On Solitons for the closed G_2-Laplacian Flow

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Background: G_2-geometry

A 3-form σ on a 7-manifold M is said to be definite if, for every nonzero tangent vector $v \in T_xM$, the 7-form $(v \lrcorner \sigma)^2 \wedge \sigma \in \Lambda^7(T^*M)$ is also nonzero. The set of definite 3-forms on M is denoted $\Omega^3_+(M)$, which is open in the sections of $\Lambda^3(T^*M)$.

If σ is definite, there is a unique Riemannian metric g_σ and orientation $*_\sigma$ such that, for all vector fields X on M, we have

$$(X \lrcorner \sigma)^2 \wedge \sigma = 6\, g_\sigma(X, X) *_\sigma 1.$$

Equivalently, a definite 3-form σ defines a G_2-structure on M, where $\mathrm{G}_2 \subset \mathrm{SO}(7)$ is the compact, connected, exceptional simple group of dimension 14.

Theorem (Fernandez–Gray (1982))

A G_2-structure σ on M^7 is g_σ-parallel if and only if $\mathrm{d}\sigma = \mathrm{d}(*_\sigma \sigma) = 0$.

The closed condition $\mathrm{d}\sigma = 0$ is $\binom{7}{4} = 35$ linear equations on σ (relatively easy to solve), while the co-closed condition $\mathrm{d}(*_\sigma \sigma) = 0$ is $\binom{7}{5} = 21$ (nonlinear) equations on σ.

However, these two systems overlap by 7 equations because of the Fundamental Identity

$$\big(*_\sigma \mathrm{d}(*_\sigma \sigma)\big) \wedge *_\sigma \sigma + (*_\sigma \mathrm{d}\sigma) \wedge \sigma = 0.$$

Thus, $\mathrm{d}\sigma = \mathrm{d}(*_\sigma \sigma) = 0$ is $49 = 35 + 21 - 7$ first-order nonlinear partial differential equations for 35 unknowns.

Theorem (B—, (1984))

1. *Up to diffeomorphism, the germs of closed and co-closed G_2-structures depend on six functions of six variables.*
2. *For the generic such σ, the associated metric g_σ is not locally symmetric or a product.*

Proof: The system $\mathrm{d}\sigma = \mathrm{d}(*_\sigma \sigma) = 0$ is involutive in Cartan's sense, and so the general local solution can be constructed via the Cartan-Kähler Theorem.

In particular, consider the the space $\mathcal{J}_0^k$ of k-jets at $0 \in \mathbb{R}^7$ of closed and coclosed G_2-structures σ for which $\sigma(0)$ is fixed and the standard coordinates are geodesic normal.

Set $\delta_k = \dim \mathcal{J}_0^k - \mathcal{J}_0^{k-1}$. Then $P(t) = \delta_0 + \delta_1\, t + \delta_2\, t + \cdots$ is

$$P(t) = t^2 \left(\frac{14}{(1-t)^2} + \frac{21}{(1-t)^3} + \frac{21}{(1-t)^4} + \frac{15}{(1-t)^5} + \frac{6}{(1-t)^6} \right).$$

In general, once one establishes the involutivity of a set of PDE, Cartan's theory provides a formula for computing the 'Poincaré series' of the space k-jets of the 'general' local solution. One can think of this as the 'flexibility' of the 'general solution', and it indicates what sorts of identities the higher derivatives of solutions will satisfy.

The G_2-decomposition of exterior forms into irreps

G_2-irrep\degree	$\Lambda^0(V)$	$\Lambda^1(V)$	$\Lambda^2(V)$	$\Lambda^3(V)$	$\Lambda^4(V)$	$\Lambda^5(V)$	$\Lambda^6(V)$	$\Lambda^7(V)$
1	$\mathbb{R}$	0	0	$\mathbb{R}\cdot\sigma$	$\mathbb{R}\cdot *\sigma$	0	0	$*_\sigma 1$
7	0	V	V	V	V	V	V	0
14	0	0	$\mathfrak{g}_2$	0	0	$\mathfrak{g}_2$	0	0
27	0	0	0	$S^2_0(V)$	$S^2_0(V)$	0	0	0

Proposition (The Fundamental Identity)

