Survey of Lagrangian mean curvature flow

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Mean curvature flow (MCF)

\[ X_t : \Sigma^n \to (N^m, h) \] satisfy \( \left( \frac{\partial X_t}{\partial t} \right) = H(X_t(x)) \)

- \( H = g^{ij} \left( \nabla x_i \frac{\partial X}{\partial x_j} \right)^+ \), \( g_{ij} = \langle \frac{\partial x}{\partial x_i}, \frac{\partial x}{\partial x_j} \rangle \)

- Negative gradient flow of area functional

- Short time existence (Cpt case, special argument for noncpt)

- Curve shortening flow

  Embedded closed plane curve: Shrink to round pts

  \( \text{(Gage-Hamilton-Grayson)} \)

- Hyper surface MCF

  Embedded convex closed hypersurface in \( \mathbb{R}^n \): Shrink to round pts

  \( \text{(Huisken)} \)
Lagrangian mean curvature flow (LMCF) \( w |_{L^n} = 0 \)

Starting from a smooth immersed Lag in KE a smooth sol of MCFl will preserve Lag condition

\( \Rightarrow \) called LMCF

- \( \nabla H \) infinitesimal symplectic v.f.

\[ \dd 
abla H |_{L} = R.c |_{L}^{KE} = 0 \]

or by max principle Smoczyk

- noncpt case needs to prove separately

\( \Rightarrow \) a potential way to find slag or min Lag

Thomas - Yau conjecture & its refinement by Joyce
\[ L^r \subset N^{2r} \]

- Difficulties for MCF with high codim:
  1. Nonlinear parabolic system not one single eq (no avoid principle)
  2. H in normal bundle, estimates involve tensor, not a scalar function (more complicated)
  3. No natural convex condition

- Good things for LMCF than general MCF with high codim:
  1. With Lag nbh Thm $\leftrightarrow$ Hamiltonian deformation can be written as the graph of $df$
     \[ \rightarrow \text{Hope it resembles hypersurface MCF} \]
  2. When in CY, we have lag angle $\Omega$
      \[ \rightarrow \text{give additional information} \]

  Good example for testing case for MCF with high codim
Existence & Singularities of MCF & LMCF

- Short time existence for cpt initial data
  Noncpt case needs to be discussed case by case
  (Short time existence for ACIMCF by W.B. Su)

- Singularities occur in general (Suppose at time \( T \))
  \( |A|^2 \) blows up and at least of the rate \( (T-t)^{-\frac{1}{2}} \)
  If \( |A|^2(x,t)(T-t) < C \) called type I singularity
  Otherwise are called type II singularity

- Big set of singularities can happen
  \( \infty \times \mathbb{R}^n \), collapsing \( M \times S^k(E) \), isoparametric submanifold

- Restrict to isolated singularities first
  Simple & more generic
• Parabolic scaling (blow up)

If \( L(x,t) \) satisfies \( MC^F \), then

\[
\tilde{L}(x,s) = \sigma \left( L(x, t_0 + \frac{s}{\sigma^2}) - y_0 \right)
\]

also satisfies \( MC^F \). (If not in \( IR^n \), scale the metric instead.)

If \( L \) defined in \([0, T]\), then \( \tilde{L} \) defined in \([t - t_0 \sigma^2, (t - t_0) \sigma^2]\)

1. Type I blow up (central \( \sigma \) blow up)

   Assume \( L \) has an isolated singularity at \((y_0, T)\)

   \( \sigma \to \infty \), \( \tilde{L}^i(x,s) = \sigma \cdot \left( L(x, T + \frac{s}{\sigma^2}) - y_0 \right) \)

   \( \exists \) a subseq.

