Complete noncompact metrics of special and exceptional holonomy: the first 40 years.

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11 September 2018
SCGP weekly talk

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Parallel transport and holonomy.

Geodesics in Euclidean space: the “straightest” paths between two points.

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e.g. $G = U(m)$ $\leftrightsquigarrow$ parallel complex structure J and 2-form ω
$G = SU(m)$ $\leftrightsquigarrow$ in addition a parallel holomorphic volume form Ω
$G = G_2$ $\leftrightsquigarrow$ parallel 3-form and 4-form of special type

What is special holonomy?

Possible holonomy groups of Riemannian manifolds[1] are extremely limited:

- 5 possible infinite families and
- 2 exceptional cases, the Lie groups G_2 and $Spin_7$ (in dims 7 and 8)

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- For the two exceptional holonomy cases we can no longer reduce to a scalar equation as in the $SU(n)$ case. The best one can do involves systems of nonlinear first-order PDEs. Many questions still remain open.

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Why special holonomy?

Special holonomy manifolds have special curvature properties:

- They are always Einstein metrics, $Ric(g) = \lambda g$. Except in one of the infinite families, in fact $\lambda = 0$, i.e. they are Ricci-flat metrics.
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- Exceptional holonomy spaces arise in M-theory as the simplest condition to guarantee *supersymmetric* compactifications from 11 to 4 dimensions. Here a characterisation in terms of *parallel spinors* is central.

The octonions, 3-forms and G_2

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Define a *vector cross-product* on $\mathbb{R}^7 = \mathrm{Im}(\mathbb{O})$

$$u \times v = \mathrm{Im}(uv)$$

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$$\varphi_0(u, v, w) := \langle u \times v, w \rangle = \langle uv, w \rangle$$

φ_0 is a generic 3-form so in fact

$$G_2 = Aut\, \mathbb{O} = \{A \in \mathrm{GL}(7, \mathbb{R}) |\ A^*\varphi_0 = \varphi_0\} \subset \mathrm{SO}(7).$$

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Engel first suggested the second viewpoint in 1886; details not worked out until 1907 PhD thesis of Reichel.

1st-order PDE system for G_2 holonomy metrics

- A G_2-structure on an oriented 7-manifold M is a choice of smoothly varying pointwise isomorphism between T_pM and $(\mathbb{R}^7, \varphi_0)$ $\forall p \in M$.

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Theorem: Let (M, φ, g_φ) be a G_2-structure; the following are equivalent

1. $\mathrm{Hol}(g_\varphi) \subset G_2$ and φ is the induced 3-form
2. $d\varphi = d^*\varphi = 0$, where d^* is defined using Hodge star $*$ w.r.t. g_φ.

Call such a G_2-structure a torsion-free G_2 structure.

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By writing equation for 3-form φ (not metric g directly) and allowing $\mathrm{Hol}(g_\varphi) \subset G_2$ we obtain *differential* (not integro-differential) equations.

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2 is a 1st-order system of 49 equations on the 35 coefficients of φ!

Why focus on noncompact complete examples?

- Some useful extra flexibility vs the compact case.
- Possible ingredients in constructions of compact examples or understanding how families of metrics on compact manifolds degenerate.
- Long history and a variety of construction methods available.
- History of fruitful interactions between mathematics and physics.
- Significant recent advances to report, but many questions still open.

In the cases of complete hyperKähler metrics and Calabi–Yau metrics we now have a wealth of examples and a variety of powerful construction techniques. Nevertheless there have been significant recent developments and open questions remain.

For the two exceptional holonomy cases until very recently we had very few complete noncompact examples and no very general construction methods.

The recent progress in the exceptional cases relies heavily on advances in the special holonomy cases.

