Algebraically Constrained Special Holonomy Metrics and Second-order Associative 3-folds

A progress report

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Part I: Curvature-Constrained Special Holonomy

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$\eta : TB \to \mathbb{R}^n$, $\theta : TB \to \mathfrak{h}$, and $R : B \to K(\mathfrak{h})$ is the curvature function, where $K(\mathfrak{h})$ is the H-representation

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Second Bianchi: $\mathrm{d}R = -\theta.R + R'(\eta)$. where

$$R' : B \to K^{(1)}(\mathfrak{h}) \subset \mathrm{Hom}\big(\mathbb{R}^n, K(\mathfrak{h})\big)$$

represents the covariant derivative of the curvature.

Example: SU(2) $\subset$ SO(4)

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$$K\big(\mathfrak{su}(2)\big) = S^2_0(\mathbb{R}^3) \simeq \mathbb{R}^5 \qquad \text{and} \qquad K^{(1)}(\mathfrak{su}(2)) \simeq \mathbb{C}^6 \simeq S^5(\mathbb{C}^2)$$

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É. Cartan (1926): SU(2)-holonomy depends on 2 functions of 3 variables.

Basic holonomy problem: For a given subgroup $H \subset \mathrm{SO}(n)$ how to classify, up to local diffeomorphism, the 'solutions' to the structure equations

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Example: $H = \mathrm{SU}(2) = \mathrm{Spin}(3) \subset \mathrm{SO}(4)$ acts on $K(\mathfrak{su}(2)) = S^2_0(\mathbb{R}^3)$ preserving the symmetric functions of the eigenvalues of $R \in S^2_0(\mathbb{R}^3)$. Specifying a relation between $\sigma_2(R)$ and $\sigma_3(R)$ defines such an invariant subset $A \subset S^2_0(\mathbb{R}^3)$.

$$\sigma_3(R)^2 + \tfrac{4}{27}\sigma_2(R)^3 \le 0.$$

Cases of interest in special holonomy

1. $\mathsf{SU}(n) \subset \mathsf{SO}(2n)$

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$$K\big(\mathfrak{so}(7)\big) \simeq V^{0,2,0}(\mathfrak{so}(7)) \simeq \mathbb{R}^{168}.$$

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We have $A \simeq K(\mathfrak{su}(2)) \simeq \mathbb{R}^5$ with

$$(s_1, s_2, s_3, s_4, s_5, s_6, s_7) = (0, 3, 2, 0, 0, 0, 0).$$

and $\dim A^{(1)} = \dim K(\mathfrak{su}(2))^{(1)} = 12 = 2s_2 + 3s_3$, so it's involutive.

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Example: The SU(2) structure equations in which $R : B \to S^2_0(\mathbb{R}^3)$ has a double eigenvalue everywhere are not involutive:

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Applying $\mathrm{d}^2 = 0$ to the equations

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with $r \neq 0$ implies that there exist u_0, u_1, u_1, u_3 for which

$$\begin{aligned} \mathrm{d}r &= 4r\,(\quad u_0\,\eta_0 + u_1\,\eta_1 + u_2\,\eta_2 + u_3\,\eta_3) \\ \theta_2 &= 2\;\;(-u_2\,\eta_0 - u_3\,\eta_1 + u_0\,\eta_2 + u_1\,\eta_3) \\ \theta_3 &= 2\;\;(-u_3\,\eta_0 + u_2\,\eta_1 - u_1\,\eta_2 + u_0\,\eta_3) \end{aligned}$$

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These are structure equations for a coframing $(\eta_0, \eta_1, \eta_2, \eta_3, \theta_1)$ with coefficients (r, u_0, u_1, u_2, u_3) that still are not involutive.

Differentiating the structure equations again yields relations of the form

$$\mathrm{d}\begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} = U(r, u_0, u_1, u_2, u_3, v_1, v_2, v_3) \begin{pmatrix} \theta_1 \\ \eta_0 \\ \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix}$$

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Differentiating these last relations yields no more relations. Coupled with

$$\mathrm{d}r = 4r\,(u_0\,\eta_0 + u_1\,\eta_1 + u_2\,\eta_2 + u_3\,\eta_3)$$

This gives 8 ‘independent’ coefficients in the structure equations for which $\mathrm{d}^2 = 0$ is an identity.

