Shifted symplectic Derived Algebraic Geometry for dummies

Dominic Joyce, Oxford University

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These slides available at
`http://people.maths.ox.ac.uk/~joyce/`.

Plan of talk:

1. Derived Algebraic Geometry
2. PTVV's shifted symplectic geometry
3. Donaldson–Thomas theory and its generalizations

1.1. Derived Algebraic Geometry for dummies

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, e.g. $\mathbb{K} = \mathbb{C}$. Work in the context of Toën and Vezzosi's theory of *Derived Algebraic Geometry* (*DAG*). This gives ∞-categories of *derived* $\mathbb{K}$*-schemes* $\mathbf{dSch}_{\mathbb{K}}$ and *derived stacks* $\mathbf{dSt}_{\mathbb{K}}$. In this talk, for simplicity, we will mostly discuss derived schemes, though the results also extend to derived stacks.

This is a very technical subject. It is not easy to motivate DAG, or even to say properly what a derived scheme is, in an elementary talk. So I will lie a little bit.

What is a derived scheme?

$\mathbb{K}$-schemes in classical algebraic geometry are geometric spaces X which can be covered by Zariski open sets $Y \subseteq X$ with $Y \cong \operatorname{Spec} A$ for A a commutative $\mathbb{K}$-algebra. General $\mathbb{K}$-schemes are very singular, but *smooth* $\mathbb{K}$*-schemes* X are very like smooth manifolds over $\mathbb{K}$, many differential geometric ideas like cotangent bundles TX, T^*X work nicely for them.

Think of a derived $\mathbb{K}$-scheme $\boldsymbol{X}$ as a geometric space which can be covered by Zariski open sets $\boldsymbol{Y} \subseteq \boldsymbol{X}$ with $\boldsymbol{Y} \simeq \operatorname{Spec} A^\bullet$ for $A^\bullet = (A, \mathrm{d})$ a commutative differential graded algebra (cdga) over $\mathbb{K}$, in degrees $\leqslant 0$.

We require $\boldsymbol{X}$ to be *locally finitely presented*, that is, we can take the $A^\bullet$ to be finitely presented, a strong condition.

Why derived algebraic geometry?

One reason derived algebraic geometry can be a powerful tool, is the combination of two facts:

(A) Many algebro-geometric spaces one wants to study, such as moduli spaces of coherent sheaves, or complexes, or representations, etc., which in classical algebraic geometry may be very singular, also have an incarnation as (locally finitely presented) derived schemes (or derived stacks).

(B) Within the framework of DAG, one can treat (locally finitely presented) derived schemes or stacks very much like smooth, nonsingular objects (Kontsevich's 'hidden smoothness philosophy'). Some nice things work in the derived world, which do not work in the classical world.

1.2. Tangent and cotangent complexes

In going from classical to derived geometry, we always replace vector bundles, sheaves, representations, ..., by *complexes* of vector bundles, ... A classical smooth $\mathbb{K}$-scheme X has a tangent bundle TX and dual cotangent bundle T^*X, which are vector bundles on X, of rank the dimension $\dim X \in \mathbb{N}$.

Similarly, a derived $\mathbb{K}$-scheme $\boldsymbol{X}$ has a *tangent complex* $\mathbb{T}_{\boldsymbol{X}}$ and a dual *cotangent complex* $\mathbb{L}_{\boldsymbol{X}}$, which are perfect complexes of coherent sheaves on $\boldsymbol{X}$, of rank the virtual dimension $\mathrm{vdim}\, \boldsymbol{X} \in \mathbb{Z}$. A complex $\mathcal{E}^\bullet$ on $\boldsymbol{X}$ is called *perfect in the interval* $[a, b]$ if locally on $\boldsymbol{X}$ it is quasi-isomorphic to a complex $\cdots 0 \to E_a \to E_{a+1} \to \cdots \to E_b \to 0 \to \cdots$, with E_i a vector bundle in position i. For $\boldsymbol{X}$ a derived scheme, $\mathbb{T}_{\boldsymbol{X}}$ is perfect in $[0, \infty)$ and $\mathbb{L}_{\boldsymbol{X}}$ perfect in $(-\infty, 0]$; for $\boldsymbol{X}$ a derived Artin stack, $\mathbb{T}_{\boldsymbol{X}}$ is perfect in $[-1, \infty)$ and $\mathbb{L}_{\boldsymbol{X}}$ perfect in $(-\infty, 1]$.

