Survey of Lagrangian mean curvature flow

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Mean curvature flow (MCF)

$X_t : \Sigma^n \longrightarrow (N^m, h)$ satisfy $\left(\frac{\partial X_t}{\partial t}\right)^\perp = H(X_t(x))$

- $H = g^{ij}\left(\nabla_{\frac{\partial X}{\partial x_i}} \frac{\partial X}{\partial x_j}\right)^\perp$, $g_{ij} = \langle \frac{\partial X}{\partial x_i}, \frac{\partial X}{\partial x_j} \rangle$
- Negative gradient flow of area functional
- short time existence (cpt case, special argument for non cpt)

- curve shortening flow

  Embedded closed plane curves shrink to round pts

  (Gage - Hamilton - Grayson)

- Hypersurface MCF

  Embeded convex closed hypersurface in $\mathbb{R}^n$ shrink to round pts

  (Huisken)

Lagrangian mean curvature flow (LMCF)

$\omega|_{L^n} = 0$

Stating from a smooth immersed Lag in KE

a smooth sol of MCF will preserve Lag condition

⇝ called LMCF

$\alpha_H =: \omega(H, \cdot)$

$d\alpha_H|_L = Ric|_L \overset{KE}{=} 0$

- ∵ H infinitesimal symplectic v.f.

  or by max principle Smoczyk

- Non cpt case neede to prove seperately.

⇝ a potential way to find SLag or min Lag

Thomas - Yau conjecture & its refinement by Joyce

LMCF is a MCF with high codim $L^n \subset N^{2n}$

- Difficulties for MCF with high codim

(a) Nonlinear parabolic system not one single eq (no avoid principle)

(b) H in normal bundle, estimates involved tensors not a scalar function (more complicated)

(c) No natural convex condition

- Good things for LMCF than general MCF with high codim

(a) with Lag nbh Thm ← Hamiltonian deformation can be written as the graph of df

→ Hope it resembles hypersurface MCF

(b) When in CY, we have Lag angle Θ

→ give additional information

Good example & testing case for MCF with high codim

Existence & Singularities of MCF & LMCF

- Short time existence for cpt initial data

  Noncpt case needs to be discussed case by case

  (Short time existence for ACLMCF by W.B. Su)

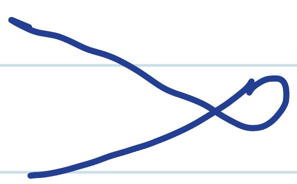
- Singularities occur in general (Suppose at time T)

  $\rightarrow$ $|A|^2$ blows up and at least of the rate $(T-t)^{-\frac{1}{2}}$

  if $|A|^2(x,t)(T-t) < C$. called type I singularity

  others are called type II singularity

  $\rightarrow$ big set of singularities can happen

   $\times \mathbb{R}^n$, collapsing $M \times S^k(\varepsilon)$. isoparametric submfd

  $\rightarrow$ Restrict to isolated singularities first

  simple & more generic

- Parabolic scaling (blow up)

$\sigma > 0$

If $L(X,t)$ satisfies MCF, then $\tilde{L}(x,s) = \sigma\left(L(x, t_0 + \frac{s}{\sigma^2}) - y_0\right)$ also satisfies MCF. (if not in $\mathbb{R}^n$, scale the metric instead)

If L defined in $[0,T)$, then $\tilde{L}$ defined in $[-t_0\sigma^2, (T-t_0)\sigma^2)$

① Type I blow up (Central pt blow up)

assume L has an isolated singularity at (y_0, T)

$\sigma_i \to \infty$. $\tilde{L}^i(x,s) = \sigma_i\left(L(x, T + \frac{s}{\sigma_i^2}) - y_0\right)$

$-\sigma_i^2 T \leq s < 0 \to$ ancient sol

$\exists$ a subseq.
$\ni \tilde{L}^i$ has Brakke flow limit $\tilde{L}^\infty$

Different subseq may have different limits

② Type II blow up (max pt blow up)

define $\tilde{L}^i(x,s) = \sigma_i \left(L(x, t_i + \frac{s}{\sigma_i^2}) - y_i \right)$, $\sigma_i \to \infty$

