let $M \to B$ be a **principal circle bundle**.

Assume that $c_1(M) \neq 0$ but $c_1(M) \cup [\omega_0] = 0 \in H^4(B)$.

Then for every $\epsilon > 0$ sufficiently small there exists an $\mathbb{S}^1$–invariant G_2–holonomy metric g_ϵ on M with the following properties

- (M, g_ϵ) is an ALC manifold: as $r \to \infty$, $g_\epsilon = g_C + \epsilon^2 \theta_\infty^2 + O(r^{-\nu})$.
- (M, g_ϵ) collapses to (B, g_0) with bounded curvature as $\epsilon \to 0$.

Comments on Theorem

- Only 4 non-trivial examples of simply connected complete non-compact G_2-manifolds were previously known:
  - three asymptotically conical examples due to Bryant–Salamon (1989);
  - an explicit example due to Brandhuber–Gomis–Gubser–Gukov (2001) moving in a 1-parameter family whose existence was rigorously established by Bogoyavlenskaya (2013).

  We produce infinitely many new examples.
- Noncompact complete examples of manifolds with special holonomy that collapse with globally bounded curvature are a **new higher-dimensional phenomenon**: the only hyperKähler 4-manifold with a triholomorphic circle action without fixed points is $\mathbb{R}^3 \times \mathbb{S}^1$.
- **Connections to physics**: Type IIA String theory compactified on AC CY 3-fold (B, ω_0, Ω_0) with Ramond–Ramond 2-form flux $d\theta$ satisfying $[d\theta] \cup [\omega_0] = 0$ and no D6-branes as the weak-coupling limit of M-theory compactified on an ALC G_2–manifold.
- **Question**: what happens when ϵ increases?

An explicit 4-dimensional example

The **Gibbons–Hawking Ansatz** (1978): local form of hyperkähler metrics in dimension 4 with a triholomorphic circle action

- h **positive harmonic function** on $U \subset \mathbb{R}^3$
- $M \to U$ principal U(1)–bundle and connection θ with $d\theta = *dh$

$$g = h\, g_{\mathbb{R}^3} + h^{-1}\theta^2 \text{ is a hyperkähler metric on } M$$

Example: ALF and ALE metrics of cyclic type

$$g_m = \left(m + \sum_{i=1}^{n} \frac{1}{2|x - a_i|} \right) dx \cdot dx + \left(m + \sum_{i=1}^{n} \frac{1}{2|x - a_i|} \right)^{-1} \theta^2$$

- $a_1, \ldots, a_n$ distinct $\Longrightarrow$ complete metric
  $a_1 = \cdots = a_{k+1} \Longrightarrow$ orbifold singularity $\mathbb{C}^2/\mathbb{Z}_k$
- m is called the mass
  - $m > 0 \Longrightarrow$ **ALF** (= ALC with flat asymptotic cone)
  - $m = 0 \Longrightarrow$ **ALE** (= AC with flat asymptotic cone)

Explicit 4-dimensional examples

$$g_m = \left(m + \sum_{i=1}^{n} \frac{1}{2|x - a_i|} \right) dx \cdot dx + \left(m + \sum_{i=1}^{n} \frac{1}{2|x - a_i|} \right)^{-1} \theta^2$$

- $m \to \infty$: **collapse** to $\mathbb{R}^3$ (with curvature blow-up at finitely many points)
- $m \to 0$: smooth convergence to **ALE limit**

- By scaling get different picture of limit $m \to 0$:

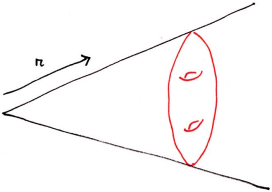
$$m\, g_m = m \left(m + \sum_{i=1}^{n} \frac{1}{2|x - a_i|} \right) dx \cdot dx + m \left(m + \sum_{i=1}^{n} \frac{1}{2|x - a_i|} \right)^{-1} \theta^2$$

$$\underset{y = m\,x}{=} \left(1 + \sum_{i=1}^{n} \frac{1}{2|y - m\, a_i|} \right) dy \cdot dy + \left(1 + \sum_{i=1}^{n} \frac{1}{2|y - m\, a_i|} \right)^{-1} \theta^2$$