Tangent complexes of moduli stacks

Suppose X is a smooth projective scheme, and $\boldsymbol{\mathcal{M}}$ is a derived moduli stack of coherent sheaves E on X. Then for each point $[E]$ in $\boldsymbol{\mathcal{M}}$ and each $i \in \mathbb{Z}$ we have natural isomorphisms

$$H^i(\mathbb{T}_{\boldsymbol{\mathcal{M}}}|_{[E]}) \cong \mathrm{Ext}^{i+1}(E, E). \tag{1}$$

In effect, the derived stack $\boldsymbol{\mathcal{M}}$ remembers the entire deformation theory of sheaves on X. In contrast, if $\mathcal{M} = t_0(\boldsymbol{\mathcal{M}})$ is the corresponding classical moduli scheme, (1) holds when $i \leqslant 1$ only. This shows that the derived structure on a moduli scheme/stack can remember useful information forgotten by the classical moduli scheme/stack, e.g. the Ext groups $\mathrm{Ext}^i(E, E)$ for $i \geqslant 2$.

If X has dimension n then (1) implies that $H^i(\mathbb{T}_{\boldsymbol{\mathcal{M}}}|_{[E]}) = 0$ for $i \geqslant n$, so $\mathbb{T}_{\boldsymbol{\mathcal{M}}}$ is perfect in $[-1, n-1]$.

Quasi-smooth derived schemes and virtual cycles

A derived scheme $\boldsymbol{X}$ is called *quasi-smooth* if $\mathbb{T}_{\boldsymbol{X}}$ is perfect in $[0,1]$, or equivalently $\mathbb{L}_{\boldsymbol{X}}$ is perfect in $[-1,0]$.

A proper quasi-smooth derived scheme $\boldsymbol{X}$ has a *virtual cycle* $[\boldsymbol{X}]_{\mathrm{virt}}$ in the Chow homology $A_*(X)$, where $X = t_0(\boldsymbol{X})$ is the classical truncation. This is because the natural morphism $\mathbb{L}_{\boldsymbol{X}}|_X \to \mathbb{L}_X$ induced by the inclusion $X \hookrightarrow \boldsymbol{X}$ is a 'perfect obstruction theory' in the sense of Behrend and Fantechi.

Most theories of invariants in algebraic geometry – e.g. Gromov–Witten invariants, Mochizuki invariants counting sheaves on surfaces, Donaldson–Thomas invariants – can be traced back to the existence of quasi-smooth derived moduli schemes.

For an (ordinary) derived moduli scheme $\boldsymbol{\mathcal{M}}$ of coherent sheaves E on X to be quasi-smooth, we need $\mathrm{Ext}^i(E,E) = 0$ for $i \geqslant 3$. This is automatic if $\dim X \leqslant 2$. For Calabi–Yau 3-folds X, you would expect a problem with $\mathrm{Ext}^3(E,E) \neq 0$, but stable sheaves E with fixed determinant have trace-free Ext groups $\mathrm{Ext}^3(E,E)_0 = 0$.