Noncompact Calabi–Yau, hyperKähler metrics

1. Continuous symmetries possible (impossible in the compact case)
   - Cohomogeneity one solutions: the isometry group has codimension 1 orbits. Permits the reduction of PDEs to ODEs, which in simple cases are explicitly solvable. Specific instances: Taub–NUT metric on $\mathbb{C}^2$, Eguchi–Hanson metric on $T^*\mathbb{P}^1$, Atiyah–Hitchin metric, Calabi's metric on $\widehat{\mathbb{C}^n}/\mathbb{Z}_n$.
   - The Gibbons–Hawking ansatz (1978): reduces problem of finding circle invariant hK 4-metrics respecting the hK structure to finding positive harmonic functions. Extremely important: Will return to this later. Specific instances: the multi-centre Eguchi–Hanson and Taub–NUT metrics.

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2. Twistor theory methods, e.g. Kronheimer's construction and classification of all ALE gravitational instantons.
3. Finite and infinite dimensional hyperKähler reduction (Hitchin, Karlhede, Lindstrom, Rocek), e.g. implies that many moduli spaces in gauge theory admit interesting hyperKähler metrics: monopole moduli spaces.

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4. Adaptations of Yau's complex Monge-Ampere methods to the noncompact setting, e.g. Tian–Yau, Hein, H–Hein–Nordstrom.
5. Gluing methods, e.g. Biquard–Minerbe's noncompact variants of Kummer construction.

Complete noncompact G_2–manifolds circa 2000

- As of 2000 only 3 complete noncompact smooth G_2–holonomy metrics on simply connected manifolds were known:

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- Common features:
  - All are total spaces of rank 4 or 3 vector bundles over Einstein 4 or 3-manifolds.
  - Metrics have a lot of symmetries: isometry group acts with codimension one orbits. Symmetry allows reduction of the torsion-free PDEs to (an explicitly solvable) system of ODEs. Each metric is rigid (up to scale).
  - Similar asymptotic geometry: volume of large balls of radius r grows like r^7. Each metric is asymptotic to a cone with smooth cross-section: $S^3 \times S^3$, $\mathbb{P}^3$ or $F_{1,2} = SU(3)/T^2$.

    Structures on cross-sections are homogeneous *nearly Kähler structures* – a special type of $SU(3)$ structure generalising the classical almost complex structure on S^6 induced by octonionic multiplication.

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    Structures on cross-sections are homogeneous *nearly Kähler structures* – a special type of $SU(3)$ structure generalising the classical almost complex structure on S^6 induced by octonionic multiplication.
  - Scaling down provides possible local models for families of smooth compact G_2-manifolds to develop isolated conical singularities.
    (Rigidity suggests such singularities could occur in 1-dimensional families).

Possible ways to generalise

- Try to deform the existing examples.
- Look for other highly symmetric (cohomogeneity one) G_2–metrics.
- Look for other asymptotically conical G_2–metrics.

  Look for other nearly Kähler structures.

  A long-standing question.

  Do inhomogeneous nearly Kähler structures on simply connected 6-manifolds exist?
- Look for other metrics via bundle constructions.

Except for the last point there has been some significant recent progress along all these lines.

Some negative results

- The standard nK structures on S^6, $S^3 \times S^3$, $F_{1,2}$ and CP^3 are the only *homogeneous* nK structures. (Butruille)
- The standard nK structures on S^6, $S^3 \times S^3$ and CP^3 are *infinitesimally rigid* as nK spaces (Moroianu–Semmelmann). So no possibility to deform away from the homogeneous structures to an inhomogeneous one.
- The standard nK structure on $F_{1,2}$ does have infinitesimal deformations. They are all *obstructed at second-order* though. (Foscolo)
- The only complete G_2–metrics asymptotic to the standard homogeneous nK structures are the Bryant–Salamon ones. (Karigiannis–Lotay)
- The $\Lambda^+_2 M^4$ bundle construction works for any compact Einstein self-dual 4-manifold M^4 with positive scalar curvature. However, Hitchin proved the only such manifolds are $\mathbb{S}^4$ and $\mathbb{P}^2$.
  There are however many Einstein self-dual 4-*orbifolds* (e.g. Galicki, Lawson, Hitchin). These can give rise to G_2 cones where the 6-dim nK cross-section is itself singular.

New nearly Kähler 6-manifolds

Theorem 1: (Foscolo–H, Annals 2017).

