The Simons Collaboration on Special Holonomy

An Overview of Its Progress and Goals

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Part I: Some Historical Background

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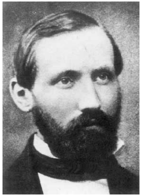

In 1854, G. F. Bernhard Riemann (1826–1866) gave the most famous job talk in mathematical history: *On the Hypotheses That Lie at the Foundations of Geometry*. His ideas about geometry, which developed into Riemannian geometry, paved the way for Einstein's theory of relativity and much else.

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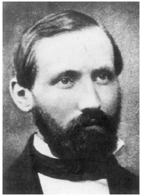

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In 1917, Tullio Levi-Civita (1873–1941) showed, given a Riemannian metric g on an n-manifold M, that one could define a notion of parallel transport along a curve $\gamma : [0,1] \to M$,

$$P^g(\gamma) : T_{\gamma(0)}M \to T_{\gamma(1)}M,$$

that was a linear isometry between the two tangent spaces.

In 1918, J. A. Schouten (1883–1971) considered the set of possible results of such parallel translations around loops γ based at $x \in M$

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This is the first known case of what are now known as metrics with special holonomy.

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is g-parallel, and the g-Laplacian Δ commutes with this decomposition, yielding the all-important corresponding type decomposition in cohomology of a Kähler manifold

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In 1957, S.-s. Chern (1911–2004) considered the generalization of Kähler geometry in which the holonomy groups H_x^g are conjugate to a subgroup $K \subset \mathrm{O}(n)$ that leaves invariant a decomposition into irreducible K-modules

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However, aside from a few simple examples (mainly local product structures and locally symmetric spaces), he did not have an application in mind.

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Berger's List:

n	**Holonomy**	**Name**
n	$\mathrm{SO}(n)$	generic
$n = 2m$	$\mathrm{U}(m) \subset \mathrm{SO}(n)$	Kähler
$n = 2m$	$\mathrm{SU}(m) \subset \mathrm{SO}(n)$	Calabi–Yau
$n = 4m$	$\mathrm{Sp}(m) \subset \mathrm{SO}(n)$	hyperKähler
$n = 4m$	$\mathrm{Sp}(m)\mathrm{Sp}(1) \subset \mathrm{SO}(n)$	Quaternion Kähler
$n = 7$	$\mathrm{G}_2 \subset \mathrm{SO}(7)$	?
$n = 8$	$\mathrm{Spin}(7) \subset \mathrm{SO}(8)$	?
$n = 16$	$\mathrm{Spin}(9) \subset \mathrm{SO}(16)$	?

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Berger's List (modified) and the corresponding parallel forms

n	**Holonomy**	**Name**	**Parallel Forms Generators**
n	$\mathrm{SO}(n)$	generic	$vol \in \Omega^n(M)$
$n = 2m$	$\mathrm{U}(m) \subset \mathrm{SO}(n)$	Kähler	$\omega_J \in \Omega^{1,1}(M)$
$n = 2m$	$\mathrm{SU}(m) \subset \mathrm{SO}(n)$	Calabi–Yau	$\omega_J \in \Omega^{1,1}(M),\ \Upsilon \in \Omega^{m,0}(M)$
$n = 4m$	$\mathrm{Sp}(m) \subset \mathrm{SO}(n)$	hyperKähler	$\omega_I, \omega_J, \omega_K \in \Omega^2(M)$
$n = 4m$	$\mathrm{Sp}(m)\mathrm{Sp}(1) \subset \mathrm{SO}(n)$	Quaternion Kähler	${\omega_I}^2 + {\omega_J}^2 + {\omega_K}^2 \in \Omega^4(M)$
$n = 7$	$\mathrm{G}_2 \subset \mathrm{SO}(7)$		$\phi \in \Omega^3(M),\ *\phi \in \Omega^4(M)$
$n = 8$	$\mathrm{Spin}(7) \subset \mathrm{SO}(8)$		$\Phi \in \Omega^4(M)$

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These developments include:

- Applications in algebraic geometry
- Connections with the theory of calibrations, e.g., *special Lagrangian geometry*
- Formulation of supersymmetric field theories
- The discovery of mirror symmetry, e.g., a formulation of duality by Strominger-Yau-Zaslow

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Meanwhile, the interest in applications of Calabi–Yau geometry has stimulated an enormous development of new mathematical tools and techniques that is still ongoing.

After the initial work of M. Berger, others (notably E. Bonan, A. Gray, D. Alexeevski, and M. Fernandez) showed that many of the features of Calabi–Yau metrics would have analogs in G_2- and $\mathrm{Spin}(7)$-holonomy metrics, if such metrics existed.