   \( \tilde{L}^i \) has Brakke flow limit \( \tilde{L}^* \)

   Different subseq may have different limits
(2) Type-II blow up (max pt. blow up)

define \( \tilde{L}^i(x, s) = \sigma_i \left( L(x, t_i + \frac{s}{\sigma_i^2}) - y_i \right), \quad \sigma_i \to \infty \)

\( y_i = L(x_i, t_i), \quad t_i \to T, \quad \sigma_i \to \infty \)

\(-t_i \sigma_i^2 < s < (T-t_i) \sigma_i^2\)

\( s \to (T-t_i) \sigma_i^2 \to \infty \) \( \Rightarrow \) eternal sol

\( \exists \) a subseq \( \Rightarrow \tilde{L}^i \) has a Brakke flow limit \( \tilde{L}^{\infty} \)

usually take \( \sigma_i \sim \max |A(x, t_i)| = A(x_i, t_i) \)

monotonicity formula of Huisken

\( \Rightarrow \) for central pt. blow up, limit is a shranker

entropy defined by back heat kernel
Monotonicity formula (Husken)

- Back heat kernel at \((y_0, t_0) \in \mathbb{R}^k \times \mathbb{R}\)

\[
\Phi_{y_0, t_0}(y, t) = \frac{1}{\sqrt{4\pi(t-t_0)}} e^{-\frac{|y-y_0|^2}{4(t-t_0)}} \quad t < t_0
\]

- If \(L_t\) satisfies \(MC F\), then

\[
\frac{d}{dt} \int_{L_t} \Phi_{y_0, t_0} d\mu_t = -\int_{L_t} \Phi_{y_0, t_0} L H^+ \frac{(y-y_0)^T}{2(t-t_0)} \frac{1}{2(t-t_0)} d\mu_t < 0
\]

when \(L_t\) non-compact, assume all integrals finite values

Type I blow up

\[
\int_{L_t} \Phi_{y_0, t} d\mu_t = \int_{L_{\hat{t}}} \Phi_{0, 0} d\mu_{\hat{t}} \quad \hat{t} = T + \frac{s}{\delta_0^2}
\]

\[
\Rightarrow \int_{L_{\hat{t}}} \Phi_{0, 0} d\mu_{\hat{t}} = \text{constant} \Rightarrow Y_{\hat{t}} = 2 \leq H . \quad s < 0
\]
Soliton Solv

1. If submfd $X$ in $\mathbb{R}^k$ satisfy $X^t = \alpha H(X)$, $\alpha \neq 0$, constant
   then $X_t = \frac{\sqrt{2}}{\alpha t} X$ satisfy MCF

   If $\alpha < 0$, $X_t$ is defined for $t < 0$. self-shrinker an ancient sol
   \[
   (X_t)^t = \frac{\sqrt{2}}{\alpha t} \alpha H(X) = \frac{2}{\alpha} H(X_t) = 2 \alpha H(X_t)
   \]
   type I blow up

   If $\alpha > 0$, $X_t$ is defined for $t > 0$ self-expander
   models for surgeries
   possible limit at $t = 0$

2. If $X$ satisfy $H(X) = T^t$ for a constant vector
   then $X_t = X + tT$ satisfy MCF
   defined for all $t$, translating sol
   (an eternal sol)
   related to type II blow up limits

RMK: For MCF in $(N^n, h)$ blow up limit is also in $\mathbb{R}^n$. We can also imbed $N$ to $\mathbb{R}^k$ isometrically & MCF becomes $(\frac{\partial X}{\partial t})^t = H + E$ in $\mathbb{R}^k$
LMCF of graded Lag in CY have no type I singularity

(Wang, Chen-Li, Neves)

Key: No non-trivial smooth graded Lag shrinker

- Recall L Lag in CY: \( \Omega_L = e^{i\theta} \text{Vol}_L \); \( \Theta: L \to \mathbb{R}/2\pi \)
  - It is called graded if \( \Theta \) can be lifted to \( \Theta: L \to \mathbb{R} \).
- Being graded is preserved under LMCF and \( \frac{d\Theta_t}{dt} = \Delta \Theta_t \)

- \( f_t \), a smooth family of function on \( L_t \)

  Then \( \frac{d}{dt} \int_{L_t} f_t \Omega(y_0, T) = \int_{L_t} \left( \frac{df_t}{dt} - \Delta f_t \right) \Omega(y_0, T) \, d\mathcal{H}^n \)

\[ -\int_{L_t} f_t \left( H + \frac{(Y - y_0)^+}{2(T - t_0)} \right)^2 \Omega(y_0, T) \, d\mathcal{H}^n \]

assume all quantities finite.
Proof: Assume $L$ a smooth graded Lag shrinker $L^\perp = -2H$

then $L_t = \sqrt{-t} L$ a smooth sol to LMCFI for $t < 0$

Take $f^\perp_t = \Theta^2_t = \Theta^2 \frac{d}{dt} \Theta^2_t = \triangle \Theta^2_t - 2H^2$

