Generating equidistant points

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Fubini-Study distance

Complex projective space $\mathbb{CP}^{d-1}$ is a compact Kähler manifold. Its Riemannian metric g arises from the standard Hermitian form

$$\langle \mathbf{w} | \mathbf{z} \rangle = \sum_{i=0}^{d-1} \overline{w}_i z_i,$$

on $\mathbb{C}^d$, which is invariant by the unitary group $U(d)$.

The associated distance δ satisfies

$$\cos^2\left(\tfrac{1}{2}\delta([\mathbf{w}],[\mathbf{z}])\right) = \frac{|\langle \mathbf{w} | \mathbf{z} \rangle|^2}{\|\mathbf{w}\|^2\|\mathbf{z}\|^2} = \frac{\langle \mathbf{w} | \mathbf{z} \rangle \langle \mathbf{z} | \mathbf{w} \rangle}{\langle \mathbf{w} | \mathbf{w} \rangle \langle \mathbf{z} | \mathbf{z} \rangle}.$$

The RHS is a cross ratio of four points $[\mathbf{w}], [\mathbf{z}], [\mathbf{w}'], [\mathbf{z}']$, and represents a *transition probability*.

Wigner's theorem (J)

Theorem. Any isometry ϕ of $\mathbb{CP}^{d-1}$ arises from a unitary or conjugate unitary transformation of $\mathbb{C}^d$.

Proof [Freed]. Consider the effect of ϕ on the acs J of $\mathbb{CP}^{d-1}$. The Levi-Civita connection has holonomy $U(d-1)$, so ∇J and $\nabla(\phi^* J)$ both vanish and $R(\phi^* J) = \nabla_1 \nabla(\phi^* J) = 0$.

It follows from the invariant nature of $R \in S^2\mathfrak{h}$ that $\phi^* J = \pm J$ and ϕ is holomorphic or anti-holomorphic. In the former case, it lifts to (the transpose of) the linear map

$$\tilde{\phi}\colon H^0(\mathbb{CP}^{d-1}, L) \longrightarrow H^0(\phi^* L) \cong H^0(L),$$

where $L = \mathcal{O}(1)$. Since ϕ is an isometry, $\tilde{\phi}$ is unitary. $\square$

Moment mapping (ω)

The 2-form $\omega_0 = i\sum dz_i \wedge d\overline{z}_i$ on $\mathbb{C}^d$ induces a symplectic form on $\mathbb{CP}^{d-1}$, regarded as a symplectic quotient of $\mathbb{C}^d$ by $U(1)$. Let

$$\mathscr{D} = \{A \in \mathbb{C}^{d,d} : A = A^*,\ \operatorname{tr} A = 1\} \ \simeq\ \mathfrak{su}(d).$$

The mapping $S^{2d-1} \to \mathscr{D}$ for which

$$\mathbf{z} \longmapsto \mathbf{z}^*\mathbf{z} = \begin{pmatrix} |z_0|^2 & \overline{z}_0 z_1 & \overline{z}_0 z_2 & \cdots \\ \overline{z}_1 z_0 & |z_1|^2 & \overline{z}_1 z_2 & \cdots \\ \overline{z}_2 z_0 & \overline{z}_2 z_1 & |z_2|^2 & \cdots \\ \cdots & \cdots & \cdots & \end{pmatrix}$$

is $U(d)$-equivariant and defines an isometric embedding

$$\mu\colon\ \mathbb{CP}^{d-1} \hookrightarrow \mathscr{D} \simeq \mathbb{R}^{d^2-1}.$$

A point of $\mathscr{D}$ represents a *mixed* quantum state.

SIC-POVM

is shorthand for *Symmetric Informationally Complete Positive Operator Valued Measure*, here it is a finite one.

Definition 1. A SIC-POVM is a subset $\{P_\alpha\}$ of d^2 rank-one projections in $\mathscr{D}$ such that

$$\sum_{\alpha=1}^{d^2} P_\alpha = d\,\mathbf{1}, \qquad \mathrm{tr}(P_\alpha P_\beta) = \begin{cases} 1 & \alpha = \beta \\ \lambda & \alpha \neq \beta. \end{cases}$$

Equivalently, by setting $P_\alpha = \mathbf{z}_\alpha^* \mathbf{z}_\alpha$ (with $\|\mathbf{z}_\alpha\| = 1$), it is a set of d^2 points $[\mathbf{z}_\alpha]$ in $\mathbb{CP}^{d-1}$ such that

$$\big|\,\langle \mathbf{z}_\alpha \,|\, \mathbf{z}_\beta \rangle\,\big|^2 = \lambda, \quad \alpha \neq \beta, \quad \text{for some fixed } \lambda \in (0,1).$$

The points represent "equiangular lines" in $\mathbb{C}^d$.