1.3. An example of nice behaviour in the derived world

Here is an example of the 'hidden smoothness philosophy'. Suppose we have a Cartesian square of smooth $\mathbb{K}$-schemes (or indeed, smooth manifolds)

$$\begin{array}{ccc} W & \xrightarrow[\quad f \quad]{} & Y \\ \downarrow{\scriptstyle e} & & {\scriptstyle h}\downarrow \\ X & \xrightarrow{\quad g \quad} & Z, \end{array}$$

with g, h transverse. Then we have an exact sequence of vector bundles on W, which we can use to compute TW:

$$0 \to TW \xrightarrow{Te \oplus Tf} e^*(TX) \oplus f^*(TY) \xrightarrow{e^*(Tg) \oplus -f^*(Th)} (g \circ e)^*(TZ) \to 0.$$

Similarly, if we have a homotopy Cartesian square of derived $\mathbb{K}$-schemes

$$\begin{array}{ccc} \boldsymbol{W} & \xrightarrow{\boldsymbol{f}} & \boldsymbol{Y} \\ \downarrow{\scriptstyle \boldsymbol{e}} & & \downarrow{\scriptstyle \boldsymbol{h}} \\ \boldsymbol{X} & \xrightarrow{\boldsymbol{g}} & \boldsymbol{Z}, \end{array}$$

with no transversality, we have a distinguished triangle on $\boldsymbol{W}$

$$\mathbb{T}_{\boldsymbol{W}} \xrightarrow{\mathbb{T}_{\boldsymbol{e}} \oplus \mathbb{T}_{\boldsymbol{f}}} \boldsymbol{e}^*(\mathbb{T}_{\boldsymbol{X}}) \oplus \boldsymbol{f}^*(\mathbb{T}_{\boldsymbol{Y}}) \xrightarrow{\boldsymbol{e}^*(\mathbb{T}_{\boldsymbol{g}}) \oplus -\boldsymbol{f}^*(\mathbb{T}_{\boldsymbol{h}})} (\boldsymbol{g} \circ \boldsymbol{e})^*(\mathbb{T}_{\boldsymbol{Z}}) \to \mathbb{T}_{\boldsymbol{W}}[+1],$$

which we can use to compute $\mathbb{T}_{\boldsymbol{W}}$. This is false for classical schemes. So, derived schemes with arbitrary morphisms, have good behaviour analogous to smooth classical schemes with transverse morphisms, and are better behaved than classical schemes, especially for things concerned with (co)tangent bundles.

2. PTVV's shifted symplectic geometry
Classical symplectic geometry

Let M be a smooth manifold. Then M has a tangent bundle and cotangent bundle T^*M. We have k-forms $\omega \in C^\infty(\Lambda^k T^*M)$, and the de Rham differential $\mathrm{d}_{dR} : C^\infty(\Lambda^k T^*M) \to C^\infty(\Lambda^{k+1} T^*M)$. A k-form ω is *closed* if $\mathrm{d}_{dR}\omega = 0$.

A 2-form ω on M is *nondegenerate* if $\omega\cdot : TM \to T^*M$ is an isomorphism. This is possible only if $\dim M = 2n$ for $n \geqslant 0$. A *symplectic structure* is a closed, nondegenerate 2-form ω on M.

Symplectic geometry is the study of symplectic manifolds (M, ω).

A *Lagrangian* in (M, ω) is a submanifold $i : L \to M$ such that $\dim L = n$ and $i^*(\omega) = 0$.

Shifted symplectic Derived Algebraic Geometry

Pantev, Toën, Vaquié and Vezzosi (arXiv:1111.3209) defined a version of symplectic geometry in the derived world.

Let $\boldsymbol{X}$ be a derived $\mathbb{K}$-scheme. The cotangent complex $\mathbb{L}_{\boldsymbol{X}}$ has exterior powers $\Lambda^p\mathbb{L}_{\boldsymbol{X}}$. The *de Rham differential* $\mathrm{d}_{dR}:\Lambda^p\mathbb{L}_{\boldsymbol{X}}\rightarrow\Lambda^{p+1}\mathbb{L}_{\boldsymbol{X}}$ is a morphism of complexes. Each $\Lambda^p\mathbb{L}_{\boldsymbol{X}}$ is a complex, so has an internal differential $\mathrm{d}:(\Lambda^p\mathbb{L}_{\boldsymbol{X}})^k\rightarrow(\Lambda^p\mathbb{L}_{\boldsymbol{X}})^{k+1}$. We have $\mathrm{d}^2=\mathrm{d}_{dR}^2=\mathrm{d}\circ\mathrm{d}_{dR}+\mathrm{d}_{dR}\circ\mathrm{d}=0$.