$\Rightarrow 0 = \frac{d}{dt} \int_{L_t} \Theta^2_t \varphi(y_0, T) d\mathcal{H}^n = -\int 2H^2 \varphi(y_0, T) d\mathcal{H}^n$

$\text{scalip inv} \Rightarrow \text{cmst}$

$\Rightarrow H = 0 \Rightarrow L^\perp = 0 \therefore \text{a smooth minimal cone}$

$\Rightarrow L \text{ a plane} \Rightarrow \text{no type I singularity}$

By white's regularity Thm

Thm (Neveu)

LMCFi of graded Lag with $101 < \infty$. When do central blow up at a singularity (it's a Type II singularity) $\Rightarrow \exists$ a rescaled subseq converges to union of integral slag cone $L_1, \ldots, L_n$ with multiplicity $m_1, \ldots, m_n$. Lag angles $\Theta_1, \ldots, \Theta_n$
• The set $\{\Theta_1, \ldots, \Theta_n\}$ does not depend on the scale of rescaling chosen.
• When $n = 2$, slag cones are planes.

\[ \text{Thm. (Never)} \]

If in the above thm, $L_0$ is almost calibrated & rational, then any convergent subsequence of connected component \( \text{of} \, B_{x_0}^{(1)} \cap L_i \) intersecting $B_{r(0)}$ converges to a slag cone with multiplicity with slag angle $\Theta$, a slag $L$ is called rational if

\[ \lambda(H, (L, \Sigma)) = \int a_2 k \pi \, k \in \mathbb{Z} \]

A: Liouville form

Convergence in the thms is:\ for every smooth cpt support func $\Phi$ and every $f \in C^2(\mathbb{R})$, and every $S < 0$

\[ \lim_{i \to \infty} \int_{L_i} \int (\Theta_i, s) \Phi \, d\mathcal{H}^n = \sum_{j=1}^N m_j f(\Theta_j) M_j(\Phi) \]
• As the singularities for graded lag in CY are type II, one may want to consider max point blow up instead. Is there any relation between central pt blow up limits (I) & max point blow up limits (II)?

Will a blow-down of (II) arise as (I)? Hard to see and also limits may depend on subsequence.

• Lambert-Lotay-Schulze can characterize ancient exact almost calibrated solns to LMCF with uniformly bounded area ratio and 0, when a blow down is a pair of multiplicity one transverse planes $P_1, P_2$. 
With my collaborators (Wang, Joyce, Tsui) we construct many soliton sole (shrinkers, expanders, and translators) for LMCF with different properties and applications.

1. Maslov class non-zero case
   - Construct Lag. shrinkers, cones, expanders (HS & non-HS)
   - Can glue together to form an eternal sol
   - Of Lag Brakke flow without mass loss

Related to Schoen-Wolfson's work mentioned in Marke talk,
For any C_p,q core we can find C_p,q and resolve C_p,q U C_p,q'
ae (*)& can also resolve a single C_p,q for p > 1, q > 1

→ C_{2.1} may be the only possibly tangent core for 2d Lag minimizers' singularities
Maslov class zero case

- Lag expanders, translating sole \( (\text{top } S^{n-1} \times \mathbb{R}) \)
- Lag angle can be arbitrarily small
- Asymptotic to two planes

For two Lag planes intersect at one pt, by changing coordinates can make
\[ P_0 = \{ (x_1, \ldots, x_n) | x_i \in \mathbb{R} \} \]
\[ P_\varphi = \{ (x_1 e^{i \varphi_1}, \ldots, x_n e^{i \varphi_n}) | x_i \in \mathbb{R} \} \]
\[ \varphi = (\varphi_1, \ldots, \varphi_n) \text{ fixed, } 0 < \varphi_0 < \pi. \]