For any G_2-structure $\sigma \in \Omega^3_+(M)$, there exist unique differential forms $\tau_0 \in \Omega^0(M)$, $\tau_1 \in \Omega^1(M)$, $\tau_2 \in \Omega^2_{14}(M,\sigma)$, and $\tau_3 \in \Omega^3_{27}(M,\sigma)$ so that the following equations hold:

$$\begin{aligned} \mathrm{d}\sigma &= \tau_0 \, *_\sigma \sigma + 3\,\tau_1 \wedge \sigma + *_\sigma \tau_3\,, \\ \mathrm{d}(*_\sigma \sigma) &= \qquad 4\,\tau_1 \wedge *_\sigma \sigma + \tau_2 \wedge \sigma\,. \end{aligned} \tag{1}$$

Solitons

One strategy for constructing solutions on a compact manifold is to consider an analog of the Kähler-Ricci flow, i.e., look at the 'heat equation'

$$\frac{\partial \sigma}{\partial t} = \Delta_\sigma \, \sigma$$

with a closed initial G_2-structure σ_0 at $t = 0$.

Theorem (B— – Xu, 2011)

The above heat equation on a compact manifold M^7 with closed initial G_2-structure has a unique solution for short time, and its volume form is pointwise increasing under the flow.

Lotay and Wei (2017) have gone further and proved Shi-type estimates for the flow and showed that the flow is unique and exists as long as the curvature remains bounded.

Just as in Ricci-flow, the notion of a *soliton* turns out to be important:

Definition
A pair $(\sigma, X) \in \Omega^3_+(M) \times \mathrm{Vect}(M)$ is a λ-soliton (where λ is a constant) if

$$\Delta_\sigma \sigma = \lambda\, \sigma + \mathsf{L}_X \sigma$$

Under the closed G_2-Laplacian flow, a λ-soliton scales and flows by a diffeomorphism on M, so it can't converge to a co-closed solution. The goal is to understand blow-ups and singularity development.

Because of the Fundamental Identity, a closed G_2-structure σ satisfies

$$\mathrm{d}(*_\sigma \sigma) = \tau \wedge \sigma$$

where $\tau \in \Omega^2_{14}(M)$, i.e., $\tau \wedge (*_\sigma \sigma) = 0$, which implies

$$\Delta_\sigma \sigma = \mathrm{d}\tau.$$

When (σ, X) is a λ-soliton, this implies

$$\mathrm{d}\tau = \Delta_\sigma \sigma = \lambda\,\sigma + \mathsf{L}_X \sigma = \lambda\,\sigma + \mathrm{d}\big(X \lrcorner\, \sigma\big).$$

Which can be written as

$$\mathrm{d}(\tau - X \lrcorner\, \sigma) = \lambda\,\sigma$$

Setting $\beta = \tau - X \lrcorner\, \sigma$, the $\Omega^2(M) = \Omega^2_{14}(M) \oplus \Omega^2_7(M)$ decomposition of β is just

$$\big(\tau,\ -X \lrcorner\, \sigma\big).$$

The G_2-decomposition implies that we can recover τ in terms of β:

$$\tau \wedge \sigma = -\tfrac{2}{3} *_\sigma \beta + \tfrac{1}{3}\,\beta \wedge \sigma$$

Thus, one has a first-order system S_λ for the pair (β,σ):

1. $0 = \mathrm{d}\beta - \lambda\,\sigma \in \Omega^3(M)$ — 35 equations
2. $0 = \mathrm{d}\sigma \in \Omega^4(M)$ — 35 equations
3. $0 = \mathrm{d}(*_\sigma\sigma) + \frac{2}{3}\,*_\sigma\beta - \frac{1}{3}\,\beta{\wedge}\sigma \in \Omega^5(M)$ — 21 equations
4. $0 = \mathrm{d}\big(*_\sigma\beta\big) \in \Omega^6(M)$ — 7 equations

The fourth equation is a first-order consequence of the first three equations. It is not *a priori* clear that there are not more first-order equations on (β,σ) that are derivable from these.

This seems to be 98 equations for $56 = 21 + 35$ unknowns, but the Fundamental Identity implies that the 2nd and 3rd set of equations overlap by 7, so it is actually only 91 equations, which turn out to be independent.

Proposition
If $(\beta,\sigma) \in \Omega^2(M^7) \times \Omega^3_+(M^7)$ satisfies the system S_λ, then $\beta = \delta_\sigma\sigma - X \lrcorner\, \sigma$, where (σ, X) is a λ-soliton.

Using the standard tautological construction on the bundle $X = \Lambda^2(T^*M) \oplus \Lambda^3_+(T^*M)$ over M, one can construct an ideal $\mathcal{I}_\lambda$ on X with the property that sections (β, σ) of $X \to M$ that are integrals of $\mathcal{I}_\lambda$ are exactly the graphs of solutions of S_λ.