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The G_2-analog of the Hodge 'diamond' in Kähler geometry

G_2-irrep\degree	$\Lambda^0(V)$	$\Lambda^1(V)$	$\Lambda^2(V)$	$\Lambda^3(V)$	$\Lambda^4(V)$	$\Lambda^5(V)$	$\Lambda^6(V)$	$\Lambda^7(V)$
1	$\mathbb{R}$	0	0	$\mathbb{R}\cdot\phi$	$\mathbb{R}\cdot{*\phi}$	0	0	$*1$
7	0	V	V	V	V	V	V	0
14	0	0	$\mathfrak{g}_2$	0	0	$\mathfrak{g}_2$	0	0
27	0	0	0	$S^2_0(V)$	$S^2_0(V)$	0	0	0

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In 1984, R. L. Bryant, inspired by the work of F. R. Harvey and H. B. Lawson, Jr. on calibrated geometry, applied É. Cartan's techniques to show that metrics with holonomy G_2 and $\mathrm{Spin}(7)$ do, indeed, exist, and he determined their local generality.

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- The ongoing development of gauge theory and symplectic topology and the study of moduli spaces and enumerative problems in algebraic geometry motivated the study of similar questions about these special holonomy manifolds and their 'distinguished' submanifolds.

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For the rest of my talk, I'll focus on G_2-structures and metrics on 7-manifolds with holonomy G_2. (The story for $\mathrm{Spin}(7)$ is analogous.) Here is some basic information:

- Let V be a 7-dimensional vector space over $\mathbb{R}$.
  - A 3-form $\phi \in \Lambda^3(V^*)$ is said to be definite if $(v \lrcorner \phi)^2 \wedge \phi \neq 0$ for all nonzero $v \in V$.
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- What features of examples will be important for applications in string theory and M-theory?
- What aspects of gauge theory and enumerative invariants, so effective in the complex cases, can be usefully generalized to the exceptional cases?

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B. A parabolic flow for closed G_2-structures

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For a compact 7-manifold M^7, let $\gamma \in H^3_{dR}(M)$ be chosen for which

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The 'gradient flow' of Hitchin's functional is the so-called closed G_2-Laplacian flow:

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- We do not know when a orientable, spinnable manifold M^7 supports a closed everywhere nondegenerate form, not even for $M = S^7$.

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On a Riemannian (M^7, g) with G_2-holonomy, the parallel 3-form ϕ and its dual $\psi = *_\phi \phi$ are calibrations (Harvey–Lawson), and they can be used to define calibrated submanifolds of dimensions 3 (associative) and 4 (co-associative), as well as G_2-instantons, which are the analogs of anti-self-dual connections ($*F = -\phi{\wedge}F$).

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Developing tools to exploit the moduli spaces of these objects in G_2-geometry is a major goal of the Collaboration.

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- Recent progress in geometric measure theory has given us a better understanding of the kinds of singularities that can arise in calibrated submanifolds, particularly the 3-dimensional associative case in G_2-geometry. R. Bryant has shown that any real-analytic curve in a G_2-manifold can be locally realized as the singular locus of a real-analytic associative subvariety, but much is still not understood concerning the possible tangent cones at singularities of associatives or co-associatives.

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- Meanwhile, D. Joyce has developed new tools to study the moduli of associative 3-folds in G_2-manifolds and G_2-instantons as part of a very general program to associate vertex algebras and Lie algebras to the homology of moduli spaces that arise in a number of contexts in differential geometry of special holonomy spaces.

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In string theory, following the work of R. C. McLean, the phenomenon of mirror symmetry in Calabi–Yau 3-folds has a conjectural formulation (due to Strominger, Yau, and Zaslow) in terms of a pair of (singular) special Lagrangian torus fibrations over a common base:

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The analog in G_2-geometry is a (singular) co-associative $K3$-fibration of a G_2-holonomy manifold (sometimes known as a Kovalev-Lefschetz (singular) fibration). The direct construction of such 'fibrations' is extremely challenging.

D. Analogs of the SYZ theory and 'mirror phenomena'

In string theory, following the work of R. C. McLean, the phenomenon of mirror symmetry in Calabi–Yau 3-folds has a conjectural formulation (due to Strominger, Yau, and Zaslow) in terms of a pair of (singular) special Lagrangian torus fibrations over a common base:

$$\begin{array}{ccccccccc} T & \to & \mathrm{M} & & & & M^* & \leftarrow & T^* \\ & & & \searrow & & \swarrow & & & \\ & & & & \mathrm{B} & & & & \end{array}$$

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S. Donaldson will speak about an approach to gaining information about such mappings by passing to an 'adiabatic limit' and also about understanding prescribed boundary data for G_2-manifolds with boundary.

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S. Salamon will explain how a symmetry reduction applied to a well-known smooth, complete G_2-manifold yields a singular SU(3)-space whose features are expected to be useful in studying a model in *M*-theory investigated by Atiyah and Witten.