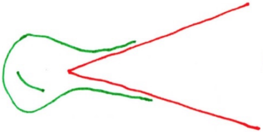
- $m \to 0$: convergence to **orbifold ALF**
- **orbifold ALF + ALE $\rightsquigarrow$ smooth ALF**

End behaviour modelled on cones

- The **Riemannian cone** $C(\Sigma)$ over a smooth manifold (Σ^{n-1}, g_Σ):
  $C(\Sigma) = \mathbb{R}^+ \times \Sigma$, $g_C = dr^2 + r^2 g_\Sigma$



- (M, g_M) has an **isolated conical singularity (CS)** at p modelled on $C(\Sigma)$ with rate $\nu > 0$ if a punctured neighbourhood of p is diffeomorphic to $(0, r_0) \times \Sigma$ and $g_M = g_C + O(r^\nu)$
- (M, g_M) has an **asymptotically conical (AC)** end modelled on $C(\Sigma)$ with rate $-\nu < 0$ if the end is diffeomorphic to $(R, \infty) \times \Sigma$ and $g_M = g_C + O(r^{-\nu})$



The known G_2-cones and AC G_2–manifolds

G_2 cones: $C = C(\Sigma)$ has torsion-free G_2–structure iff Σ is a nearly Kähler 6-manifold.

- 4 examples of homogeneous nearly Kähler 6-manifolds
  - round 6-sphere $\mathbb{S}^6$
  - flag manifold $F_{1,2} = \mathrm{SU}(3)/T^2$ (twistor space of $\mathbb{P}^2$)
  - $\mathbb{P}^3 = Sp(2)/Sp(1)\mathrm{U}(1)$ (twistor space of S^4)
  - $S^3 \times S^3 = \mathrm{SU}(2)^3/\Delta\mathrm{SU}(2)$
- Free quotients of homogeneous examples exist only in $S^3 \times S^3$ case.
- Existence of inhomogeneous smooth nearly Kähler 6-manifolds was an open question for many years.
- Foscolo–H 2017 (**Annals**): There exists an inhomogeneous nearly Kähler structure on S^6 and on $S^3 \times S^3$.

Asymptotically conical G_2–manifolds

Theorem: Bryant–Salamon 1989. For each of the 3 non spherical homogeneous nearly Kähler manifolds there is an asymptotically conical G_2–manifold M with end modelled on Σ. The AC G_2–metric on M is rigid (up to scale). $M = S^3 \times \mathbb{R}^4$ or $\Lambda^+\mathbb{S}^4$ or $\Lambda^+\mathbb{P}^2$.

No further AC G_2–manifolds have been constructed since. We will prove the existence of **infinitely many new AC G_2–manifolds**.

Cohomogeneity one manifolds

A Riemannian manifold (M, g) has **cohomogeneity one** if there is a Lie group G acting by isometries, such that a generic orbit has codimension 1.

- The orbit space M/G is an interval I
- Interior points of I correspond to generic (principal) orbits.
- Stabilisers of points in generic (principal) orbits are all conjugate, say $K \subset G$.
- Boundary points of I correspond to higher codimension orbits.
- If $H \subset G$ is the stabiliser of a point in a special orbit, then $K \subset H \subset G$ and if the metric extends smoothly over the singular orbit G/H then necessarily H/K must be a sphere.

Assuming existence of a cohomogeneity 1 action reduces the PDEs for a torsion-free G_2–structure to a nonlinear system of ODEs.

Basic problem: Do solutions to this ODE with certain initial conditions

- extend (backwards) smoothly across a singular orbit?
- extend to a forward complete solution?
- blow up (becoming incomplete without extending smoothly)?