A *p-form of degree k* on $\boldsymbol{X}$ for $k\in\mathbb{Z}$ is an element $[\omega^0]$ of $H^k(\Lambda^p\mathbb{L}_{\boldsymbol{X}},\mathrm{d})$. A *closed p-form of degree k* on $\boldsymbol{X}$ is an element

$$[(\omega^0,\omega^1,\ldots)]\in H^k\bigl(\textstyle\bigoplus_{i=0}^{\infty}\Lambda^{p+i}\mathbb{L}_{\boldsymbol{X}}[i],\mathrm{d}+\mathrm{d}_{dR}\bigr).$$

There is a projection $\pi:[(\omega^0,\omega^1,\ldots)]\mapsto[\omega^0]$ from closed p-forms $[(\omega^0,\omega^1,\ldots)]$ of degree k to p-forms $[\omega^0]$ of degree k.

Nondegenerate 2-forms and symplectic structures

Let $[\omega^0]$ be a 2-form of degree k on $\boldsymbol{X}$. Then $[\omega^0]$ induces a morphism $\omega^0 : \mathbb{T}_{\boldsymbol{X}} \to \mathbb{L}_{\boldsymbol{X}}[k]$, where $\mathbb{T}_{\boldsymbol{X}} = \mathbb{L}_{\boldsymbol{X}}^\vee$ is the tangent complex of $\boldsymbol{X}$. We call $[\omega^0]$ *nondegenerate* if $\omega^0 : \mathbb{T}_{\boldsymbol{X}} \to \mathbb{L}_{\boldsymbol{X}}[k]$ is a quasi-isomorphism.

If $\boldsymbol{X}$ is a derived scheme then the complex $\mathbb{L}_{\boldsymbol{X}}$ lives in degrees $(-\infty, 0]$ and $\mathbb{T}_{\boldsymbol{X}}$ in degrees $[0, \infty)$. So $\omega^0 : \mathbb{T}_{\boldsymbol{X}} \to \mathbb{L}_{\boldsymbol{X}}[k]$ can be a quasi-isomorphism only if $k \leqslant 0$, and then $\mathbb{L}_{\boldsymbol{X}}$ lives in degrees $[k, 0]$ and $\mathbb{T}_{\boldsymbol{X}}$ in degrees $[0, -k]$. If $k = 0$ then $\boldsymbol{X}$ is a smooth classical $\mathbb{K}$-scheme, and if $k = -1$ then $\boldsymbol{X}$ is quasi-smooth.

A closed 2-form $\omega = [(\omega^0, \omega^1, \ldots)]$ of degree k on $\boldsymbol{X}$ is called a *k-shifted symplectic structure* if $[\omega^0] = \pi(\omega)$ is nondegenerate.

Although the details are complex, PTVV are following a simple recipe for translating some piece of geometry from smooth manifolds/smooth classical schemes to derived schemes:

(i) replace manifolds/smooth schemes X by derived schemes $\boldsymbol{X}$.

(ii) replace vector bundles $TX, T^*X, \Lambda^p T^*X, \ldots$ by complexes $\mathbb{T}_{\boldsymbol{X}}, \mathbb{L}_{\boldsymbol{X}}, \Lambda^p\mathbb{L}_{\boldsymbol{X}}, \ldots$.

(iii) replace sections of $TX, T^*X, \Lambda^p T^*X, \ldots$ by cohomology classes of the complexes $\mathbb{T}_{\boldsymbol{X}}, \mathbb{L}_{\boldsymbol{X}}, \Lambda^p\mathbb{L}_{\boldsymbol{X}}, \ldots$, in degree $k \in \mathbb{Z}$.

(iv) replace isomorphisms of vector bundles by quasi-isomorphisms of complexes.