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3. $H = \mathrm{Spin}(7) \subset \mathrm{SO}(8)$: $s_7 = 12$ is last nonzero character.

Curvature restrictions in the $\mathrm{SU}(2) \subset \mathrm{SO}(4)$ case

The $\mathrm{SU}(2)$-invariants on $K(\mathfrak{su}(2)) \simeq S_0^2(\mathbb{R}^3) \simeq \mathbb{R}^5$ are generated by $\sigma_2, \sigma_3 : S_0^2(\mathbb{R}^3) \to \mathbb{R}$, satisfying

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3. $(\sigma_3(R))^2 + \tfrac{4}{27}\,(\sigma_2(R))^3 = 0$.
This is the 'double eigenvalue case', with nontrivial stabilizer $S^1 \subset \mathrm{SU}(2)$.
Not involutive, but prolongation yields a 2-parameter family of solutions, not all of which are complete, but some are.

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The most promising candidate to date is the subset $S \subset K(\mathfrak{h})$ that consists of the curvatures that have nontrivial H-stabilizers. It is not a smooth manifold, but it can be stratified into smooth pieces according to the stabilizer type, and these can be analyzed.

Curvatures in $K(\mathfrak{h})$ with nontrivial H-stabilizers

Classifying the general H-invariant $A \subset K(\mathfrak{h})$ for which the corresponding H-structures have nontrivial solutions is probably intractable.

However, it seems likely that one can classify the constraints $A \subset K(\mathfrak{h})$ that are either involutive or their first prolongation is involutive.

The most promising candidate to date is the subset $S \subset K(\mathfrak{h})$ that consists of the curvatures that have nontrivial H-stabilizers. It is not a smooth manifold, but it can be stratified into smooth pieces according to the stabilizer type, and these can be analyzed.

This is the project that I have been engaged in.

Table: Stablized curvatures for subgroups of SU(3) and Generality

G	$\dim\big(K(\mathfrak{su}(3))\big)^G$	G-splitting of $\mathbb{C}^3$	Generality
U(2)	1	$\mathbb{C} \oplus \mathbb{C}^2$	1 const. (known)
SU(2)	1	$\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{C}^2$	1 const. (known)
SO(3)	1	$\mathbb{R}^3 \oplus \mathbb{R}^3$	does not exist
$\mathbb{T}^2$	3	$\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$	8 constants
$S^1(p/q)^\dagger$	3	$\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$	8 constants
$S^1(0)$	5	$\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{C} \oplus \mathbb{C}$	$s_1 = 2$??
$S^1(1)$	7	$\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$	$s_1 = 4$??

† $p/q \neq 0, 1$, where $S^1(p/q)$ is the circle of diagonal matrices $\mathrm{diag}(e^{ipt}, e^{iqt}, e^{-i(p+q)t})$.

Table: Stabilized curvatures of subgroups of G_2

G	$\dim\left(K(\mathfrak{g}_2)\right)^G$	G-splitting of $\mathbb{R}^7$	Generality
SU(3)	0	$\mathbb{R}^1 \oplus \mathbb{C}^3$	only flat
SO(4)	1	$\mathbb{R}^3 \oplus \mathbb{R}^4$	only $\Lambda^2_+(S^4)$
$\mathrm{U}(2)_1$	2	$\mathbb{R}^3 \oplus \mathbb{R}^4$	only $\Lambda^2_+(\mathbb{CP}^2)$
$\mathrm{U}(2)_2$	2	$\mathbb{R}^1 \oplus \mathbb{R}^2 \oplus \mathbb{R}^4$	DNE
$\mathbb{T}^2$	5	$\mathbb{R} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$	'only' consts.
$\mathrm{SU}(2)_1$	3	$\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{C}^2$	$s_2 = 0$
$\mathrm{SU}(2)_2$	6	$\mathbb{R}^3 \oplus \mathbb{R}^4$	???
$\mathrm{SO}(3)_1$	1	$\mathbb{R} \oplus \mathbb{R}^3 \oplus \mathbb{R}^3$	DNE
$\mathrm{SO}(3)_2$	1	$\mathbb{R}^7$	'only' consts.
$S^1(p/q)^\dagger$	5	$\mathbb{R} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$	'only' consts.
$S^1(1/2)$	7	$\mathbb{R} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$	$s_1 = 2$??
$S^1(1)$	9	$\mathbb{R} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$	$s_1 = 4$??
$S^1(0)$	13	$\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{C} \oplus \mathbb{C}$	???