Since $P_\alpha = \mu([\mathbf{z}_\alpha])$, we have

Definition 2. A SIC consists of a regular simplex in $\mathfrak{su}(d)$ whose d^2 vertices lie in the orbit $\mathbb{CP}^{d-1}$.

Lemma. d^2 is the maximum number of mutually equidistant points possible in $\mathbb{CP}^{d-1}$, and in this case λ must equal $\frac{1}{d+1}$.

Proof. Given an equidistant set $\{P_\alpha\}$, set $Q_\beta = P_\beta - \lambda \mathbf{1}$. Then

$$\operatorname{tr}(P_\alpha Q_\beta) = \begin{cases} 1-\lambda & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases}$$

So $\{P_\alpha\}$ is linearly independent in $i\,\mathfrak{u}(d)$.
Applying $\operatorname{tr}(Q_\beta \bullet)$ to $\mathbf{1} = \sum c_\alpha P_\alpha$, gives $1 - \lambda d = c_\alpha(1-\lambda)\ \forall \alpha$.
Then $d = d^2 c_\alpha$, and so $d(1-\lambda d) = 1-\lambda$. $\square$

Existence

Following pioneering work of Zauner [PhD 1999], SIC-POVM's are known to exist for all $d \leqslant 21$ and in a few higher dimensions. There is strong numerical evidence for their existence $\forall\, d \leqslant 151$.

There are many papers [since 2004] on the subject by Appleby, Bengtsson, Caves, Flammia, Grassl, Renes, Scott, Zhu, and others.

Appleby-Flammia-McConnell-Yard [in 2016] establish links with number theory. A preprint [2018] by Kopp advances this.

However, all known SIC sets are orbits of a finite group. With one exception, that group is conjugate to the Heisenberg (WH or GP) group that acts on $\mathbb{CP}^{d-1}$ as $(\mathbb{Z}/d\mathbb{Z})^2$.

Conjecture. For every d, one can always find $\mathbf{z} \in \mathbb{C}^d$ whose orbit $WH \cdot [\mathbf{z}]$ is a SIC set in $\mathbb{CP}^{d-1}$. Such a $\mathbf{z}$ is called a *fiducial vector*.

Clifford-Weyl-Heisenberg

Fix $d \geqslant 3$. Set $\omega = e^{2\pi i/d}$. Consider the cyclic groups W and H generated by the 'shift' and 'clock' isometries

$$\begin{array}{ccc} [z_0, z_1, \dots, z_{d-1}] & \xrightarrow{\;h\;} & [z_0, \omega z_1 \dots, \omega^{d-1} z_{d-1}] \\ w \big\downarrow & & \\ [z_{d-1}, z_0, \dots, z_{d-2}] & & \end{array}$$

which commute on $\mathbb{CP}^{d-1}$. The normalizer $C(d)$ of $W \times H$ in $U(d)$ is the *Clifford group*. There is a representation

$$SL(2, \mathbb{Z}_d) \ltimes (\mathbb{Z}_d)^2 \rightsquigarrow C(d)/U(1),$$

which can be extended to include complex conjugation. It is an isomorphism if d is odd.

Example. If $d = 5$ the order is $15000/5$ [Horrocks-Mumford].

Low dimensions

Case $d = 2$. A SIC set is a regular tetrahedron inscribed in $S^2 = \mathbb{CP}^1$ and $SO(3) = SU(2)/\mathbb{Z}_2$. An example is

$$[1+r, 1+i], \quad [1+r, -1-i], \quad [1+i, 1+r], \quad [1+i, -1-r]$$

where $r = \sqrt{3}$, not the simplest.

Case $d = 3$. An example is

$$[0, 1, -1],\ [0, 1, -\omega],\ [0, 1, -\omega^2],\ \text{and cyclic permutations,}$$

where $\omega = e^{2\pi i/3}$. Easy to extend to a one-parameter family.

