Quotients and hypersurfaces of model metrics

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- PART 2. Atiyah-Witten's example of an S^1 quotient of the G_2 metric on $\mathbb{CP}^3 \times \mathbb{R}^+$. Identifying the induced $SU(3)$ structure. Joint work in progress with B. Acharya and R. Bryant.

- PART 3. Various quotients of metrics with holonomy $Spin(7)$, starting with an example built up from $SU(3)$ holonomy. Joint work in progress with U. Fowdar.

The emphasis will be on S^1 quotients of conical metrics with exceptional holonomy defined on vector bundles over S^4, using common techniques that are (in a nutshell) generalizations of the 1978 Gibbons-Hawking ansatz.

PART 1. REVIEW

Construction of metrics with special holonomy (parallel spinors) on manifolds of dimension $6, 7, 8$ typically involves G-structures with

$$\begin{array}{ccccccccc} G = & SU(2) & \subset & SU(3) & \subset & G_2 & \subset & Spin(7) \\ & M^5 & & M^6 & & M^7 & & M^8 \end{array}$$

Theorem. If g has 'weak holonomy' (Killing spinor) on M^n then

- $dr^2 + r^2 g$ has reduced holonomy on M^{n+1} [Bär]
- $dr^2 + (\sin r)^2 g$ has weak holonomy on M^{n+1} [Acharya]

Arbitrary hypersurfaces of reduced holonomy spaces satisfy much weaker conditions: 'hypo' on M^5 (more generally in a Calabi-Yau space), 'half-flat' on M^6, 'co-calibrated' on M^7.

Hypersurfaces

Consider $M^n \subset M^{n+1}$. The second fundamental form of M^n lies in $S^2(\mathbb{R}^n)^*$ at each point. Instead

$$\Lambda^2(\mathbb{R}^n)^* \cong \mathfrak{so}(n) = \mathfrak{g} \oplus \mathfrak{g}^\perp,$$

and the failure of holonomy to reduce is parametrized pointwise by an 'intrinsic torsion' tensor in $(\mathbb{R}^n)^* \otimes \mathfrak{g}^\perp$.

If M^{n+1} has reduced holonomy then the 2nd ff of M^n can be identified with the reduced torsion, neatly via spinor connections.

dim M	5	6	7	8
2nd ff	15	21	28	(36)
dim $\mathfrak{g}^\perp$	$\underset{=1+1+1+4}{7}$	$\underset{=1+6}{7}$	7	7
torsion	35	42	49	56

. . . and quotients

If M^7 has holonomy G_2 then $d\varphi = 0$ and $d*\varphi = 0$. It induces an $SU(3)$ structure on a hypersurface $i(M^6)$ with

$$\psi^+ = i^*\varphi, \qquad \tfrac{1}{2}\omega^2 = i^*(*\varphi).$$

The closure of these forms is the half-flat condition.

In a dual way, if instead S^1 acts freely on M^7 with $\mathscr{L}_X\varphi = 0$, then $\varpi = X \lrcorner\, \varphi$ is closed and determines an almost-Kähler structure on M^6/S^1. There is a connection 1-form η such that

$$\begin{aligned} \varphi &= \eta \wedge \varpi + t^{3/2}\psi^+ \\ *\varphi &= \eta \wedge (t^{1/2}\psi^-) + \tfrac{1}{2}(t\varpi)^2, \end{aligned}$$

where $t = \|\eta\|$, so $N = \|X\|^2 = t^{-2}$. Note that $d(t^{1/2}\psi^-) = 0$, and the torsion is determined by dt and $d\psi^+$.

Kähler reduction

Since ψ^- is 'stable' (and its stabilizer in $GL(6,\mathbb{R})$ is $SL(3,\mathbb{C})$), it determines the almost complex structure J. An associated $(3,0)$ form is $\Psi = \psi^+ + i\psi^-$ and J is integrable iff $d(t^{1/2}\psi^+) = 0$.

Theorem [Apostolov-S]. In this case,

- the Ricci form of the Kähler metric equals $\frac{1}{2} i\partial\overline{\partial}\log t$,
- a new Killing vector field U is defined by $U \lrcorner\, \varpi = -dt$, and one can further quotient to 4 dimensions.