Note that in (iii), we can specify the degree $k \in \mathbb{Z}$ of the cohomology class (e.g. $[\omega] \in H^k(\Lambda^p\mathbb{L}_{\boldsymbol{X}})$), which doesn't happen at the classical level.

2.1. Calabi–Yau moduli schemes and moduli stacks

PTVV prove that if Y is a Calabi–Yau m-fold over $\mathbb{K}$ and $\boldsymbol{\mathcal{M}}$ is a derived moduli scheme or stack of (complexes of) coherent sheaves on Y, then $\boldsymbol{\mathcal{M}}$ has a $(2-m)$-shifted symplectic structure ω.

This suggests applications — lots of interesting geometry concerns Calabi–Yau moduli schemes, e.g. Donaldson–Thomas theory.

We can understand the associated nondegenerate 2-form $[\omega^0]$ in terms of *Serre duality*. At a point $[E] \in \boldsymbol{\mathcal{M}}$, we have $h^i(\mathbb{T}_{\boldsymbol{\mathcal{M}}})|_{[E]} \cong \mathrm{Ext}^{i+1}(E,E)$ and $h^i(\mathbb{L}_{\boldsymbol{\mathcal{M}}})|_{[E]} \cong \mathrm{Ext}^{1-i}(E,E)^*$. The Calabi–Yau condition gives $\mathrm{Ext}^i(E,E) \cong \mathrm{Ext}^{m-i}(E,E)^*$, which corresponds to $h^{i-1}(\mathbb{T}_{\boldsymbol{\mathcal{M}}})|_{[E]} \cong h^{i-1}(\mathbb{L}_{\boldsymbol{\mathcal{M}}}[2-m])|_{[E]}$. This is the cohomology at $[E]$ of the quasi-isomorphism $\omega^0 : \mathbb{T}_{\boldsymbol{\mathcal{M}}} \to \mathbb{L}_{\boldsymbol{\mathcal{M}}}[2-m]$.

2.2. Lagrangians and Lagrangian intersections

Let $(\boldsymbol{X},\omega)$ be a k-shifted symplectic derived scheme or stack. Then Pantev et al. define a notion of *Lagrangian* $\boldsymbol{L}$ in $(\boldsymbol{X},\omega)$, which is a morphism $\boldsymbol{i}:\boldsymbol{L}\rightarrow\boldsymbol{X}$ of derived schemes or stacks together with a homotopy $i^*(\omega)\sim 0$ satisfying a nondegeneracy condition, implying that $\mathbb{T}_{\boldsymbol{L}}\simeq\mathbb{L}_{\boldsymbol{L}/\boldsymbol{X}}[k-1]$.

If $\boldsymbol{L},\boldsymbol{M}$ are Lagrangians in $(\boldsymbol{X},\omega)$, then the fibre product $\boldsymbol{L}\times_{\boldsymbol{X}}\boldsymbol{M}$ has a natural $(k-1)$-shifted symplectic structure.

If (S,ω) is a classical smooth symplectic scheme, then it is a 0-shifted symplectic derived scheme in the sense of PTVV, and if $L,M\subset S$ are classical smooth Lagrangian subschemes, then they are Lagrangians in the sense of PTVV. Therefore the (derived) Lagrangian intersection $L\cap M=L\times_S M$ is a -1-shifted symplectic derived scheme.

Examples of Lagrangians

Let $(\boldsymbol{X},\omega)$ be k-shifted symplectic, and $\boldsymbol{i}_a:\boldsymbol{L}_a\rightarrow\boldsymbol{X}$ be Lagrangian in $\boldsymbol{X}$ for $a=1,\ldots,d$. Then Ben-Bassat (arXiv:1309.0596) shows

$$\boldsymbol{L}_1\times_{\boldsymbol{X}}\boldsymbol{L}_2\times_{\boldsymbol{X}}\cdots\times_{\boldsymbol{X}}\boldsymbol{L}_d\longrightarrow(\boldsymbol{L}_1\times_{\boldsymbol{X}}\boldsymbol{L}_2)\times\cdots\times(\boldsymbol{L}_{d-1}\times_{\boldsymbol{X}}\boldsymbol{L}_d)\times(\boldsymbol{L}_d\times_{\boldsymbol{X}}\boldsymbol{L}_1)$$

is Lagrangian, where the r.h.s. is $(k-1)$-shifted symplectic by PTVV. This is relevant to defining 'Fukaya categories' of complex symplectic manifolds.