Case $d = 4$. Let $s = \sqrt{2}$ and $t = \sqrt{2+\sqrt{5}}$. A fiducial vector is

$$\Big[-t - i(s+t),\ 1 - s + i,\ t + i(t-s),\ 1 + s + i\Big].$$

NB. No SIC set in $\mathbb{CP}^3$ can project to only 4 points in $\mathbb{HP}^1 = S^4$.

Maximal torus

Having fixed d and coordinates on $\mathbb{C}^d$, let

$$\mathbb{T} = \{\mathrm{diag}(e^{it_0}, \ldots, e^{it_{d-1}}) : \sum t_i = 0\}$$

be the standard $(d-1)$-torus in $SU(d)$. The associated moment map $\tau\colon \mathbb{CP}^{d-1} \to \mathfrak{t}^*$ is μ composed with a projection, and can be identified with

$$[\mathbf{z}] \mapsto \frac{1}{\|\mathbf{z}\|^2}(|z_0|^2, \ldots, |z_{d-1}|^2) = \mathrm{diag}(\mathbf{z}^*\mathbf{z}).$$

Given $\mathbf{z} \in S^{2d-1}$, we set $x_i = |z_i|^2$, so that the image of τ is the standard simplex

$$\Delta = \left\{(x_0, \ldots, x_{d-1}) : \sum x_i = 1,\ x_i \geqslant 0\right\} \subset \mathbb{R}^d.$$

Correct separation

Fix $d \geqslant 3$, and set $\omega = e^{2\pi i/d}$.
Recall that $H = \langle h \rangle$ where $h = \mathrm{diag}(1, \omega, \omega^2, \ldots, \omega^{d-1})$.

Definition. A point $[\mathbf{z}] \in \mathbb{CP}^{d-1}$ is *H-fiducial* if all points in its H-orbit are 'correctly separated', i.e.

$$\left| \langle \mathbf{z} \,|\, h^i \cdot \mathbf{z} \rangle \right|^2 = \frac{1}{d+1} \|\mathbf{z}\|^4, \qquad 1 \leqslant i \leqslant d-1.$$

Since this concept is $\mathbb{T}$-invariant, we say that $\mathbf{x} \in \Delta$ is H-fiducial if the same is true for any $[\mathbf{z}] \in \mu^{-1}(\mathbf{x})$. Let $\mathscr{H}$ denote the subset of Δ consisting of H-fiducial points.

Trivial example. When $d = 2$, μ is the height function on S^2 and $\mathscr{H}$ consists of two points, each the image of a small circle.

More tori

Given

$$\mu\colon\ \mathbb{CP}^{d-1} \longrightarrow \Delta \subset \mathbb{R}^d,$$

we can identify the H-fiducial subset $\mathscr{H}$ of Δ:

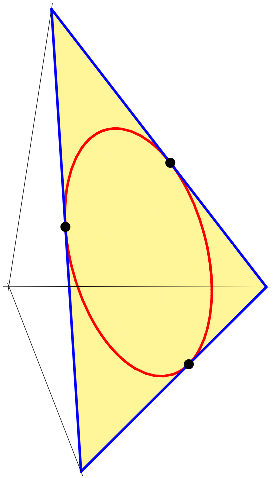
Theorem [Dixon-S]. $\mathscr{H}$ lies in a sphere of radius $\sqrt{\frac{2}{d+1}}$ centre the origin. Moreover,

- if $d = 2n+1$, then $\mathscr{H} = \Delta \cap T$ where T is an n-dimensional torus of revolution;
- if $d = 2n$ then $\mathscr{H} = \Delta \cap (T' \sqcup T'')$ where T', T'' are $(n-1)$-tori of revolution in parallel hyperplanes.

Observation. The centre of an n-face of Δ is a distance $\frac{1}{\sqrt{n+1}}$ from the origin, so the sphere will touch such faces if $d = 2n+1$.

The case d=3

In this case, $\mathscr{H}$ will touch the 1-faces, and is an incircle that we later parametrize by $\left(\cos^2\phi,\ \cos^2(\phi+\frac{2\pi}{3}),\ \cos^2(\phi+\frac{4\pi}{3})\right)$.