Other reductions leading to triples of 2-forms and Monge-Ampère equations can be imposed with extra symmetry [Donaldson].

In general, the curvature $F = d\eta$ of the S^1 bundle is constrained by the residual torsion:

$$\begin{aligned} F \wedge \varpi &= -\tfrac{3}{2} t^{1/2} dt \wedge \psi^+ - t^{3/2} d\psi^+ \\ F \wedge \psi^- &= -t^{1/2} dt \wedge \varpi^2. \end{aligned}$$

PART 2. An explicit S^1 quotient

If M^6 is nearly-Kähler then the conical metric on $M \times \mathbb{R}^+$ has a metric with holonomy $\subseteq G_2$. In particular, this is true when M is the twistor space $(\mathbb{CP}^3, J_2)$ over S^4, with isometry group $SO(5)$.

Let $S^1 = SO(2)$ be the subgroup acting on $S^4 \subset \mathbb{R}^2 \oplus \mathbb{R}^3$, and by extension on $\mathbb{CP}^3$.

Problem. Describe the structure of the quotient of $\mathbb{CP}^3 \times \mathbb{R}^+$

We shall see that the resulting T^2 action on $\mathbb{C}^4$ is equivalent to $S^1 \times S^1$ acting by left multiplication on some $\mathbb{H} \oplus \mathbb{H}$, so that the quotient is essentially $\mathbb{R}^3 \oplus \mathbb{R}^3 = \mathbb{R}^6$. Also, S^1 fixes an S^2 in S^4, covered by two disjoint S^2's in $\mathbb{CP}^3$, giving rise to a singular locus $\mathbb{R}^3 \cup \mathbb{R}^3$ of two 3-spaces meeting at the origin in $\mathbb{R}^6$.

Right actions on $\mathbb{R}^8 = \mathbb{H}^2$

Euclidean coordinates $x_1, \ldots, x_8$, vector fields $\partial_i = \frac{\partial}{\partial x_i}$.

Radius squared $R = \sum x_i^2$, metric $g = \sum dx_i \otimes dx_i$.

Right multiplication by $Sp(1)$ gives Killing vector fields

$$\begin{aligned}
Y_1 &= x_2\partial_1 - x_1\partial_2 - x_4\partial_3 + x_3\partial_4 + x_6\partial_5 - x_5\partial_6 - x_8\partial_7 + x_7\partial_8 \\
Y_2 &= x_3\partial_1 - x_1\partial_3 - \qquad \cdots \qquad - x_6\partial_8 + x_8\partial_6 \\
Y_3 &= x_4\partial_1 - x_1\partial_4 - \qquad \cdots \qquad - x_7\partial_6 + x_6\partial_7
\end{aligned}$$

tangent to the fibres of

$$S^7 = \frac{Sp(2)}{Sp(1)} \to \frac{Sp(2)}{Sp(1)Sp(1)} = S^4.$$

Consider the dual 1-forms $\alpha_i = Y_i \lrcorner\, g$, such as

$$\alpha_1 = x_2 dx_1 - x_1 dx_2 - \cdots - x_8 dx_7 + x_7 dx_8,$$

with $-\frac{1}{2} d\alpha_1 = dx_{12} - dx_{34} + dx_{56} - dx_{78}$ 'anti-self-dual'.

The form $\hat{\alpha}_i = \alpha_i / R$ is invariant by $\mathbb{R}^*$, and is a connection on an S^1 bundle $S^7 \to \mathbb{CP}^3$. Fix $i = 1$ and $\mathbb{CP}^3 = S^7/U(1)_1$. Then the curvature $d\hat{\alpha}_1$ is a Kähler form for the Fubini-Study metric on the twistor space $(\mathbb{CP}^3, J_1)$.

Using $\mathscr{L}_{Y_2} \hat{\alpha}_3 = \hat{\alpha}_1$ etc, one sees that the 2-forms

$$\begin{cases} \tau_1 &= d\hat{\alpha}_1 - 2\hat{\alpha}_{23} \\ \tau_2 &= d\hat{\alpha}_2 - 2\hat{\alpha}_{31} \\ \tau_3 &= d\hat{\alpha}_3 - 2\hat{\alpha}_{12} \end{cases}$$

pass to S^4, and form a basis of ASD forms there.