$y_i = L(x_i, t_i)$, $t_i \to T$ & $y_i \to y_0$

$-t_i \sigma_i^2 \le s < (T - t_i)\sigma_i^2$

$\nexists \ (T-t_i)\sigma_i^2 \to \infty$ ⇝ eternal sol

$\exists$ a subseq $\ni$ $\tilde{L}^i$ has a Brakke flow limit $\tilde{L}^\infty$

usually take $\sigma_i \sim \max \|A(x, t_i)\| = A(x_i, t_i)$

montonicity formula of Huisken

$\Rightarrow$ for central pt blow up, limit is a shrinker

entropy defined by back heat kernel

Monotonicity formula (Huisken)

- Back heat kernel at $(y_0, t_0) \in \mathbb{R}^k \times \mathbb{R}$

$$\Phi_{y_0,t_0}(y,t) = \frac{1}{\sqrt{4\pi(t_0-t)}^{\,n}} e^{-\frac{|y-y_0|^2}{4(t_0-t)}} \qquad t < t_0$$

- If L_t satisfies MCF, then

$$\frac{d}{dt}\int_{L_t} \Phi_{y_0,t_0}\, d\mu_t = -\int_{L_t} \Phi_{y_0,t_0} \left| H + \frac{(Y-y_0)^{\perp}}{2(t_0-t)} \right|^2 d\mu_t < 0$$

when L_t noncompact, assume all integrals finite values

Type I blow up $\quad \int_{L_t} \Phi_{y_0,T}\, d\mu_t = \int_{\tilde{L}^i_s} \Phi_{0,0}\, d\mu^{\sigma_i}_s \qquad t = T + \frac{s}{\sigma_i^2}$

$$\Rightarrow \int_{\tilde{L}^\infty_s} \Phi_{0,0}\, d\mu^\infty_s = \text{constant} \Rightarrow Y_s^{\perp} = 2sH. \qquad s<0$$

Soliton sols

① If submfd X in $\mathbb{R}^k$ satisfies $X^\perp = \alpha H(X)$, $\alpha \neq 0$. constant

then $X_t = \sqrt{\frac{2}{\alpha}t}\, X$ satisfies MCF

If $\alpha < 0$, X_t is defined for $t < 0$. self-shrinker an ancient sol

$$(X_t)^\perp = \sqrt{\frac{2}{\alpha}t}\, X^\perp = \sqrt{\frac{2}{\alpha}t}\, \alpha H(X) = \frac{2}{\alpha}t \cdot \alpha H(X_t) = 2t H(X_t)$$

type I blow up limits

If $\alpha > 0$. X_t is defined for $t > 0$ self-expander

· models for surgeries

· possible limit at $t = \infty$

② If X satisfies $H(X) = T^\perp$ for a constant vector

then $X_t = X + tT$ satisfies MCF (an eternal sol)

defined for all t, translating sol

related to type II blow up limits

RMK: For MCF in (N^m, h). blow up limit is also in $\mathbb{R}^n$. We can also imbed N to $\mathbb{R}^k$ isometrically & MCF becomes $\left(\frac{\partial X}{\partial t}\right)^\perp = H + E$ in $\mathbb{R}^k$

LMCF of graded Lag in CY has no type I singularity

(Wang, Chen-Li, Neves)

Key: No non-trivial smooth graded Lag shrinker

$-J\nabla\theta = H$

- Recall L Lag in CY. $\Omega|_L = e^{i\theta}\,\mathrm{Vol}_L$; $\theta : L \mapsto \mathbb{R}/2\pi$

  it is called graded if θ can be lifted to $\theta : L \to \mathbb{R}$.

- being graded is preserved under LMCF & $\dfrac{d\theta_t}{dt} = \Delta\theta_t$

- f_t, a smooth family of function on L_t. $\left(\dfrac{dL_t}{dt}\right)^{\perp} = H$

  Then $$\frac{d}{dt}\int_{L_t} f_t\, \Phi(y_0,T) = \int_{L_t} \left(\frac{df_t}{dt} - \Delta f_t\right) \Phi(y_0,T)\, d\mathcal{H}^n$$
  $$- \int_{L_t} f_t \left| H + \frac{(Y-y_0)^{\perp}}{2(T-t_0)} \right|^2 \Phi(y_0,T)\, d\mathcal{H}^n$$

  assume all quantities finite.