Let Y be a Calabi–Yau m-fold, so that the derived moduli stack $\boldsymbol{\mathcal{M}}$ of coherent sheaves (or complexes) on Y is $(2-m)$-shifted symplectic by PTVV, with symplectic form ω. Then

$$\boldsymbol{\mathcal{E}xact}\xrightarrow{\pi_1\times\pi_2\times\pi_3}(\boldsymbol{\mathcal{M}},\omega)\times(\boldsymbol{\mathcal{M}},-\omega)\times(\boldsymbol{\mathcal{M}},\omega)$$

is Lagrangian, where $\boldsymbol{\mathcal{E}xact}$ is the derived moduli stack of short exact sequences in $\mathrm{coh}(Y)$ (or distinguished triangles in $D^b\,\mathrm{coh}(Y)$). This is relevant to Cohomological Hall Algebras.

2.3. A 'Darboux theorem' for shifted symplectic schemes

Theorem 2.1 (Brav, Bussi and Joyce arXiv:1305.6302)

Suppose $(\boldsymbol{X},\omega)$ is a k-shifted symplectic derived $\mathbb{K}$-scheme for $k<0$. If $k\not\equiv 2 \mod 4$, then each $x\in\boldsymbol{X}$ admits a Zariski open neighbourhood $\boldsymbol{Y}\subseteq\boldsymbol{X}$ with $\boldsymbol{Y}\simeq\mathrm{Spec}(A,\mathrm{d})$ for (A,d) an explicit cdga generated by graded variables x_j^{-i},y_j^{k+i} for $0\leqslant i\leqslant -k/2$, and $\omega|_{\boldsymbol{Y}}=[(\omega^0,0,0,\ldots)]$ where x_j^l,y_j^l have degree l, and

$$\omega^0=\textstyle\sum_{i=0}^{[-k/2]}\sum_{j=1}^{m_i}\mathrm{d}_{dR}y_j^{k+i}\mathrm{d}_{dR}x_j^{-i}.$$

Also the differential d in (A,d) is given by Poisson bracket with a Hamiltonian H in A of degree $k+1$.

If $k\equiv 2 \mod 4$, we have two statements, one étale local with ω^0 standard, and one Zariski local with the components of ω^0 in the degree $k/2$ variables depending on some invertible functions.

The case of -1-shifted symplectic derived schemes

When $k=-1$ the Hamiltonian H in the theorem has degree 0. Then Theorem 2.1 reduces to:

Corollary 2.2

Suppose $(\boldsymbol{X},\omega)$ is a -1-shifted symplectic derived $\mathbb{K}$-scheme. Then $(\boldsymbol{X},\omega)$ is Zariski locally equivalent to a derived critical locus $\mathbf{Crit}(H:U\to\mathbb{A}^1)$, for U a smooth classical $\mathbb{K}$-scheme and $H:U\to\mathbb{A}^1$ a regular function. Hence, the underlying classical $\mathbb{K}$-scheme $X=t_0(\boldsymbol{X})$ is Zariski locally isomorphic to a classical critical locus $\mathrm{Crit}(H:U\to\mathbb{A}^1)$.

Corollary 2.3

Let Y be a Calabi–Yau 3-fold over $\mathbb{K}$ and $\mathcal{M}$ a classical moduli $\mathbb{K}$-scheme of coherent sheaves, or complexes of coherent sheaves, on Y. Then $\mathcal{M}$ is Zariski locally isomorphic to the critical locus $\mathrm{Crit}(H:U\to\mathbb{A}^1)$ of a regular function on a smooth $\mathbb{K}$-scheme.

