Algebraically Constrained Special Holonomy Metrics and Second-order Associative 3-folds
A progress report

Robert L. Bryant — Duke University

The Simons Center for Geometry and Physics

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Part I: Curvature-Constrained Special Holonomy

$(M^n, g)$ Riemannian with holonomy $H \subset O(n)$, with Lie algebra $\mathfrak{h} \subset \mathfrak{so}(n)$. 

The structure equations on the $H$-bundle $B \to M$:

$$d\eta = -\theta \wedge \eta \quad \text{and} \quad d\theta = -\theta \wedge \theta + R(\eta \wedge \eta)$$

$\eta: TB \to \mathbb{R}^n$, $\theta: TB \to \mathfrak{h}$, and $R: B \to K(\mathfrak{h})$ is the curvature function, where $K(\mathfrak{h})$ is the $H$-representation $0 \to K(\mathfrak{h}) \to S^2(\mathfrak{h}) \to \Lambda^4(\mathbb{R}^n)$.

Second Bianchi:

$$dR = -\theta.$$ 

$R': B \to K(1)(\mathfrak{h}) \subset \text{Hom}(\mathbb{R}^n, K(\mathfrak{h}))$ represents the covariant derivative of the curvature.
Part I: Curvature-Constrained Special Holonomy

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The **structure equations** on the \(H\)-bundle \(B \to M\):

\[
d\eta = -\theta \wedge \eta \quad \text{and} \quad d\theta = -\theta \wedge \theta + R(\eta \wedge \eta).
\]

\(\eta : TB \to \mathbb{R}^n\), \(\theta : TB \to \mathfrak{h}\), and \(R : B \to K(\mathfrak{h})\) is the **curvature function**, where \(K(\mathfrak{h})\) is the \(H\)-representation

\[
0 \longrightarrow K(\mathfrak{h}) \longrightarrow S^2(\mathfrak{h}) \overset{\wedge}{\longrightarrow} \Lambda^4(\mathbb{R}^n).
\]
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The structure equations on the \(H\)-bundle \(B \to M\):

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\[
0 \to K(\mathfrak{h}) \to S^2(\mathfrak{h}) \overset{\wedge}{\to} \Lambda^4(\mathbb{R}^n).
\]

Second Bianchi: \(dR = -\theta.R + R'(\eta)\). where

\[
R' : B \to K^{(1)}(\mathfrak{h}) \subset \text{Hom}(\mathbb{R}^n, K(\mathfrak{h}))
\]

represents the covariant derivative of the curvature.
Example: \( \text{SU}(2) \subset \text{SO}(4) \)

\[
\begin{pmatrix}
\text{d}\eta_0 \\
\text{d}\eta_1 \\
\text{d}\eta_2 \\
\text{d}\eta_3
\end{pmatrix}
= -\begin{pmatrix}
0 & \theta_1 & \theta_2 & \theta_3 \\
-\theta_1 & 0 & -\theta_3 & \theta_2 \\
-\theta_2 & \theta_3 & 0 & -\theta_1 \\
-\theta_3 & -\theta_2 & \theta_1 & 0
\end{pmatrix}
\wedge
\begin{pmatrix}
\eta_0 \\
\eta_1 \\
\eta_2 \\
\eta_3
\end{pmatrix}
\]
Example: \( \text{SU}(2) \subset \text{SO}(4) \)

\[
\begin{pmatrix}
\frac{\mathrm{d}\eta_0}{\mathrm{d}t} \\
\frac{\mathrm{d}\eta_1}{\mathrm{d}t} \\
\frac{\mathrm{d}\eta_2}{\mathrm{d}t} \\
\frac{\mathrm{d}\eta_3}{\mathrm{d}t}
\end{pmatrix}
=
-\begin{pmatrix}
0 & \theta_1 & \theta_2 & \theta_3 \\
-\theta_1 & 0 & -\theta_3 & \theta_2 \\
-\theta_2 & \theta_3 & 0 & -\theta_1 \\
-\theta_3 & -\theta_2 & \theta_1 & 0
\end{pmatrix}
\wedge
\begin{pmatrix}
\eta_0 \\
\eta_1 \\
\eta_2 \\
\eta_3
\end{pmatrix}
\]

\[
\begin{pmatrix}
\frac{\mathrm{d}\theta_1}{\mathrm{d}t} \\
\frac{\mathrm{d}\theta_2}{\mathrm{d}t} \\
\frac{\mathrm{d}\theta_3}{\mathrm{d}t}
\end{pmatrix}
=
-\begin{pmatrix}
2 \theta_2 \wedge \theta_3 \\
2 \theta_3 \wedge \theta_1 \\
2 \theta_1 \wedge \theta_2
\end{pmatrix}
+
\begin{pmatrix}
R_{11} & R_{12} & R_{13} \\
R_{21} & R_{22} & R_{23} \\
R_{31} & R_{32} & R_{33}
\end{pmatrix}
\begin{pmatrix}
\eta_0 \wedge \eta_1 - \eta_2 \wedge \eta_3 \\
\eta_0 \wedge \eta_2 - \eta_3 \wedge \eta_1 \\
\eta_0 \wedge \eta_3 - \eta_1 \wedge \eta_2
\end{pmatrix},
\]

where \( R_{ij} = R_{ji} \) with \( R_{11} + R_{22} + R_{33} = 0 \).
Example: \( SU(2) \subset SO(4) \)