† $p/q \neq 0, \frac{1}{2}, 1$

Part II: Second order associative 3-folds

Associative submanifolds $M^3 \subset \mathbb{R}^n$ can be defined by the condition that their tangent spaces belong to the 8-dimensional *associative Grassmannian*

$$\mathrm{Assoc}\ (\simeq \mathrm{G}_2/\mathrm{SO}(4)) = \mathrm{G}_2 \cdot \mathbb{R}^3 \subset \mathrm{Gr}_3^+(\mathbb{R}^7).$$

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$$\Sigma\ (\simeq G/H) = G \cdot \mathbb{R}^m \subset \mathrm{Gr}_m^+(\mathbb{R}^n), \qquad \text{where } H = \mathrm{Stab}(G, \mathbb{R}^m)$$

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one can speak of Σ-manifolds $M^m \subset \mathbb{R}^n$, whose tangent spaces belong to Σ. The Gauss map $\gamma_M : M \to \Sigma$ given by $\gamma_M(x) = T_x M \in \Sigma$ has a derivative

$$\gamma_M'(x) : T_x M \to T_{\gamma_M(x)}\Sigma \subset N_{\gamma_M(x)} \otimes T_x^* M \simeq \mathbb{R}^{n-m} \otimes (\mathbb{R}^m)^*$$

that satisfies (because of symmetry of second partials),

$$\mathrm{II}_x = \gamma_M'(x) \in T_{\gamma_M(x)}\Sigma \otimes T_x^* M \cap \left(N_{\gamma_M(x)} \otimes S^2(T_x^* M)\right).$$

All of the spaces $\mathbb{R}^m$, $(\mathbb{R}^m)^\perp = \mathbb{R}^{n-m}$, and $T_{\mathbb{R}^m}\Sigma \simeq \mathfrak{g}/\mathfrak{h}$ are H-modules, and so is the space

$$\mathrm{II}(\mathfrak{g},\mathfrak{h}) = \left(T_{\mathbb{R}^m}\Sigma \otimes \mathbb{R}^m\right) \cap \left((\mathbb{R}^m)^\perp \otimes S^2(\mathbb{R}^m)\right),$$

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Example: Special Lagrangian submanifolds. $G = \mathrm{SU}(m)$, $H = \mathrm{SO}(m)$. Then

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In 2000, I analyzed the case $m = 3$ of special Lagrangian 3-folds whose second fundamental forms (harmonic cubic forms) had a nontrivial symmetry, and found many integrable cases. (*Second order families of special Lagrangian* 3-*folds*, arXiv:math/0007128.)

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My student, Marianty Ionel, did a similar analysis of the case $m = 4$ in 2002 and also found many integrable cases.

Associative 3-folds: $G = \mathrm{G}_2$, $H = \mathrm{SO}(4)$. $(\mathrm{Spin}(4) = \mathrm{Sp}(1) \times \mathrm{Sp}(1).)$

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Thus, one can think of the second fundamental form of an associative 3-fold as a quintic polynomial in two complex variables

$$p(z_1, z_2) = a_5\, z_1^5 + 5a_4\, z_1^4 z_2 + 10a_3\, z_1^3 z_2^2 + 10a_2\, z_1^2 z_2^3 + 5a_1\, z_1 z_2^4 + a_0\, z_2^5.$$

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One can interpret a second fundamental form of an associative 3-manifold (up to real scalar multiples) as a degree ≤ 5 rational mapping

$$P : \mathbb{CP}^1 \to \mathbb{CP}^1, \qquad P(z_1, z_2) = \left[p(z_1, z_2), \overline{p(-\overline{z_2}, \overline{z_1})}\right].$$

up to (independent) isometric rotations in the domain and range 2-spheres.