$\mathrm{SU}(2)^2 \times \mathrm{U}(1)$–invariant G_2–metrics

- **Bryant–Salamon** (1989): first examples of complete G_2–metrics
  - AC metric on spinor bundle of S^3 $(= S^3 \times \mathbb{R}^4)$: symmetry group is $\mathrm{SU}(2)^3$
- Brandhuber–Gomis–Gubser–Gukov (2001): a single explicit complete example on $S^3 \times \mathbb{R}^4$ with $\mathrm{SU}(2)^2 \times \mathrm{U}(1)$–symmetry and ALC geometry. Conjectured existence of a 1-parameter family of such ALC metrics.
- Other **conjectural families** of complete examples with symmetry $\mathrm{SU}(2)^2 \times \mathrm{U}(1)$ (Brandhuber, Cvetič–Gibbons–Lü–Pope, 2001–2002)
- Existence of one of the BGGG conjectural family of examples **established rigorously** by Bogoyavlenskaya (2013)
- Related work by Dancer–Wang (2004), Madsen–Salamon (2013)
- **Today**: existence of all conjectural families of complete metrics and study their limits $\rightsquigarrow$ new results not anticipated in the physics literature

  In particular, $\exists$ infinitely many complete AC G_2–manifolds and infinitely many 1-parameter families of complete ALC G_2–manifolds.

Main results

Theorem (Foscolo–H–Nordström, 2018)

- There exists a (unique up to scale) G_2–metric g_0 on $M_0 = (0,\infty) \times S^3 \times S^3$ such that
  - (M, g_0) has an **isolated conical singularity** modelled on the G_2–cone over the homogeneous nearly Kähler structure over $S^3 \times S^3$;
  - (M, g_0) has a **complete ALC end**.
- For every pair of coprime positive integers m, n there exists a **complete AC** G_2–metric (unique up to scale) on the (simply connected) total space $M_{m,n}$ of the circle bundle over $K_{\mathbb{P}^1 \times \mathbb{P}^1}$ with first Chern class $(m, -n)$.

- $M_{m,n}$ is asymptotic to the cone over $S^3 \times S^3/\mathbb{Z}_{2(m+n)}$.
  Karigiannis–Lotay: Bryant–Salamon unique AC asymptotic to cone over $S^3 \times S^3$
- (M_0, g_0) can be desingularised using the AC metrics to produce families of complete ALC metrics.

Cohomogeneity 1 AC CY 3-folds

- The simplest Calabi–Yau cone (Candelas–de la Ossa): the **conifold** $\{z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0\} \subset \mathbb{C}^4$
- AC Calabi–Yau 3-folds modelled on the conifold:
  - the **smoothing** of the conifold: T^*S^3 (tip of the cone replaced by a round totally geodesic special Lagrangian S^3)
  - the **small resolution** of the conifold: total space of $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$ (tip of the cone replaced by a round totally geodesic holomorphic S^2)
  - $K_{\mathbb{P}^1 \times \mathbb{P}^1}$ with **Calabi's metric** and its deformations: asymptotic to conifold$/\mathbb{Z}_2$ (tip of the cone replaced by an exceptional divisor $\mathbb{P}^1 \times \mathbb{P}^1$)
- The conifold itself and its asymptotically conical CY desingularisations are **cohomogeneity one**: $\mathrm{SU}(2) \times \mathrm{SU}(2)$ acts isometrically with generic orbit of codimension one

Complete cohomogeneity one G_2–manifolds

- Specialise **FHN** construction of highly collapsed ALC G_2–metrics to case where AC CY base B has cohomogeneity one.
  - (B, ω_0) small resolution of the conifold or $K_{\mathbb{P}^1\times\mathbb{P}^1}$ with AC CY metric ω_0
  - $M \to B$ be a **principal circle bundle** with $c_1(M) \neq 0$ but $c_1(M) \cup [\omega_0] = 0$