Summary of the story so far

- Derived schemes behave better than classical schemes in some ways – they are analogous to smooth schemes, or manifolds. So, we can extend stories in smooth geometry to derived schemes. This introduces an extra degree $k \in \mathbb{Z}$.
- PTVV define a version of ('k-shifted') symplectic geometry for derived schemes. This is a new geometric structure.
- 0-shifted symplectic derived schemes are just classical smooth symplectic schemes.
- Calabi–Yau m-fold moduli schemes and stacks are $(2-m)$-shifted symplectic. This gives a *new geometric structure* on Calabi–Yau moduli spaces – relevant to Donaldson–Thomas theory and its generalizations.
- One can go from k-shifted symplectic to $(k-1)$-shifted symplectic by taking intersections of Lagrangians.

3. Donaldson–Thomas theory and its generalizations

Let X be a Calabi–Yau 3-fold. The *Donaldson–Thomas invariants* $DT^\alpha(\tau)$ in $\mathbb{Z}$ or $\mathbb{Q}$ 'count' moduli schemes $\mathfrak{M}_{\rm ss}^\alpha(\tau)$ of τ-semistable coherent sheaves on X in Chern class α. They were defined by Thomas 1998 when τ-stable$=\tau$-semistable using Behrend–Fantechi obstruction theories and virtual classes, and by Joyce–Song 2008 in general. In String Theory, I believe they are 'numbers of BPS states'.

In the τ-stable$=\tau$-semistable case, $\mathfrak{M}_{\rm ss}^\alpha(\tau)$ is the classical truncation of a derived moduli scheme $\boldsymbol{\mathfrak{M}}_{\rm ss}^\alpha(\tau)$, which is quasi-smooth. PTVV say that $\boldsymbol{\mathfrak{M}}_{\rm ss}^\alpha(\tau)$ is -1-shifted symplectic, so BBJ say that $\mathfrak{M}_{\rm ss}^\alpha(\tau)$ is Zariski-locally a critical locus.

There is some interesting geometry associated with critical loci:

- Perverse sheaves of vanishing cycles.
- Motivic Milnor fibres.
- Categories of matrix factorizations.

We can use these to generalize Donaldson–Thomas theory.

3.1. Orientation data and perverse sheaves

Definition (based on Kontsevich and Soibelman 2008)

Let $(\boldsymbol{S},\omega)$ be a -1-shifted symplectic derived scheme or stack. Then the cotangent complex $\mathbb{L}_{\boldsymbol{S}}$ is a perfect complex on $\boldsymbol{S}$, and $K_{\boldsymbol{S}}=\det\mathbb{L}_{\boldsymbol{S}}$ is a line bundle on $\boldsymbol{S}$. *Orientation data* for $(\boldsymbol{S},\omega)$ is a choice of square root line bundle $K_{\boldsymbol{S}}^{1/2}$.

Some notion of orientation data is needed for most generalizations of Donaldson–Thomas theory of Calabi–Yau 3-folds.

Theorem (Joyce–Upmeier arXiv:2001.00113, 2020)

Let X be a compact Calabi–Yau 3-fold and $(\mathfrak{M},\omega)$ the -1-shifted symplectic derived moduli stack of coherent sheaves (or complexes) on X. Then canonical orientation data exists for $(\mathfrak{M},\omega)$.

Question

What is the meaning of orientation data in String Theory?

Theorem (Ben-Bassat-Bussi–Brav–Dupont–Joyce–Szendrői)

Let $(\boldsymbol{S},\omega)$ be a -1-shifted symplectic derived scheme or stack with orientation data $K_{\boldsymbol{S}}^{1/2}$. Then we can construct a natural perverse sheaf $P_{\boldsymbol{S}}^{\bullet}$ on $\boldsymbol{S}$. The hypercohomology $\mathbb{H}^*(P_{\boldsymbol{S}}^{\bullet})$ is a graded vector space. When $\boldsymbol{S}$ is a Calabi–Yau 3-fold derived moduli scheme $\mathfrak{M}_{\rm ss}^{\alpha}(\tau)$ with τ-stable$=\tau$-semistable, we have

$$DT^{\alpha}(\tau)=\sum_{i\in\mathbb{Z}}(-1)^i\dim\mathbb{H}^i(P^{\bullet}_{\mathfrak{M}_{\rm ss}^{\alpha}(\tau)}).$$