Lemma. The nearly-Kähler structure of $(\mathbb{CP}^3, J_2)$ is given by

$$\begin{aligned} \omega &= -\hat{\alpha}_{23} + \tau_1 = d\hat{\alpha}_1 - 3\hat{\alpha}_{23} \\ \Upsilon &= (\hat{\alpha}_2 - i\hat{\alpha}_3) \wedge (\tau_2 + i\tau_3) \end{aligned}$$

Using the fact that $\tau_i^2 = 2e_{1234}$ is independent of i, one verifies

$$\begin{cases} d\omega &= 3\,\mathrm{Im}\Upsilon, \\ d\Upsilon &= 2\omega^2. \end{cases}$$

The conical G_2 structure on $\mathbb{CP}^3 \times \mathbb{R}^+ = \mathbb{H}^2_*/S^1$ then has

$$\begin{aligned} \varphi &= dR \wedge R^2\omega + R^3\,\mathrm{Im}\Upsilon &&= d(\tfrac{1}{3}R^3\omega), \\ *\varphi &= dR \wedge R^3\,\mathrm{Re}\Upsilon + \tfrac{1}{2}(R^2\omega)^2 &&= d(\tfrac{1}{4}R^4\,\mathrm{Re}\Upsilon). \end{aligned}$$

A 2-torus action on $\mathbb{R}^8$

Left multiplication by S^1 on $\mathbb{R}^8$ gives a Killing vector field

$$X = X_1 = x_2\partial_1 - x_1\partial_2 + x_4\partial_3 - x_3\partial_4 + \cdots + x_8\partial_7 - x_7\partial_8,$$

Observe that

$$\begin{aligned} \tfrac{1}{2}(X + Y_1) &= x_2\partial_1 - x_1\partial_2 + x_6\partial_5 - x_5\partial_6, \\ \tfrac{1}{2}(X - Y_1) &= x_4\partial_3 - x_3\partial_4 + x_8\partial_7 - x_7\partial_8, \end{aligned}$$

so that $X \pm Y_1$ define standard S^1 actions on the two summands $\langle 1,2,5,6\rangle \oplus \langle 3,4,7,8\rangle = \mathbb{H}^2$, and indeed

$$\frac{\mathbb{CP}^3 \times \mathbb{R}^+}{S^1} \overset{*}{=} \frac{\mathbb{H}}{S^1} \times \frac{\mathbb{H}}{S^1} = \mathbb{R}^3 \times \mathbb{R}^3.$$

Moment maps

The action of $X+Y_1$ on $\mathbb{H}=\langle 1,2,5,6\rangle$ is tri-holomorphic for the hyperkähler structure with 2-forms

$$dx_{12}-dx_{56},\quad dx_{15}-dx_{62},\quad dx_{16}-dx_{25}.$$

Using the quaternion $q=x_1+x_2i+x_5j+x_6k$, the components

$$\left.\begin{array}{ll} u_1 & = x_1^2+x_2^2-x_5^2-x_6^2 \\ u_2 & = 2(-x_1x_6+x_2x_5) \\ u_3 & = \;2(x_1x_5+x_2x_6) \end{array}\right\} \text{ of } \;\overline{q}\,i\,q=u_1i+u_2j+u_3k$$

are invariant by the $U(1)$ action $q\rightsquigarrow e^{i\theta}q$. We set

$$u_0=x_1^2+x_2^2+x_5^2+x_6^2,\quad \text{so}\quad u_0^2=\sum_{i=1}^{3}u_i^2=|\mathbf{u}|^2.$$

Similarly, the hyperkähler moment map for $X - Y_1$ on $\langle 3,4,7,8\rangle$ equals (v_1, v_2, v_3), where

$$\left\{\begin{array}{ll} v_1 & = x_3^2 + x_4^2 - x_7^2 - x_8^2 \\ v_2 & = 2(-x_3x_8 + x_4x_7) \\ v_3 & = 2(x_3x_7 + x_4x_8). \end{array}\right.$$

We also set $v_0 = |\mathbf{v}|$, so that $u_0 + v_0 = \sum_{i=1}^{8} x_i^2 = R$.