  For every $0 < \epsilon \ll 1$ there exists a **complete G_2–holonomy metric** g_ϵ on M with the following properties.
  - (M, g_ϵ) is an ALC manifold: as $r \to \infty$, $g_\epsilon = g_C + \epsilon^2\theta_\infty^2 + O(r^{-\nu})$.
  - As $\epsilon \to 0$, (M, g_ϵ) collapses with bounded curvature to (B, ω_0).
  - (M, g_ϵ) has a **cohomogeneity one** action of $SU(2) \times SU(2) \times U(1)$.
- Brandhuber–Gomis–Gubser–Gukov (2001), Bogoyavlenskaya (2013): 1-parameter family up to scale of $SU(2) \times SU(2) \times U(1)$–invariant ALC G_2–metrics on $S^3 \times \mathbb{R}^4 \to T^*S^3 = S^3 \times \mathbb{R}^3$. At one end of the parameter space we have **collapse to the AC CY metric on the smoothing T^*S^3 of the conifold with curvature concentrating along the zero section**.

- **Question**: what happens when ϵ increases?

Step 1: $\mathrm{SU}(2)\times\mathrm{SU}(2)$–invariant G_2–structures

- A **G_2–structure** on a 7-manifold is a 3-form φ such that

$$\tfrac{1}{6}(u\lrcorner\varphi)\wedge(v\lrcorner\varphi)\wedge\varphi = g_\varphi(u,v)\,\mathrm{vol}_{g_\varphi}.$$

  (M^7,φ) is a **G_2–manifold** if $d\varphi = 0 = d*_\varphi\varphi$. Then $\mathrm{Hol}(g_\varphi)\subseteq \mathrm{G}_2$.

Work on $(0,\infty)\times\mathrm{SU}(2)\times\mathrm{SU}(2)/K_0$, where $K_0=\{1\}$ or $K_0=\mathbb{Z}_{2(n+m)}$

- Any **closed invariant** G_2–structure can be written in the form

$$\varphi = p\,e_1\wedge e_2\wedge e_3 + q\,e_1'\wedge e_2'\wedge e_3' + d\left(a_1\,e_1\wedge e_1' + a_2\,e_2\wedge e_2' + a_3\,e_3\wedge e_3'\right)$$

  where $p,q\in\mathbb{R}$ represent the choice of a cohomology class on $\mathrm{SU}(2)\times\mathrm{SU}(2)/K_0$, e_i, e_i' are a basis of left invariant 1-forms and

$$\dot a_i > 0, \qquad \Lambda(a_1,a_2,a_3)<0, \qquad 2\dot a_1\dot a_2\dot a_3 = \sqrt{-\Lambda(a_1,a_2,a_3)}$$

  for a certain quartic polynomial $\Lambda=\Lambda_{p,q}$.

Step 1: $SU(2) \times SU(2)$–invariant G_2–structures

- When $a_1 = a_2 = a_3 = a$, the metric g_φ admits the additional isometric right action of $\triangle SU(2)$ on $SU(2) \times SU(2)$.
- Moreover φ is automatically coclosed!
  - The solution with $p = 0 = q$ is the **G_2–cone** over the homogeneous nearly Kähler structure over $SU(2) \times SU(2)$: $a(s) = s^3$
  - Only when $(p, q) = (r_0^3, -r_0^3), (-r_0^3, 0)$ or $(0, r_0^3)$ for some $r_0 > 0$, φ extends smoothly at $s = 0$ over a 3-sphere and defines a complete AC G_2–metric on $S^3 \times \mathbb{R}^4$. Up to scale and (non-equivariant) diffeomorphisms, this metric coincides with the **Bryant–Salamon** metric on the spinor bundle of S^3.