Theorem (B —)

All of the horizontal integral elements of $\mathcal{I}_\lambda$ are Kähler-regular, and the Cartan characters are given by $(s_0, s_1, \ldots, s_7) = (0, 0, 1, 3, 7, 15, 23, 7)$.

Proof Idea: The 'highest order' part of the equations can be grouped as $\mathrm{d}\sigma = \mathrm{d}(*_\sigma \sigma) = 0$ (compare with the EDS for closed and co-closed σ, with characters (0,0,0,1,4,10,13,7)) and $\mathrm{d}\beta = \mathrm{d}(*_\sigma \beta) = 0$ (compare with the EDS for β harmonic with respect to g_σ, with characters (0,0,1,2,3,5,10,0)). The independence of the 91 equations implies that Cartan's Test is satisfied.

Corollary

The diffeomorphism classes of germs of λ-solitons depend on 16 $(= 23 - 7)$ functions of 6 variables. Every $C^{1,\alpha}$ solution (β, σ) is real-analytic in g_σ-harmonic coordinates.

In the study of Ricci-flow, the important solitons (g, X) turn out to be the gradient solitons, i.e., the vector field X is actually the g-gradient of a function f. These are much more special than general Ricci-flow solitons.

Example

In dimension 3, the Ricci-flow solitons depend on 6 functions of 2 variables, while the Ricci-flow *gradient* solitons depend on only 2 functions of 2 variables.

Definition

A λ-soliton (σ, X) is a *gradient* λ-soliton if X is a g_σ-gradient vector field, i.e., $\mathrm{d}(X^\flat) = 0$.

Many examples of gradient λ-solitons are now known to exist: Lotay–Wei (2017), Fowdar (2017), Lauret (2017), Ball (2020), Haskins–Nordström (2021), and recent work by Haskins–Khan–Payne. However, the generality of the set of gradient λ-solitons is not yet known.

To see the difficulty, start with the identity of 5-forms

$$(X \lrcorner \sigma) \wedge \sigma = 2\, X^\flat \wedge *_\sigma \sigma$$

Taking the exterior derivative of the left hand side, we get

$$\mathrm{d}\big((X \lrcorner \sigma) \wedge \sigma\big) = \mathrm{d}(X \lrcorner \sigma) \wedge \sigma = (\mathrm{d}\tau - \lambda\, \sigma) \wedge \sigma = \mathrm{d}\tau \wedge \sigma = \mathrm{d}(\tau \wedge \sigma) = 0$$

since $\tau \wedge \sigma = \mathrm{d}(*_\sigma \sigma)$. Thus, we must have

$$0 = \mathrm{d}(X^\flat \wedge *_\sigma \sigma) = \mathrm{d}(X^\flat) \wedge *_\sigma \sigma - X^\flat \wedge \tau \wedge \sigma.$$

Hence $\mathrm{d}(X^\flat) = 0$ implies $X^\flat \wedge \tau \wedge \sigma = 0$, which is equivalent to the algebraic condition.

$$X \lrcorner \tau = 0.$$

The vanishing of this 1-form is 7 algebraic equations on (β, σ). The remaining 14 = 21 - 7 equations implied by $\mathrm{d}(X^\flat) = 0$ are new first-order equations on (β, σ), independent from the 91 first-order equations that define a λ-soliton.

One can enlarge S_λ by adding $\mathrm{d}(X^\flat) = 0$, which, in terms of β and σ, is equivalent to

$$\mathrm{d}\big(*_\sigma(\beta \wedge *_\sigma \sigma)\big) = 0.$$

to get S_λ^+, and, similarly, one can enlarge $\mathcal{I}_\lambda$ to $\mathcal{I}_\lambda^+$ by adding the 2-form $\mathrm{d}\big(*_\sigma(\beta\wedge *_\sigma \sigma)\big)$.

Unfortunately, $\mathcal{I}_\lambda^+$ is not involutive! (I.e., there are no Kähler-regular integral elements.)

Cartan's general method for dealing with non-involutive systems is to prolong the system, essentially by adding the 'free' derivatives as new variables and test the new differential system for involutivity.

Unfortunately, after two successive prolongations of $\mathcal{I}_\lambda^+$, the resulting system is still not involutive, and the calculations become unwieldy. Thus, we still do not yet understand the 'generality' of gradient λ-solitons. In particular, we do not understand the possible 'higher Bianchi' identities that they might satisfy.

Thanks for your attention!