Let m_1, m_2, m_3 be the midpoints, and set $C_i = \mu^{-1}(m_i) \subset \mathbb{CP}^2$:

There exists a 1-parameter family $\{\mathbb{M}_t\}$ of SIC sets with

$$\mathbb{M}_t \subset C_1 \sqcup C_2 \sqcup C_3.$$

Proof, exemplified by d=5

Let $\mathbf{z} = (z_0, \ldots, z_4) \in \mathbb{C}^5$. Set $x_i = |z_i|^2$ and suppose $\sum x_i = 1$ so

$$\mu[\mathbf{z}] = (x_0, x_1, \ldots, x_4).$$

Set $\omega = e^{2\pi i/5}$ and $\alpha_i = \langle h^i \cdot \mathbf{z} | \mathbf{z} \rangle$ for $i = 0, 1, 2$:

$$\begin{aligned}
\alpha_0 &= x_0 + x_1 + x_2 + x_3 + x_4 &&= 1 \\
\alpha_1 &= x_0 + \omega x_1 + \omega^2 x_2 + \omega^3 x_3 + \omega^4 x_4 &&= \tfrac{1}{\sqrt{6}} e^{i\theta} \\
\alpha_2 &= x_0 + \omega^2 x_1 + \omega^4 x_2 + \omega^6 x_3 + \omega^8 x_4 &&= \tfrac{1}{\sqrt{6}} e^{i\phi}.
\end{aligned}$$

Taking the moduli squared and rearranging gives three quadrics:

$$\begin{aligned}
Q_0 :& \quad x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 &&= \tfrac{1}{3} = \tfrac{2}{d+1} \\
Q_1 :& \quad x_0x_1 + x_1x_2 + x_2x_3 + x_3x_4 + x_4x_0 &&= \tfrac{1}{6} \\
Q_2 :& \quad x_0x_2 + x_1x_3 + x_2x_4 + x_3x_0 + x_4x_1 &&= \tfrac{1}{6},
\end{aligned}$$

the first a sphere.

Parametrizing the intersection

It follows that $\mathscr{H}$ lies in the intersection of 2 quadrics with the 4-dimensional hyperplane $\sum x_i = 1$ in $\mathbb{R}^5$. It can now be parametrized by the very equations that define the quadrics.

Converting the equations for α_i into real and imaginary parts,

$$\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 1 & \cos\zeta & \cos 2\zeta & \cos 3\zeta & \cos 4\zeta \\ 0 & \sin\zeta & \sin 2\zeta & \sin 3\zeta & \sin 4\zeta \\ 1 & \cos 2\zeta & \cos 4\zeta & \cos 6\zeta & \cos 8\zeta \\ 0 & \sin 2\zeta & \sin 4\zeta & \sin 6\zeta & \sin 8\zeta \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ \cos\theta \\ \sin\theta \\ \cos\phi \\ \sin\phi \end{pmatrix},$$

where $\zeta = 2\pi/5$ and ($\sqrt{\frac{2}{5}}$ times) the 5×5 matrix is orthogonal!

Therefore, $Q_1 \cap Q_2 \cap Q_3$ defines a torus of revolution.

The case of d even

If $d = 6$, we'll have 3 quadrics, Q_1, Q_2 as before (6 terms each), together with

$$Q_3: \quad 2(x_0x_3 + x_1x_4 + x_2x_5) \;=\; \tfrac{1}{7}.$$

Rearranging their equations, ones sees that

$$(x_0 - x_1 + x_2 - x_3 + x_4 - x_5)^2 = \tfrac{1}{7}.$$

If $d = 2n$, $\mathscr{H}$ will lie in the union of the two hyperplanes

$$\sum_{j=0}^{d-1} (-1)^j x_j = \pm \frac{1}{\sqrt{d+1}}.$$

The remaining $n-1$ moment maps to $\mathbb{C}$ define the two tori.

Conjugation

Let $C = \langle g \rangle$ be a cyclic subgroup of $W \times H$ of order d. Let T be a maximal torus of $SU(d)$ containing C. The modulus of

$$\mathrm{tr}(g^k \cdot \mathbf{z}^* \mathbf{z}) = \langle g^k \cdot \mathbf{z} | \mathbf{z} \rangle$$

equals $\frac{1}{\sqrt{d+1}}$ whenever $\mathbf{z} \in S^{2d-1}$ is a fiducal vector.