Identifications $\mathbb{R}^4/S^1 \cong \Lambda^2_- \mathbb{R}^4$ arise from the Gibbons-Hawking ansatz. Indeed, $f = 1/v_0$ is harmonic and the connection form $\eta = f(X - Y_1) \lrcorner\, g$ satisfies

$$\begin{aligned} 4g &= f^{-1}\eta^2 + f \sum_{i=1}^{3} dv_i^2 \\ d\eta &= -\tfrac{1}{2} f^3\{\mathbf{v}, d\mathbf{v}, d\mathbf{v}\} = *df. \end{aligned}$$

Returning to $\mathbb{R}^8/T^2$, we next compute the symplectic form

$$\varpi = X \lrcorner \varphi = X \lrcorner d(\tfrac{1}{3}R^3\omega) = d\sigma,$$

where $\sigma = -\frac{1}{3}R^3 X \lrcorner \omega = \frac{1}{3}R^3 X \lrcorner (d\hat{\alpha}_1 - 3\hat{\alpha}_{23})$. This can be expressed in terms of the functions $\mu_j = X \lrcorner Y_j$ with

$$\begin{cases} \mu_1 &= x_1^2 + x_2^2 - x_3^2 - x_4^2 + x_5^2 + x_6^2 - x_7^2 - x_8^2 \\ \mu_2 &= 2(-x_1x_4 + x_2x_3 - x_5x_8 + x_6x_7) \\ \mu_3 &= 2(x_1x_3 + x_4x_2 + x_5x_7 + x_6x_8). \end{cases}$$

Computation yields

Proposition [Bryant]. $\sigma = \frac{1}{6}R\big[u_0 dv_0 - v_0 du_0 - 3(\mathbf{u}\cdot d\mathbf{v} - \mathbf{v}\cdot d\mathbf{u})\big]$

So ϖ is not in standard form on $\mathbb{R}^3 \oplus \mathbb{R}^3$.

Invariant functions

The G_2 structure on $\mathbb{CP}^3 \times \mathbb{R}^+$ is invariant by $Sp(2)$. Recall that the double covering $Sp(2) \to SO(5)$ is given by $\Lambda^2_0(\mathbb{C}^4) \stackrel{\cong}{\to} \mathbb{C}^5$.

The diagonal S^1 in $Sp(2)$ commutes with $Sp(1) = SU(2)$, which acts as $SO(3)$ as follows:

- trivially on the first factor of $\mathbb{R}^2 \oplus \mathbb{R}^3 \supset S^4$
- diagonally on the quotient $\mathbb{R}^6 = \mathbb{R}^3 \oplus \mathbb{R}^3$.

The induced $SU(3)$ structure on $\mathbb{R}^6$ can be expressed in terms of $SO(3)$ invariant quantities manufactured from the coordinates $(\mathbf{u}, \mathbf{v}) = (u_1, u_2, u_3; v_1, v_2, v_3)$ using scalar and triple products. These include the radii $(u_0, v_0) = (|\mathbf{u}|, |\mathbf{v}|)$, which are not smooth on $\mathbb{R}^3 \cup \mathbb{R}^3$. More examples follow.

The antilinear involution j on $\mathbb{CP}^3$ generates the $\mathbf{u} \leftrightarrow \mathbf{v}$ symmetry.

Recall that $t^{1/2}\psi^- = X \lrcorner (*\varphi) = d\tau$, where $\tau = -\frac{1}{4}R^4 X \lrcorner \mathrm{Re}\Upsilon$.