Both S^6 and $S^3 \times S^3$ admit complete inhomogeneous nearly Kähler structures. Both these new inhomogeneous structures admit a cohomogeneity one action of $\mathrm{SU}(2) \times \mathrm{SU}(2)$.

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Conjecturally, these 2 new structures are the unique cohomogeneity one simply connected nearly Kähler structures.

Open questions.
1. Find complete asymptotically conical G_2–metrics asymptotic to these two new nearly Kähler structures.
2. Understand the physical significance (if any) of these two new nearly Kähler structures.

I won't discuss these two examples further here since we don't yet get new complete G_2–metrics from them. We discovered them here at SCGP though!

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Some concrete developments in the noncompact setting.

- In 2001 Brandhuber–Gomis–Gubser–Gukov constructed a new explicit complete G_2–metric on $\$S^3$ and suggested that their example should be a member of a 1-parameter family of such complete G_2–metrics.
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- Numerical studies by Brandhuber and Cvetic–Gibbons–Lu–Pope supported that belief, but explicit solutions or existence proofs were lacking.
- The BGGG example has cohomogeneity one, but a smaller symmetry group than that of the Bryant–Salamon metric on $\$S^3$: $SU(2)^2 \times U(1)$ versus $SU(2)^3$.
- The asymptotic geometry of the BGGG example is also different from the Bryant–Salamon metric. Its volume growth is r^6 not r^7. CGLP coined the term *ALC (asymptotically locally conical)* to describe its asymptotic geometry (generalising ALF spaces like the Taub–NUT metric.)

Input from physics II

- Further work by CGLP, and Hori–Hosomichi–Page–Rabadan–Walcher (2005) suggested there should be four 1-parameter families of complete cohomogeneity one ALC G_2–metrics: $\mathbb{A}_7$, $\mathbb{B}_7$, $\mathbb{C}_7$, $\mathbb{D}_7$.
- BGGG example belongs to the $\mathbb{B}_7$ family. In 2013 Bogoyavlenskaya proved existence of the whole 1-parameter family by qualitative ODE methods.
- No existence proof was available for any members of the $\mathbb{A}_7$, $\mathbb{C}_7$ and $\mathbb{D}_7$ families. Challenge is to control which solutions of ODE system give rise to complete solutions, even though explicit general solutions are lacking.

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- In physics the $\mathbb{A}_7$ and $\mathbb{B}_7$ families were viewed as deriving from certain 4-dimensional ALF hyperKähler manifolds: the Atiyah–Hitchin metrics and Taub–NUT metrics respectively. The meaning of this was not clear (to mathematicians!)
- It is natural/important to understand the behaviour of these metrics at the two extremes of the possible range of parameters. Here a simpler hyperKähler anology is illustrative.

The Gibbons–Hawking ansatz

The **Gibbons–Hawking Ansatz** (1978): local form of hyperkähler metrics in dimension 4 with a *triholomorphic* circle action

- h **positive harmonic function** on $U \subset \mathbb{R}^3$
- $M \to U$ principal U(1)–bundle and connection θ with $d\theta = *dh$

$$g = h\, g_{\mathbb{R}^3} + h^{-1}\theta^2 \text{ is a hyperkähler metric on } M$$

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Example: ALF and ALE metrics of cyclic type

$$g_m = \left(m + \sum_{i=1}^{n} \frac{1}{2|x - a_i|} \right) dx \cdot dx + \left(m + \sum_{i=1}^{n} \frac{1}{2|x - a_i|} \right)^{-1} \theta^2$$

- $a_1, \ldots, a_n$ distinct $\Longrightarrow$ complete metric
  $a_1 = \cdots = a_{k+1} \Longrightarrow$ orbifold singularity $\mathbb{C}^2/\mathbb{Z}_k$
- m is called the mass
  - $m > 0 \Longrightarrow$ **ALF** (= ALC with flat asymptotic cone)
  - $m = 0 \Longrightarrow$ **ALE** (= AC with flat asymptotic cone)

Limits of ALF geometries

We can see three different limits

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- $m \to 0$: convergence to **orbifold ALF**
- **orbifold ALF + ALE $\rightsquigarrow$ smooth ALF**