Thus the graded vector spaces $\mathbb{H}^*(P^{\bullet}_{\mathfrak{M}_{\rm ss}^{\alpha}(\tau)})$ categorify Donaldson–Thomas invariants. I expect these are interpreted in String Theory as *vector spaces of BPS states*.

Future work: for Calabi–Yau 3-fold moduli stacks $\mathfrak{M}$, one should make $\mathbb{H}^*(P^{\bullet}_{\mathfrak{M}})$ into an associative algebra, a *Cohomological Hall Algebra* as in Kontsevich–Soibelman 2010. Algebras of BPS states?

There is also a theory of *motivic D–T invariants* I won't talk about.

3.2. Donaldson–Thomas style invariants of C–Y 4-folds

There is also an interesting story for Calabi–Yau 4-folds, which uses PTVV -2-shifted symplectic structures on C–Y 4 moduli spaces.

Definition (Borisov–Joyce 2015)

Let $(\boldsymbol{S},\omega)$ be a -2-shifted symplectic derived scheme or stack (for example, a Calabi–Yau 4-fold moduli space), and set $K_{\boldsymbol{S}}=\det\mathbb{L}_{\boldsymbol{S}}$. Then ω induces a natural isomorphism $\iota:K_{\boldsymbol{S}}^{\otimes^2}\rightarrow\mathcal{O}_{\boldsymbol{S}}^{\otimes^2}$. An *orientation* of $(\boldsymbol{S},\omega)$ is an isomorphism $\jmath:K_{\boldsymbol{S}}\rightarrow\mathcal{O}_{\boldsymbol{S}}$ with $\jmath^{\otimes^2}=\iota$.

Theorem (Cao–Gross–Joyce 2019)

Let X be a compact Calabi–Yau 4-fold and $(\mathfrak{M},\omega)$ the -2-shifted symplectic derived moduli stack of coherent sheaves (or complexes) on X. Then an orientation exists for $(\mathfrak{M},\omega)$.

Using the BBJ Darboux Theorem, Theorem 2.1, we prove:

Theorem (Borisov–Joyce 2015)

Let $(\boldsymbol{S},\omega)$ be a proper, oriented -2-shifted symplectic derived scheme over $\mathbb{C}$. Then we can construct a natural virtual class $[\boldsymbol{S}]_{\rm virt}$ in $H_*(S,\mathbb{Z})$.
When $\boldsymbol{S}$ is a Calabi–Yau 4-fold derived moduli scheme $\mathfrak{M}_{\rm ss}^{\alpha}(\tau)$ with τ-stable=τ-semistable, this allows us to define Donaldson–Thomas style 'DT4 invariants' of Calabi–Yau 4-folds.

See also work by Cao–Leung and Richard Thomas (talk at this conference).

Question

What is the interpretation of DT4 invariants in String Theory?

Jet travel and climate change

- UK adult average emissions per year: 9 tonnes CO_2.
- Return flight London-New York: 1.67 tonnes CO_2.
- Return flight London-San Francisco: 2.59 tonnes CO_2.
- Carbon offsetting makes you feel better, but does not work. One EU study said 85% of carbon offset projects do not reduce emissions.
- 'Binge flyers' have a disproportionate damaging effect. In one study 4% of travellers took 30% of the flights.
- On short-haul flights (e.g. London–Paris), going by train is circa 11% of the carbon cost of jet travel, and by car 30%, per mile. Train travel is clearly the most environmentally friendly.
- This Simons grant could have been designed to maximize jet travel. 4 conferences per year. 29 people flew to this conference.

Question

What should we be doing, as individuals and as a profession, to reduce the environmental damage from flying to conferences?