- We consider the case where g_φ admits the additional isometric right action of $\triangle U(1)$: $a_1 = a_2 = a$, $a_3 = b$.
- Coclosedness of $\varphi \Longleftrightarrow (a, b)$ satisfies the 2nd-order ODE (the Euler–Lagrange equation for Hitchin's volume functional)

$$2F(a'b'' - b'a'') - a'b'\,(a'F_a - 2b'F_b) = 0,$$

$$F(a, b) := -\Lambda(a, a, b) = 4a^2(b - p)(b + q) - (b^2 + pq)^2.$$

Step 2: construction of local solutions

- Construct **local solutions in a neighbourhood of the singular orbit** $\mathrm{SU}(2) \times \mathrm{SU}(2)/K$ where K is one of

$$\triangle \mathrm{SU}(2), \qquad \{1\} \times \mathrm{SU}(2), \qquad K_{m,n} = \{(e^{i\theta_1}, e^{i\theta_2}) \,|\, e^{i(m\theta_1 + n\theta_2)} = 1\}$$

- Construct 1-parameter family of **local CS solutions** with rate $\nu_0 = \frac{\sqrt{145}-7}{2} \approx 2.5$

$$a(s) = s^3 \left(1 + c\, s^{\nu_0} + \dots\right), \qquad b(s) = s^3 \left(1 - 2c\, s^{\nu_0} + \dots\right)$$

- For all $p, q \in \mathbb{R}$ construct a 1-parameter family of **AC ends**

$$s^{-3} a(s) = 1 + O(s^{-3}), \qquad s^{-3} b(s) = 1 + O(s^{-3}),$$
$$b(s) - a(s) = c\, s^{-\nu_0 - 4} + \dots$$

This uses analysis of a singular IVP of the form

$$s y' = L(y) + Q(s, y), \qquad y(0) = 0.$$

Solutions as generalised power series in powers of $s, s^{\nu_1}, \dots, s^{\nu_k}$, where $\nu_1, \dots, \nu_k$ are positive eigenvalues of the linearisation L satisfying **non-resonance** conditions.

Step 3: (in)/completeness and ALC ends

- **Criterion for long-time existence** (Böhm, Wilking): forward complete G_2–metric $\Longleftrightarrow$ mean curvature $l(s)$ of hypersurfaces $\{s = \text{const}\}$ remains strictly positive
- Appropriate choice of range of parameters for the local solutions that
  - close smoothly on a singular orbit $SU(2) \times SU(2)/\triangle SU(2)$,
  - close smoothly on a singular orbit $SU(2) \times SU(2)/\{1\} \times SU(2)$,
  - have an isolated conical singularity,

  together with the ODE system imply preserved inequalities that force $l(s) > 0$ as long as the solution exists $\Longrightarrow$ **solutions exist for all time**!
- Can modify arguments to prove that local solutions that don't belong to the parameter ranges above are **incomplete**.
- **ALC behaviour as $s \to \infty$**:
  - **model ALC end**

$$\varphi_\infty = d\left(s^3(e_1 \wedge e_1' + e_2 \wedge e_2') + \ell s^2 e_3 \wedge e_3'\right)$$

  with decaying torsion: $d\varphi = 0$ and $d * \varphi_\infty = O(s^{-3})$
  - need to control ratio $\frac{a^2}{b^3}$ and show that $a \approx s^3$ and $b \approx \ell s^2$ in C^1

A G_2–flop

- A 1-parameter family $\mathbb{B}_7$ up to scale of complete ALC G_2–metrics with the smoothing of the conifold as collapsed limit (BGGG, Bogoyavlenskaya).
- Two 1-parameter families $\mathbb{D}_7^+$ and $\mathbb{D}_7^-$ up to scale of complete ALC G_2–metrics with the two small resolutions of the conifold as collapsed limits (FHN in the collapsed limit; numerics by Brandhuber, CGLP).
- The **CS ALC** manifold (M_0, g_0) of the main theorem.