Lemma. The moment mapping

$$\mu_T \colon \mathbb{CP}^{d-1} \longrightarrow \mathfrak{t}^*$$

can be identified with the vector-valued function

$$[\mathbf{z}] \mapsto \Big(\mathrm{tr}(g^k \cdot \mathbf{z}^* \mathbf{z}) : \ 1 \leqslant k \leqslant \lfloor d/2 \rfloor \Big).$$

This has values in $\mathbb{C}^n$ if $d = 2n+1$ and $\mathbb{C}^{n-1} \oplus \mathbb{R}$ if $d = 2n$, and the same theorem applies to the image of C-fiducial vectors.

Overlap phases

In general, set $\tau = -e^{\pi i/d}$. The unitary matrices

$$D_{p,q} = \tau^{pq} w^p h^q, \qquad 0 \leqslant p, q \leqslant d-1,$$

determine a basis of $i\,\mathfrak{u}(d)$. They satisfy

$$D_{p_1,q_1} D_{p_2,q_2} = \tau^{p_1 q_2 - p_2 q_1} D_{p_1+q_1, p_2+q_2},$$

imposing a symplectic pairing.

If $\mathbf{z} \in S^{2d-1}$, consider the $d \times d$ matrix

$$\Phi_{\mathbf{z}} := \left(\sqrt{d+1}\,\mathrm{tr}(D_{p,q}\mathbf{z}^*\mathbf{z})\right).$$

- its entries describe the image of moment maps of tori;
- $[\mathbf{z}]$ is a fiducial vector iff a.e. entry has modulus one;
- $\Phi_{\mathbf{z}}$ determines $[\mathbf{z}]$.

The case d=4

For $d = 4$, entries $(0,1),(1,2)$ from the first row of $\Phi_{\mathbf{z}}$ tell us that $\mathscr{H}$ consists of two circular arcs

$$f(\theta) = \frac{1}{2\sqrt{5}}(\sigma + \cos\theta, \sigma' + \sin\theta, \sigma - \cos\theta, \sigma' - \sin\theta),$$

where $\sigma, \sigma' = \frac{1}{2}(\sqrt{5} \pm 1)$ in either order. So let

$$\mathbf{z} = (ae^{i\alpha}, be^{i\beta}, ce^{i\gamma}, de^{i\delta})$$

be a WH-fiducial vector with $a, b, c, d > 0$. Entries $(2,0),(2,2)$ corresponding to elements w^2, w^2h^2 of order 2 give

$$\cos(\alpha - \gamma) = 0 \quad \text{or} \quad \cos(\beta - \delta) = 0.$$

W.l.o.g, assume $\beta - \delta = \pm\frac{\pi}{2}$ and $\alpha = 0$, which implies

$$2ac\cos\gamma = \frac{1}{\sqrt{5}}.$$

Conclusions for d=4

Using entries from the second row of $\Phi_{\mathbf{z}}$, computing traces with $D_{1,0} - D_{1,2}$ and $D_{1,1} - D_{1,3}$ enables us to eliminate β and γ:

$$(ad-bc)(ad+bc)(ab-cd)(ab+cd) = 2a^2b^2c^2d^2\cos^2\gamma = \tfrac{1}{5}b^2d^2$$
$$\Rightarrow\ \cos\theta = \pm\tfrac{1}{2}\sqrt{1+\sqrt{5}}, \quad \sin\theta = \pm\tfrac{1}{2}\sqrt{3-\sqrt{5}} = \pm\tfrac{\sigma'}{\sqrt{2}},$$

giving four points on each of the base circles over each of which one can find 2^5 fiducial vectors [LoraLamia-S].

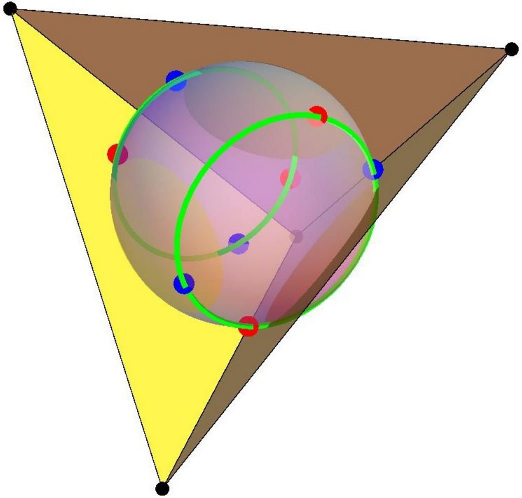
The matrix of overlap phases is remarkably simple:

$$\Phi_{\mathbf{z}} = \begin{pmatrix} \cdot & u & 1 & \overline{u} \\ u & \overline{u} & -\overline{u} & \overline{u} \\ -1 & -u & -1 & \overline{u} \\ \overline{u} & u & u & u \end{pmatrix},$$

where $u = e^{i\theta} = \frac{1}{2}\sqrt{1+\sqrt{5}} + i\frac{\sqrt{5}-1}{2\sqrt{2}}$ [Bengtsson].