By comparison, for co-associative submanifolds $M^4 \subset \mathbb{R}^7$, the coassociative Grassmannian is also $\mathrm{G}_2/\mathrm{SO}(4) \subset \mathrm{Gr}_4^+(\mathbb{R}^7)$, so again,

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For Cayley submanifolds $M^4 \subset \mathbb{R}^8$, the Cayley Grassmannian is $\mathrm{Spin}(7)/H \subset \mathrm{Gr}_4^+(\mathbb{R}^8)$, where $H = \big(\mathrm{Sp}(1)\times\mathrm{Sp}(1)\times\mathrm{Sp}(1)\big)/\mathbb{Z}_2$. Then

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The classification of the stabilizer types in the associative case can now be worked out.

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(N.B. There are some inequalities among a, b, u, v in the above cases in order to ensure no larger symmetry. Also, the three circles in Cases 1–3 are not conjugate in $\mathrm{SO}(4)$.)

Proposition 1: The associative 3-folds in $\mathbb{R}^7$ whose second fundamental forms have type $p = a\,{z_1}^5$

- are all ruled,
- depend on $s_1 = 6$ functions of 1 variable, and
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Remark: The ruled associative 3-folds in $\mathbb{R}^7$ can be regarded as surfaces in $\Lambda(\mathbb{R}^7) \simeq TS^6$, the space of lines in $\mathbb{R}^7$. There is a unique almost complex structure on $\Lambda(\mathbb{R}^7)$ such that these surfaces are the pseudoholomorphic curves in $\Lambda(\mathbb{R}^7)$. Thus, they locally depend on $s_1 = 12$ functions of 1 variable.

Proposition 2: There are no associative 3-folds in $\mathbb{R}^7$ whose second fundamental forms have type $p = 5a\, {z_1}^4 z_2$ (with $a \neq 0$).

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Proposition 3: The only associative 3-folds in $\mathbb{R}^7$ whose second fundamental forms have type $p = 10a\, {z_1}^3 {z_2}^2$ are actually special Lagrangian in a $\mathbb{R}^6 \subset \mathbb{R}^7$ and are the Harvey-Lawson examples with an $\mathrm{SO}(3)$ symmetry.

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Proposition 2: There are no associative 3-folds in $\mathbb{R}^7$ whose second fundamental forms have type $p = 5a\, {z_1}^4 z_2$ (with $a \neq 0$).

Proposition 3: The only associative 3-folds in $\mathbb{R}^7$ whose second fundamental forms have type $p = 10a\, {z_1}^3 {z_2}^2$ are actually special Lagrangian in a $\mathbb{R}^6 \subset \mathbb{R}^7$ and are the Harvey-Lawson examples with an $\mathrm{SO}(3)$ symmetry.

Proposition 4: $(\mathbb{Z}_4)$ The only associative 3-folds in $\mathbb{R}^7$ whose second fundamental forms have type $p = a\, {z_1}^5 + 5b\, z_1 {z_2}^4$ must actually have either $a = 0$ or $b = 0$ (and so have continuous symmetry).

Proposition 5: $(\mathbb{Z}_3)$ The only associative 3-folds in $\mathbb{R}^7$ whose second fundamental forms have type $p = a\, {z_1}^5 + 5ib\, {z_1}^2 {z_2}^3$ are actually special Lagrangian in a $\mathbb{R}^6 \subset \mathbb{R}^7$. (These were classified (B—) 2000.)

Proposition 6: $(\mathbb{Z}_2)$ In progress.