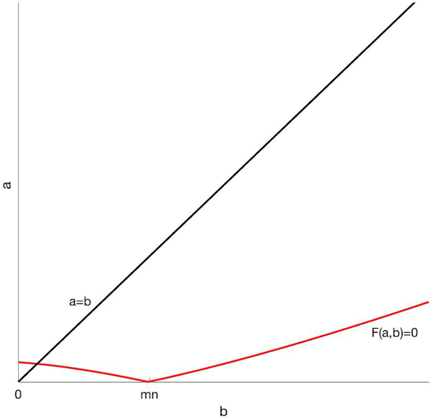
Theorem (Foscolo–H–Nordström, 2018)

CS ALC $+$ **Bryant–Salamon AC** metric $\rightsquigarrow$ $\mathbb{B}_7$, $\mathbb{D}_7^+$ and $\mathbb{D}_7^-$ families

- Extension of Karigiannis's desingularisation results (2009) from the compact setting to the ALC one. Obstructions present in the compact case vanish in the non-compact setting.
- Metric realisation of a geometric transition in G_2–geometry whose physical significance was discussed by Atiyah–Maldacena–Vafa and Acharya (2001) in the context of large N duality.

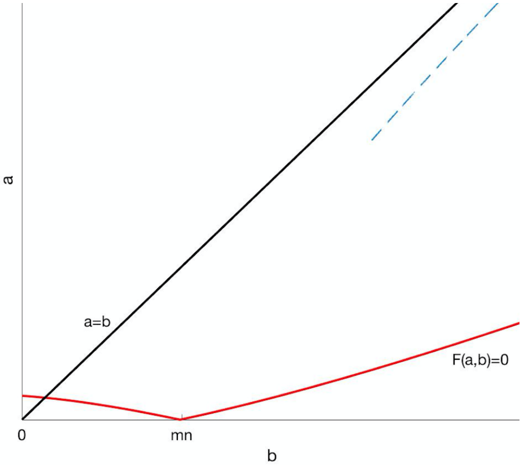
Step 4: extending AC ends backward

- Smooth extension over singular orbit $S^2 \times S^3 = \mathrm{SU}(2) \times \mathrm{SU}(2)/K_{m,n}$
  $\Longrightarrow (p, q) = r_0^3(-m^2, n^2)$
- AC(r_0, c): AC end with $b(s) - a(s) \approx c\, s^{-\nu_0 - 4}$



Step 4: extending AC ends backward

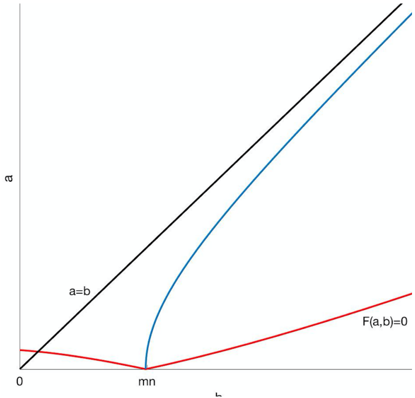
- AC(r_0, c): AC end with $b(s) - a(s) \approx c\, s^{-\nu_0 - 4}$
- $r_0, c > 0 \Longrightarrow$ AC(r_0, c) extends **backward** until $F(a, b) \to 0$



Step 4: extending AC ends backward

- For all $r_0 > 0$, $\exists!\ c_* > 0$ such that $\mathrm{AC}(r_0, c_*)$ hits $(0, mnr_0^3)$
  - **smooth extension over the singular orbit** $S^2 \times S^3$: for some $\beta_* > 0$

$$a = \beta_* s + O(s^3), \qquad b = mnr_0^3 + O(s^2)$$



Step 4: extending AC ends backward

- **Comparison with the AC solution**: local solutions closing smoothly on the singular orbit $S^2 \times S^3$ with $\beta > \beta_*$ are **complete ALC** metrics