Proposition. $8\tau/R$ equals

$$3\{\mathbf{u}, d\mathbf{v}, d\mathbf{v}\} - \frac{u_0}{v_0}\{\mathbf{v}, d\mathbf{v}, d\mathbf{v}\} + 2\{\mathbf{u}, d\mathbf{u}, d\mathbf{v}\} + \text{swapping } \mathbf{u}, \mathbf{v}$$

To find ψ^+ and the $(3,0)$-form $\Psi = \psi^+ + i\psi^-$, one needs to compute the connection 1-form η and its G_2 norm, the reciprocal of $N = t^{-2} = \|X_1\|^2$. Knowledge of

$$N = \frac{7\varphi \wedge \varpi \wedge \varpi \wedge \alpha_1}{6\varphi \wedge (*\varphi) \wedge \alpha_1} = 6u_0 v_0 - 2(u_1 v_1 + u_2 v_2 + u_3 v_3)$$

allows us to compute the induced metric $\tilde{g}$ on $\mathbb{R}^6$ by subtracting the contribution from the S^1 fibres.

One can also compute $\tilde{g}$ (or more easily, its conformal structure) from the formula

$$\tilde{g}(W,Z)\varpi^3 = -2(W \lrcorner \varpi)\wedge(Z \lrcorner \psi^-)\wedge\psi^-,$$

The restriction of $\tilde{g}$ to the plane (u,v) plane with $u = u_1$, $v = v_1$ and $u_2 = u_3 = v_2 = v_3 = 0$ equals

$$ds^2 = \left(1+\frac{v}{4u}\right)du^2 + \tfrac{3}{2}\,du\,dv + \left(1+\frac{u}{4v}\right)dv^2$$

in the quadrant $u, v > 0$.

Corollary. The coordinate 3-planes in $(\mathbb{R}^6, \tilde{g})$ meet at an angle θ, where $\cos\theta = 3t/\sqrt{(1+4t)(4+t)}$ and $t = v_0/u_0$.

. . . to be continued

PART 3. Starting from Spin(7)

Suppose, by way of transition, that (M^6, ϖ, Ψ) has holonomy in $SU(3)$ and N^7 is an S^1-bundle with connection 1-form η over satisfying $d\eta = -\omega$.

Naïvely, without rescaling, the total space N^7 has a G_2-structure with

$$\begin{aligned} \varphi &= \eta \wedge \varpi + \mathrm{Re}\Psi, & d\varphi &= \varpi^2, \\ *\varphi &= \eta \wedge \mathrm{Im}\Psi + \tfrac{1}{2}\varpi^2, & d*\varphi &= 0. \end{aligned}$$

Even more trivially, the Riemannian product $M^6 \times \mathbb{R}$ has holonomy a *subgroup* of G_2:

$$\begin{array}{ccc} N^7 & & N^7 \times \mathbb{R} \\ \downarrow & & \downarrow \\ M^6 & \subset & M^6 \times \mathbb{R}. \end{array}$$

Then $N^7 \times \mathbb{R}$ acquires a metric with holonomy in $SU(4)$ with

$$\begin{aligned} \varpi' &= x^2\varpi + \eta \wedge 2x\,dx, \\ \Psi' &= \Psi \wedge (-i\eta + 2x^7 dx), \\ g' &= x^2 g_M + x^{-6}\eta^2 + (2x^4 dx)^2. \end{aligned}$$

This can be made explicit if $M = T^6$ and N is a nilmanifold.

The associated $Spin(7)$ closed 4-form

$$\begin{aligned} \Omega &= \tfrac{1}{2}(\varpi')^2 + \mathrm{Re}\Psi' \\ &= \eta \wedge (\varpi \wedge 2x^3 dx + \mathrm{Im}\Psi) + x^4(\tfrac{1}{2}\varpi^2 + \mathrm{Re}\Psi \wedge 2x^3 dx) \\ &= \eta \wedge \phi + x^4 * \phi, \end{aligned}$$

does induce the product metric on $M^6 \times \mathbb{R}$. Why does this work?

More generally, suppose that

$$\Omega = \eta \wedge \phi + x^4 * \phi$$

defines a metric with holonomy in $Spin(7)$ on the total space of an S^1 bundle (with $\|\alpha\| = |x|^3$ non-constant), inducing a metric on the base with holonomy in G_2. Then

$$0 = d\Omega = F \wedge \phi + 4x^3 dx \wedge *\phi.$$

This implies that $F \in \Lambda^2_7$ can be identified with $4x^3 dx$ modulo G_2 invariants, so η is an *anti*-instanton. Moreover, $\nabla dx = 0$ (because ∇dx can be extracted from $dF \in \Lambda^3 \cong \mathbb{R}^7 \oplus S^2(\mathbb{R}^7)$).