Proof: Assume L a smooth graded Lag shrinker $L^{\perp} = -2H$

then $L_t = \sqrt{-t}\, L$ a smooth sol to LMCF for $t<0$

Take $f_t = \theta_t^2 = \theta^2$ $\qquad \frac{d}{dt}\theta_t^2 = \Delta\theta_t^2 - 2|H|^2$

$$\Rightarrow 0 = \underbrace{\frac{d}{dt}\int_{L_t} \theta_t^2\, \Phi(y_0, T)\, d\mathcal{H}^n}_{\text{scaling inv} \Rightarrow \text{const}} = -\int 2|H|^2\, \Phi(y_0, T)\, d\mathcal{H}^n$$

$\Rightarrow H = 0 \Rightarrow L^{\perp} = 0 \quad \therefore$ a smooth minimal cone

$\Rightarrow L$ a plane. $\Rightarrow$ no type I singularity

By White's regularity Thm

Thm (Neves)

LMCF of graded Lag with $|\theta| < \infty$. When do central blow up at a singularity (it's a Type II singularity) $\Rightarrow \exists$ a rescaled subseq converges to union of integral sLag cones $L_1, \cdots L_N$ with multiplicity $m_1, \cdots, m_N$, Lag angles $\theta_1, \cdots \theta_N$

- The set $\{\theta_1, \dots, \theta_N\}$ does not depend on the seq of rescaling chosen
- when $n=2$. slag cones are planes

Thm (Neves)

If in the above thm, L_0 is almost calibrated ($\cos\theta > \varepsilon_0$) & rational, then any convergent subsequence of connected component (of $B_{4R}(0) \cap L_s^i$ intersecting $B_R(0)$) converges to a slag cone with multiplicity with Lag angle $\bar{\theta}$

a Lag L is called rational if
$\lambda(H_1(L, \mathbb{Z})) = \{a\, 2k\pi \mid k \in \mathbb{Z}\}$ λ: Liouville form

Convergence in the thms is: for every smooth cpt support func φ and every f in $C^2(\mathbb{R})$, and every $s<0$

$$\lim_{i\to\infty} \int_{L_s^i} f(\theta_{i,s})\,\phi \, d\mathcal{H}^n = \sum_{j=1}^{N} m_j f(\bar{\theta}_j)\,\mu_j(\phi)$$

- As the singularities for graded Lag in CY are type II, one may want to consider max point blow up instead.

  Is there any relation between central pt blow up limits (I) & max point blow up limits (II)?

  Will a blow-down of (II) arise as (I)? Hard to see and also limits may depend on subsequences

- Lambert – Lotay – Schulze can characterize ancient exact almost calibrated sols to LMCF with uniformly bounded area ratios and θ, when a blow down is a pair of multiplicity one transverse planes $P_1 \cup P_2$

With my collaborators (Wang, Joyce, Tsui) we construct many soliton sols (shrinkers, expanders, and translators) for LMCF with different properties and applications.

① Maslov class non-zero case

- Construct Lag. shrinkers, cones, expanders — different topologies (HS & non-HS)
- Can glue together to form an eternal sol (*) of Lag Brakke flow without mass loss

Related to Schoen-Wolfson's work mentioned in Mark's talk,

For any $C_{p,q}$ cone we can find $C_{p',q'}$. and resolve $C_{p,q} \cup C_{p',q'}$ as (*), and can also resolve a single $C_{p,q}$. for $p>1$. $q>1$

→ $C_{2,1}$ may be the only possibly tangent cone for 2d Lag minimizers' singularities

② Maslov class zero case

- Lag expanders, translating sols $\left(\begin{matrix}\text{top } S^{n-1}\times\mathbb{R}\\ \mathbb{R}^n\end{matrix}\right)$
- Lag angle can be arbitrarily small
- asymptotic to two planes

- For two Lag planes intersect at one pt, by changing coordinates can make

$$P_0 = \{(x_1, \ldots, x_n) \mid x_i \in \mathbb{R}\}$$

$$P_\varphi = \{(x_1 e^{i\varphi_1}, \ldots x_n e^{i\varphi_n}) \mid x_i \in \mathbb{R}\}$$

$\varphi = (\varphi_1, \ldots \varphi_n)$ fixed , $0 < \varphi_i < \pi$.