G_2 analogue of the Gibbons–Hawking ansatz

- There exists a G_2 analogue of the Gibbons–Hawking ansatz for circle-invariant torsion-free G_2–structures, the *Apostolov–Salamon equations* (also written down by Kaste–Minasian–Petrini–Tomasiello).
- These equations are a *system of coupled nonlinear* equations for a (generally **nonintegrable**) SU(3)–structure on a 6-manifold B and a monopole (h, θ) on a circle bundle over B.
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- Constructing complete solutions to the Apostolov–Salamon equations directly is difficult.
- Motivated by the first limit above ($m \to \infty$) consider a collapsing family of G_2–metrics with circle fibres becoming everywhere of order ϵ.
- Formally in the limit as $\epsilon \to 0$ we get an SU(3)–structure on B that is now torsion-free, Calabi–Yau.
- We can *linearise* the Apostolov–Salamon equations about the collapsed limit: we get the *Calabi–Yau monopole equation* on B^6, together with a coupled linear equation for a stable 3-form ρ — this describes the first-order correction to the initial torsion-free SU(3) structure
- We can deal with these linearised equations analytically.

ALC G_2 metrics via AC CY metrics

Developing this idea leads to a new general analytic method to construct ALC G_2 manifolds close to the collapsed Calabi–Yau limit

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Theorem 2: Foscolo–H–Nordström arXiv:1709.04904

Let $(B, g_0, \omega_0, \Omega_0)$ be an **asymptotically conical Calabi–Yau** 3**–fold** asymptotic to a Calabi–Yau cone (C, g_C) and let $M \to B$ be a **principal circle bundle**.

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Assume that $c_1(M) \neq 0$ but $c_1(M) \cup [\omega_0] = 0 \in H^4(B)$.

Then for every $\epsilon > 0$ sufficiently small there exists an $\mathbb{S}^1$–invariant G_2–holonomy metric g_ϵ on M with the following properties

- (M, g_ϵ) is an ALC manifold: as $r \to \infty$, $g_\epsilon = g_C + \epsilon^2 \theta_\infty^2 + O(r^{-\nu})$.
- (M, g_ϵ) collapses to (B, g_0) with bounded curvature as $\epsilon \to 0$.

Initial comments on Theorem 2

- Noncompact complete examples of manifolds with special holonomy that collapse with globally bounded curvature are a **new higher-dimensional phenomenon**: the only hyperKähler 4-manifold with a triholomorphic circle action without fixed points is $\mathbb{R}^3 \times \mathbb{S}^1$.

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- Noncompact complete examples of manifolds with special holonomy that collapse with globally bounded curvature are a **new higher-dimensional phenomenon**: the only hyperKähler 4-manifold with a triholomorphic circle action without fixed points is $\mathbb{R}^3 \times \mathbb{S}^1$.
- **Connections to physics**: Type IIA String theory compactified on AC CY 3-fold (B, ω_0, Ω_0) with Ramond–Ramond 2-form flux $d\theta$ satisfying $[d\theta] \cup [\omega_0] = 0$ and no D6-branes as the weak-coupling limit of M-theory compactified on an ALC G_2–manifold.

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- The construction leads to infinitely many new noncompact complete G_2–manifolds, including infinitely many diffeomorphism types. We can construct examples with only a circle symmetry (because we know AC Calabi–Yau metrics without continuous symmetries).
- If we specialise to the cohomogeneity one case then we obtain members of the $\mathbb{C}_7$ and $\mathbb{D}_7$ families by taking the base AC CY 3-fold to be $K_{\mathbb{P}^1 \times \mathbb{P}^1}$ or the small resolution of the conifold respectively.

Further developments

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- The $\mathbb{A}_7$ family collapses to a quotient of the Stenzel metric by an antiholomorphic involution ι. Now there is no global isometric circle action. Again the curvature collapse is bounded except close to the quotient of the zero section.

  The appropriate adiabatic limit in this case is a family of Atiyah–Hitchin metrics over S^3. (Atiyah–Hitchin also does not have a global (triholomorphic) circle action, though asymptotically it is described up to an exponentially decaying error by the Gibbons–Hawking ansatz.)