Corollary [Oliveira]. The base holonomy must reduce to $SU(3)$.

Hitchin flow

The $SU(4)$ structure on $N^7 \times \mathbb{R}$ can instead be realized via the evolution equation

$$\frac{\partial(*\varphi)}{\partial x} = -d\varphi$$

for the co-calibrated G_2 structure on the hypersurfaces $x = \text{const}$.

In this picture, N^7 has a 'hypo' $SU(3)$ structure (ω, η, ψ) such that $d\omega = 0$ and $d(\eta \wedge \psi) = 0$ [Conti-Fino].

A number of Lie groups with co-calibrated G_2 structures have such structures:

Theorem [Freibert]. If N^7 arises from a 'close to abelian' Lie algebra with an invariant co-calibrated structure, this evolves to $SU(4)$ (but not $Sp(2)$) holonomy.

Model metrics over S^4

The (negative) spin bundle Δ_- over S^4 has fibre $\mathbb{H}$, and a 'left' hyperkähler structure with complex structures I_1, I_2, I_3. Choose a Killing vector field $X = X_1$ generating I_1. Fibrewise, this preserves a 'right' hyperkähler structure defined by a local basis (γ_i) of 2-forms that are ASD on $\mathbb{H}$.

Then Δ_- admits a metric with holonomy equal to *Spin*(7), with closed 4-form

$$\Omega_{\mathrm{BS}} \;\sim\; r^{-8/5} dx_{1234} + r^{2/5} \sum_{i=1}^{3} \gamma_i \wedge \varepsilon_i + r^{12/5} e_{1234},$$

where r is the radius on $\mathbb{H}$, (ε_i) is a local basis of ASD forms on S^4 matching the γ_i's, and e_{1234} is the volume form on the sphere ($\sim$ means we are ignoring universal constants in all terms).

The quotient of Δ_- by $\{e^{l_1 t}\}$ can be identified with $\Lambda^2_- T^* S^4$, allowing one to realize a *fibrewise* Gibbons-Hawking ansatz, even though the action is not hyperkähler on Δ_-. We obtain

$$\begin{aligned} \varphi_{\mathrm{GH}} = X \lrcorner \Omega &\sim r^{-8/5}(X \lrcorner dx_{1234}) + r^{2/5} \sum (X \lrcorner \gamma_i) \wedge \varepsilon_i \\ &\sim r^{-18/5} du_{123} + r^{2/5} \sum du_i \wedge \varepsilon_i \end{aligned}$$

$$\boxed{\varphi_{\mathrm{GH}} \sim R^{-9/5}\, du_{123} + R^{1/5}\, d\mathcal{T}}$$

where $R = r^2$ is the radius on $\mathbb{R}^3$, and $\mathcal{T}$ the tautological 2-form.

This contrasts with the 3-form of the known metric

$$\boxed{\varphi_{\mathrm{BS}} \sim R^{-3/2}\, du_{123} + R^{1/2}\, d\mathcal{T}}$$

with holonomy equal to G_2.

The 'Gibbons-Hawking' quotient can't have holonomy G_2. Indeed,

$$*\varphi_{\rm GH} \sim R^{8/5}e_{1234} + R^{-2/5}\mathcal{U},$$

(unit 1-forms have weight $R^{-3/5}$ vertically and $R^{2/5}$ horizontally).

$$\begin{aligned} d*\varphi_{\rm GH} &\sim R^{3/5}dR\wedge e_{1234} + R^{-3/5}dR\wedge\mathcal{U} + R^{-2/5}d\mathcal{U} \\ &\sim R^{3/5}dR\wedge e_{1234} + R^{7/5}du_{123}\wedge\mathcal{T} \end{aligned}$$

We know that

$$d^*\varphi_{\rm GH} \sim R^{2/5}F + R^{7/5}\mathcal{T}$$

lies in the 'instanton subspace' $\Lambda^2_{14}\cong\mathfrak{g}_2$, imposing a condition on the S^1 curvature $F = d\eta$.

To what extent can one extend the analysis in PARTS 2 and 3 to complete metrics?