Picture for d=4

Here $\tau(\mathbb{CP}^3) = \Delta \subset \mathbb{R}^4$ and fiducial vectors map to 8 points in $\mathscr{H} = \Delta \cap (S^1 \sqcup S^1)$:



Moreover, each set of 4 beads of the same colour is the shadow of 8 SIC sets in $\mathbb{CP}^3$. The 8 beads form an orbit of the dihedral group on Δ.

Arithmetic

The overlap phases are all generated by the unit

$$u = \frac{1}{2}\sqrt{1+\sqrt{5}} + i\,\frac{\sqrt{5}-1}{2\sqrt{2}},$$

which satisfies a monic polynomial of degree 8 over $\mathbb{Z}$. The coordinates of fiducial vectors lie in a splitting field E which has degree 2 over $\mathbb{Q}(u)$, and thus degree 8 over $K = \mathbb{Q}(\sqrt{5})$:

$$\begin{array}{ccc} \boxed{\mathrm{E}} & \qquad & \{1\} \\ \cup & & \triangle \\ K & & (\mathbb{Z}_2)^3 \\ \cup & & \triangle \\ \mathbb{Q} & & \mathbb{Z}_2 \times D_8 \end{array}$$

There are 256 possible fiducial vectors that form an orbit of the extended Clifford group, and there are 16 SIC sets.

Abelian extensions

Suppose that $\mathbf{z} = (z_0, z_1, \ldots, z_{d-1}) \in S^{2d-1}$ is the fiducial vector for a SIC set $\mathbb{S} = WH \cdot [\mathbf{z}]$ in $\mathbb{CP}^{d-1}$, and let $\tau = -e^{\pi i/d}$. Let

$$E = \mathbb{Q}(\tau,\ \bar{z}_i z_j)$$

be the field generated by τ and the entries of the matrix $\mathbf{z}^*\mathbf{z}$. Let $D = (d-3)(d+1) = (d-1)^2 - 4$, and set

$$K = \mathbb{Q}(\sqrt{D}).$$

This field contains a unit $\dfrac{d-1+\sqrt{D}}{2}$ of order 3 modulo d.

$\mathrm{Gal}(E/K)$ will act on the set of WH-fiducials and therefore SIC sets, *permuting* orbits of $ExC(d)$. There is strong empirical evidence for asserting that E is always an *abelian* extension of K.

Isometries of order 3

Example. Let $\tau = (1-i)/\sqrt{2}$. Then

$$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \rightsquigarrow \frac{1}{2}\begin{pmatrix} 1 & \tau & -1 & \tau \\ 1 & \tau^3 & 1 & \tau^7 \\ 1 & \tau^5 & -1 & \tau^5 \\ 1 & \tau^7 & 1 & \tau^3 \end{pmatrix} = X \in C(4),$$

indicating that $X^{-1}wX = h^{-1}$ and $X^{-1}hX = wh^{-1}$. Moreover, $X^3 = \tau\mathbf{1}$. An analogue exists for all d:

Conjecture [Zauner]. Let $\tau = -e^{\pi i/d}$. A fiducial vector $\mathbf{z}$ can always be found in an eigenspace of the matrix

$$\frac{1}{\sqrt{d}}\left(\tau^{2ij+j^2}\right) \in C(d).$$

All elements of $SL(2,d)$ of trace -1 give rise to unitary matrices in $C(d)$ of order 3 if $d > 3$ [Appleby].

An exceptional group

Using the identification $\mathbb{C}^8 = \mathbb{H}^4$, consider the groups

- V_1, right multiplication by $1, i, j, k \in Sp(1)$
- V_2, double sign changes in $\mathbb{H}^4$
- V_3, double transpositions of the coordinates.

Then $A = V_1 \times V_2 \times V_3 \cong (\mathbb{Z}_2)^6$ is a subgroup of $Sp(4) \subset SU(8)$.

Fix unit quaternions $p = \frac{1}{2}(1 + i + j - k),\ q = \frac{1}{2}(1 + i - j - k)$.