Liu-Luo vol minimizes \( \sum \varphi_i \geq \pi \) (Nance, Lawler, \ldots)
• When $\Sigma \Phi_i = \pi$, $\exists$ a $\Phi$ ag called Lawlor neck $L$ asymptotic to $P_0 - P_\phi$ of explicit form $-P_\phi$: reverse orientation
\[
\{(Z_1(s) X_1, \ldots Z_n(s) X_n) \mid \Sigma X_i^2 = 1, Z_i(s): \mathbb{R} \to \mathbb{C} \setminus \{0\}\}
\]

• In the non-vol-minimizing case $\Sigma \Phi_i < \pi$

  We find a family of $\Phi$ ag expanders asymptotic to $P_0 - P_\phi$

  (Joyce - L - Tsui, JLT-expander, $\Theta$ can be arbitrarily small)

• Lawlor neck can be used to desingularize intersection $pt$. (Butscher, Joyce, L-, \ldots )

• JLT-expanders can be used to resolve singularity of LMC $F$. (Begley - Moore)
smooth exact slag asymptotic to $P_{u-P_p}$ are unique up to scaling for $n \geq 3$. If embedded, it must be Lannier neck, if immersed $P_{u-P_p}$

Rmk: For workers on smoothing slags with intersecting points mentioned above, one needs to require angle criterion $\sum q_i = \pi$. This uniqueness result shows that there is no slag smoothing if without angle criterion.

- Lag expander asymptotic to $P_{u-P_p}$ must be JLT ($\text{Lotay - Neuer}, n=2$, Imagis-Joyce-Dos Santos, $n \geq 3$)

- These uniqueness results are used essentially in the works of Begley-Moore and Lambert-Lotay-Schulze
Dynamical Stability of MCF

(2) For the graph of maps between ept. mfd x M & N, suitable conditions on the curvature of N & N, and the singular value of initial map $\rightarrow$ long-time existence of MCF & convergence (to a trivial map) \(\text{Wang, Tsui.}\)

(3) Initial submfd in a C' neighborhood of a ept. smooth mini submfd $\Sigma$ with strong stability \(\text{(R - A > 0)}\) \(\text{Tsai-Wang}\)

$\rightarrow$ long-time existence & converge to $\Sigma$

uniqueness of minimal submfd in that nbh

$\rightarrow$ extended to enhanced Brakke flow and have the uniqueness among stationary varifolds \(\text{(Lotay-Schulze)}\)
Noncompact case

Long time existence & convergence to central mfd for equiv.

Almost-calibrated AC LMCF with suitable condition (Su)

How about if we do not have a minimal (lag) submfd before hand?

H2. Lc

L cpt Lag (immersed) in complete KE. For any $V_0, \Lambda_0 > 0$, there exists $\varepsilon_0$, such that if $\text{Vol}(L) \leq V_0, |\Lambda_1| \leq \Lambda_0, \int_c H_1^2 \leq \varepsilon_0$

$\Rightarrow$ LMCF of L will converge exponentially to a min Lag

if (a) when the ambient $\widehat{\mathcal{R}} < 0$

(b) $\widehat{\mathcal{R}} \geq 0$, and $\lambda_1(L) \geq \frac{\widehat{\mathcal{R}}}{2n} + \delta_0$ for given $\delta_0$

A: 2nd. f.f., $\lambda_1$: the first eigenvalue, $\varepsilon_0 = \varepsilon_0 (V_0, \Lambda_0, \delta, \widehat{\mathcal{R}}, k_5, \hat{\varepsilon}_0)$

$k_5 = \frac{5}{6} \sup |\O(\text{Rm})| < \infty$, $\gamma(H) \geq \hat{\varepsilon}_0 > 0$
• The conditions are hard to satisfy in general, but it has interesting applications when change ambient structures.

Q: Whether the existence of cpt min Lag (slag) is stable under the change of ambient KE metric?

Yes. For 2d. with c, < 0 (L-) local stability for smooth. 3d. with c, < 0 (lotay-Tommaso,)

(Collins-Jacob-Lin)

Y: del Pezzo surface or rational elliptic surface. D: smooth anti-canonical divisor. Then there exists a slag fibration on X = Y | D

The min. Lag (slag) is no longer min when change ambient, but with good bound & \( \sum H^2 \) small. Use LMCF to min Lag (slag)
Thank You!