- $L_1 \cup L_\varphi$ vol minimizing iff $\sum \varphi_i \geq \pi$ (Nance, Lawlor, ...)

- When $\sum \varphi_i = \pi$. $\exists$ a sLag called Lawlor neck L asymptotic to $P_0 \cup -P_\varphi$ of explicit form

$-P_\varphi$: reverse orientation

$\{ (z_1(s) x_1, \cdots z_n(s) x_n) \mid \sum x_i^2 = 1, \quad z_i(s) : \mathbb{R} \to \mathbb{C} \setminus \{0\} \}$

- For the non-vol-minimizing case $\sum \varphi_i < \pi$.

We find a family of Lag expander asymptotic to $P_0 \cup -P_\varphi$

(Joyce - L - Tsui, JLT-expander, θ can be arbitraily small)

- Lawlor neck can be used to desingularize intersection pt. (Butscher, Joyce, L-, ...)

- JLT-expanders can be used to resolve singularity of LMCF. (Begley - Moore)

- smooth exact sLag asymptotic to $P_0 \cup -P_\varphi$ are unique up to scaling for $n \geq 3$. If embedded, it must be Lawlor neck, if immersed $P_0 \cup -P_\varphi$ Imagi – Joyce – Dos Santos

Rmk: For works on smoothing sLags with intersecting points mentioned above, one needs to require angle criterion $\Sigma \varphi_i = \pi$. This uniqueness result shows that there is no sLag smoothing if without angle criterion.

- Lag expander asymptotic to $P_0 \cup -P_\varphi$ must be JLT (Lotay – Neves, $n=2$, Imagi – Joyce – Dos Santos, $n \geq 3$)
- These uniqueness results are used essentially in the works of Begley – Moore and Lambert – Lotay – Schulze

Dynamical stability of MCF

ⓐ For the graph of maps between cpt mfds M & N, suitable conditions on the curvature of M & N, and the singular value of initial map $\to$ Long time existence of MCF & convergence (to a trivial map) Wang, Tsui. ----

ⓑ Initial submfd in a C^1 neighborhood of a cpt smooth min submfd Σ with strong stability ($\mathcal{R} - A > 0$)

(Tsai – Wang)

$\to$ long time existence & converges to Σ

uniqueness of minimal submfd in that Nbh

$\to$ Extended to enhanced Brakke flow and have the uniquess among stationary varifolds (Lotay – Schulze)

(c) Noncompact case

long time existence & convergence to central mfd for equiv. almost-calibrated AC LMCF with suitable condition (Su)

How about if we do not have a minimal (Lag) submfd beforehand?

HZ. Li

L cpt Lag (immersed) in complete KE. For any $V_0, \Lambda_0 > 0$, there exists ε_0, such that if $\mathrm{Vol}(L) \leq V_0$, $|A| \leq \Lambda_0$, $\int_L |H|^2 \leq \varepsilon_0$

$\Rightarrow$ LMCF of L will converge exponentially to a min Lag

if (a) when the ambient $\tilde{R} < 0$

(b) $\tilde{R} \geq 0$, and $\lambda_1(L) \geq \frac{\bar{R}}{2n} + \delta_0$ for given δ_0

A: 2nd. f.f., λ_1: the first eigenvalue, $\varepsilon_0 = \varepsilon_0(V_0, \Lambda_0, \delta, \tilde{R}, K_5, i_0)$

$K_5 = \sum_{i=0}^{5} \sup |\bar{\nabla}^i \overline{Rm}| < \infty$, $\mathrm{inj}(M) \geq i_0 > 0$

- The conditions are hard to satisfy in general, but it has interesting applications when change ambient structures.

Q: Whether the existence of cpt min Lag (SLag) is stable under the change of ambient KE metric?

Yes. for 2d. with $c_1 < 0$ (L–)

local stability for smooth. Rd with $c_1 < 0$ (Lotay – Tommaso, or from above)

(Collins – Jacob – Lin)

Y: del Pezzo surface or rational elliptic surface. D: smooth anti-canonical divisor. Then there exists a SLag fibration on $X = Y \setminus D$

The min. Lag (SLag) is no longer min when change ambient. but with good bound & $\int |H|^2$ small. Use LMCF to min Lag (SLag)

Thank You !