Proposition [Hoggar, 1998]. $A \cdot [0, p, q, j]$ is a SIC set $\mathbb{S}$ in $\mathbb{CP}^7$.

Observation [Zhu, Stacey]. Let O denote the root lattice of E_8 (the set of 'octavians'). Then the stabilizer of $\mathbb{S}$ in $U(8)$ is isomorphic to $\text{Aut}(O) \cong G_2(2)$ and has order 12096.

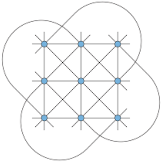
Classification for d=3

Let $\omega = e^{2\pi i/3}$. The set $\mathbb{M}_t$ of 9 points

$$\begin{array}{ccc} [0,1,-1] & [0,1,-\omega] & [0,1,-\omega^2] \\ [1,0,-1] & [1,0,-\omega] & [1,0,-\omega^2] \\ [e^{it},-1,0] & [e^{it},-\omega,0] & [e^{it},-\omega^2,0] \end{array}$$

is a 1-parameter family of SIC-POVM's in $\mathbb{CP}^2$, each conjugate to an orbit of $W \times H$.

Observation. $\mathbb{M}_0$ is the set of flexes of the cubic $x^3 + y^3 + z^3 - 3cxyz = 0$ with $c \notin \{1, \omega, \omega^3\}$, forming the Hesse configuration:



Theorem [Hughston-S, Szöllősi]. *Any* unordered set of nine mutually equidistant points in $\mathbb{CP}^2$ is isometric to $\mathbb{M}_t$ for some t.

Proof, a pinched torus

Let $\mathbb{S}$ be a SIC set in $\mathbb{CP}^2$. Up to isometries, we are free to assume that $\mathbb{S}$ contains the two points

$$[\mathbf{z}_1] = [0, 1, -\omega], \qquad [\mathbf{z}_2] = [0, 1, -\omega^2] \quad \in C_1.$$

Any other point $[\mathbf{z}]$ of $\mathbb{S}$ lies a distance $\frac{2\pi}{3}$ apart from these two.

Lemma. $\mu([\mathbf{z}])$ belongs to the incircle $\mathscr{H}$, and

$$[\mathbf{z}] = \mathbf{Z}[\sigma, \phi] := \left[e^{i\sigma}\cos\phi,\ \cos(\phi + \tfrac{2\pi}{3}), \cos(\phi + \tfrac{4\pi}{3})\right]$$

for some $-\pi < \sigma \leqslant \pi$ and $-\frac{\pi}{2} < \phi \leqslant \frac{\pi}{2}$.

Thus, $[\mathbf{z}]$ lies in a 2-torus $\mathscr{T}$, pinched at the point where $\phi = \frac{\pi}{2}$.

Equilateral triangles in $\mathscr{T}$

Given

$$\mathbf{z}[\sigma,\phi],\quad \mathbf{z}[\tau,\psi],\quad \mathbf{z}[\upsilon,\chi]\quad \in \mathscr{T}$$

all $2\pi/3$ apart in $\mathbb{CP}^2$, set $x=\tan\phi$, $y=\tan\psi$, $z=\tan\chi$ and

$$p=x+y+z,\quad q=yz+zx+xy,\quad r=xyz.$$

Lemma. $F(p,q,r)=0$, where

$$\begin{gathered}F=9-22p^2+9p^4+87q-126p^2q+27p^4q+298q^2-226p^2q^2\\ +24p^4q^2+414q^3-138p^2q^3+189q^4+27q^5-3pr-50p^3r-15p^5r\\ +88pqr-48p^3qr+234pq^2r+18p^3q^2r-144pq^3r+81pq^4r+189r^2\\ -480p^2r^2-153p^4r^2+1398qr^2-306p^2qr^2+2736q^2r^2-486p^2q^2r^2\\ +810q^3r^2+243q^4r^2-558pr^3-486p^3r^3+2376pqr^3-810pq^2r^3\\ +567r^4-162p^2r^4+6399qr^4+486q^2r^4+1701pr^5+2187r^6\end{gathered}$$

(cf. $p^2-3q=\frac{3}{4}$ in $\mathbb{R}^2$). The proof is completed by adding a 4th point to get 4 equations in 4 unknowns and finding a Gröbner basis for a quotient ideal to eliminate the known solutions. $\square$

Thank you for your patience!