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Physically these two situations correspond to the introduction of a $D6$-**brane** or an **orientifold** respectively.

Uncollapsing the collapsed limits

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In general this is currently intractable, but if the Calabi–Yau base is *cohomogeneity one* we can answer this using cohomogeneity one methods.

In the $\mathbb{B}_7$ **family** one can prove the whole 1-parameter family of ALC solutions exists and the large circle limit is the original (asymptotically conical) Bryant–Salamon metric on $\$S^3$.

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The most interesting case is $B = K_{CP^1 \times CP^1}$: it has a 2-dimensional Kähler cone, so we have many circle bundles that lead to simply connected 7-manifolds. We were therefore led to conjecture the following:

There exists an infinite family of new complete asymptotically conical G_2–metrics $M_{m,n}$. The cross-section of the asymptotic cone of $M_{m,n}$ is the quotient of the standard nK structure on $S^3 \times S^3$ by a freely acting cyclic subgroup $\mathbb{Z}_{2(m+n)}$ of the group of isometries of $S^3 \times S^3$.

AC metrics and conically singular ALC manifolds

Theorem 3 (Foscolo–H–Nordström, arixv:1805.02612)

- For every pair of coprime positive integers m, n there exists a **complete AC** G_2–metric (unique up to scale) on the (simply connected) total space $M_{m,n}$ of the circle bundle over $K_{\mathbb{P}^1\times\mathbb{P}^1}$ with first Chern class $(m, -n)$.
- $M_{m,n}$ is asymptotic to the cone over $S^3 \times S^3/\mathbb{Z}_{2(m+n)}$.
- There is a 1-parameter family of ALC G_2–metrics on $M_{m,n}$ that collapses to a Calabi–Yau metric on $K_{CP^1\times CP^1}$ at one extreme and "opens up" at the other extreme to the unique AC G_2–metric on $M_{m,n}$.

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Theorem 4 (Foscolo–H–Nordström, arixv:1805.02612)

- There exists a (unique up to scale) G_2–metric g_0 on $M_0 = (0, \infty) \times S^3 \times S^3$ such that
- (M, g_0) has an **isolated conical singularity** modelled on the G_2–cone over the homogeneous nearly Kähler structure over $S^3 \times S^3$;
- (M, g_0) has a **complete ALC end**.

(M_0, g_0) can be also be desingularised by analytic methods using the AC metrics. This gives an analytic construction of smooth ALC metrics that degenerate to (M_0, g_0).

The Apostolov–Salamon equations

- **Apostolov–Salamon** (2004): Any circle-invariant G_2 structure φ on $M^7 \to B^6$ a principal circle bundle can be written as

$$\varphi = \theta \wedge \omega + h^{\frac{3}{4}} \mathrm{Re}\Omega,$$

where (ω, Ω) is an SU(3)–structure on B, h is a positive function on B and θ is a connection 1-form. The induced metric is

$$g = \sqrt{h}\, g_M + h^{-1}\theta^2$$

and the 4-form is given by

$$*_\varphi \varphi = -h^{-1/4}\theta \wedge \mathrm{Im}\Omega + \frac{1}{2} h\, \omega^2.$$

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  φ is an $\mathbb{S}^1$-invariant torsion-free G_2-structure iff

  $$d\omega = 0, \qquad d\left(h^{\frac{3}{4}} \mathrm{Re}\Omega\right) = -d\theta \wedge \omega,$$
  $$d\left(h^{\frac{1}{4}} \mathrm{Im}\Omega\right) = 0, \qquad \tfrac{1}{2} dh \wedge \omega^2 = h^{\frac{1}{4}} d\theta \wedge \mathrm{Im}\Omega.$$

Collapsing the circle fibres

Let φ_ϵ be a family of S^1–invariant torsion-free G_2–structures on $M \to B$ with circle fibres shrinking to zero length as $\epsilon \to 0$. By rescaling along the fibres we write

$$\varphi_\epsilon = \epsilon\, \theta_\epsilon \wedge \omega_\epsilon + (h_\epsilon)^{\frac{3}{4}} \mathrm{Re}\Omega_\epsilon.$$

with induced metric

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where g_ϵ is the metric on B induced by $(\omega_\epsilon, \Omega_\epsilon)$.
The previous Apostolov–Salamon system is equivalent to

$$d\omega_\epsilon = 0, \qquad \tfrac{1}{2} dh_\epsilon \wedge \omega_\epsilon^2 = \epsilon\, (h_\epsilon)^{\frac{1}{4}} d\theta_\epsilon \wedge \mathrm{Im}\Omega_\epsilon, \qquad d\theta_\epsilon \wedge \omega_\epsilon^2 = 0,$$
$$d\mathrm{Re}\Omega_\epsilon = -\tfrac{3}{4} h_\epsilon^{-1} dh_\epsilon \wedge \mathrm{Re}\Omega_\epsilon - \epsilon (h_\epsilon)^{-\frac{3}{4}} d\theta_\epsilon \wedge \omega_\epsilon,$$
$$d\mathrm{Im}\Omega_\epsilon = -\tfrac{1}{4} h_\epsilon^{-1} dh_\epsilon \wedge \mathrm{Im}\Omega_\epsilon.$$

In the formal limit where $\epsilon \to 0$ the second equation implies $h_0 = \lim h_\epsilon$ is constant. Wlog $h_0 = 1$ and then (ω_0, Ω_0) is Calabi–Yau, i.e. $d\omega_0$ and $d\Omega_0$ both vanish.

The adiabatic limit of Apostolov–Salamon eqns

- *Linearised* equations over the collapsed limit: a **CY monopole** on B

$$*dh = d\theta \wedge \mathrm{Re}\,\Omega_0, \qquad d\theta \wedge \omega_0^2 = 0$$

together with a stable 3–form ρ satisfying a coupled linear equation

$$d\rho = -\frac{3}{4} dh \wedge \mathrm{Re}\Omega_0 - d\theta \wedge \omega_0, \qquad d\hat{\rho} = -\tfrac{1}{4} dh \wedge \mathrm{Im}\,\Omega_0.$$

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A solution of these equations yields a 1-parameter family of highly collapsed **closed** ALC G_2 structures

$$\varphi_\epsilon^{(1)} = \epsilon\,\theta \wedge \omega_0 + \mathrm{Re}\Omega_0 + \epsilon\rho$$

with torsion of order $O(\epsilon^2)$.

Proof strategy for Theorem 2

- Understand how to solve the linearised Apostolov–Salamon equations on AC manifold B in weighted Holder spaces for appropriate choice of weights.
- Understand appropriate gauge-fixing conditions to apply.
- Construct successive higher–order approximations $\varphi_\epsilon^{(k)}$ to torsion-free structure with torsion of order $O(\epsilon^{k+1})$.

  This requires a full understanding of the mapping properties of the linearisation of the Apostolov–Salamon equations.
- Construct a formal power series solution to the Apostolov–Salamon equations.
- Prove convergence of this formal power series solution for ϵ sufficiently small.

Current and Future Research

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Patch these 3 models together to obtain highly collapsed almost G_2 holonomy metrics; perturb to exact G_2 holonomy metrics. All 3 regions needed to avoid constructing a metric with a nontrivial Killing field.

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To construct highly collapsed **compact** G_2 holonomy spaces new issues arise: we need to consider 3 different types of region; each brings its own analytic and geometric character.

1. (The bulk) Construct models for sufficiently collapsed solutions on regions where collapse is occurring with bounded curvature.
2. (Geometry close to a smooth compact *D-brane*). Construct models for where collapse occurs in a circle-invariant way, but with unbounded curvature on certain compact codimension 4 submanifolds.
3. (Geometry close to a smooth compact *orientifold*). Construct models for where collapse with unbounded curvature occurs in codimension 4, but in a NON circle-invariant way.

Patch these 3 models together to obtain highly collapsed almost G_2 holonomy metrics; perturb to exact G_2 holonomy metrics. All 3 regions needed to avoid constructing a metric with a nontrivial Killing field.

Extend to the compact *singular* setting, using *singular ALC G_2 holonomy metrics* that we construct by a different method.