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\begin{pmatrix}
    d\eta_0 \\
    d\eta_1 \\
    d\eta_2 \\
    d\eta_3
\end{pmatrix}
= - \begin{pmatrix}
    0 & \theta_1 & \theta_2 & \theta_3 \\
    -\theta_1 & 0 & -\theta_3 & \theta_2 \\
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\end{pmatrix}
\wedge
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    \eta_1 \\
    \eta_2 \\
    \eta_3
\end{pmatrix}
\]

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\begin{pmatrix}
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    d\theta_3
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\end{pmatrix}
\begin{pmatrix}
    \eta_0 \wedge \eta_1 - \eta_2 \wedge \eta_3 \\
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    \eta_0 \wedge \eta_3 - \eta_1 \wedge \eta_2
\end{pmatrix},
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where \( R_{ij} = R_{ji} \) with \( R_{11} + R_{22} + R_{33} = 0 \).

\[
K(\mathfrak{su}(2)) = S_0^2(\mathbb{R}^3) \sim \mathbb{R}^5 \quad \text{and} \quad K^{(1)}(\mathfrak{su}(2)) \sim \mathbb{C}^6 \sim S^5(\mathbb{C}^2)
\]
Example: $\text{SU}(2) \subset \text{SO}(4)$

\[
\begin{pmatrix}
 d\eta_0 \\
 d\eta_1 \\
 d\eta_2 \\
 d\eta_3
\end{pmatrix}
= -
\begin{pmatrix}
 0 & \theta_1 & \theta_2 & \theta_3 \\
 -\theta_1 & 0 & -\theta_3 & \theta_2 \\
 -\theta_2 & \theta_3 & 0 & -\theta_1 \\
 -\theta_3 & -\theta_2 & \theta_1 & 0
\end{pmatrix}
\wedge
\begin{pmatrix}
 \eta_0 \\
 \eta_1 \\
 \eta_2 \\
 \eta_3
\end{pmatrix}

\begin{pmatrix}
 d\theta_1 \\
 d\theta_2 \\
 d\theta_3
\end{pmatrix}
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 2\theta_2 \wedge \theta_3 \\
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\end{pmatrix}
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\begin{pmatrix}
 R_{11} & R_{12} & R_{13} \\
 R_{21} & R_{22} & R_{23} \\
 R_{31} & R_{32} & R_{33}
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 \eta_0 \wedge \eta_1 - \eta_2 \wedge \eta_3 \\
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\end{pmatrix},
\]

where $R_{ij} = R_{ji}$ with $R_{11} + R_{22} + R_{33} = 0$.

\[
K(\mathfrak{su}(2)) = S_0^2(\mathbb{R}^3) \cong \mathbb{R}^5 \quad \text{and} \quad K^{(1)}(\mathfrak{su}(2)) \cong \mathbb{C}^6 \cong S^5(\mathbb{C}^2)
\]

É. Cartan (1926): $\text{SU}(2)$-holonomy depends on 2 functions of 3 variables.
Basic holonomy problem: For a given subgroup $H \subset \text{SO}(n)$ how to classify, up to local diffeomorphism, the ‘solutions’ to the structure equations

$$d\eta = -\theta \wedge \eta$$
$$d\theta = -\theta \wedge \theta + R(\eta \wedge \eta)$$

$(\eta, \theta) : TB \to \mathbb{R}^n \oplus \mathfrak{h}$ is a coframing and $R : B \to K(\mathfrak{h})$. 
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**Algebraically special solutions:** $H$ does not act transitively on $K(\mathfrak{h})$. A geometrically natural condition on solutions is to require that $R : B \rightarrow K(\mathfrak{h})$ take values in an $H$-invariant subset $A \subset K(\mathfrak{h})$. 

\[\sigma_3(\mathfrak{R})^2 + 4\sigma_2(\mathfrak{R})^3 \leq 0.\]
**Basic holonomy problem:** For a given subgroup $H \subset \text{SO}(n)$ how to classify, up to local diffeomorphism, the ‘solutions’ to the structure equations

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**Example:** $H = \text{SU}(2) = \text{Spin}(3) \subset \text{SO}(4)$ acts on $K(\mathfrak{su}(2)) = S_0^2(\mathbb{R}^3)$ preserving the symmetric functions of the eigenvalues of $R \in S_0^2(\mathbb{R}^3)$. Specifying a relation between $\sigma_2(R)$ and $\sigma_3(R)$ defines such an invariant subset $A \subset S_0^2(\mathbb{R}^3)$.

$$\sigma_3(R)^2 + \frac{4}{27} \sigma_2(R)^3 \leq 0.$$
Cases of interest in special holonomy

1. $\text{SU}(n) \subset \text{SO}(2n)$

   \[ K(\mathfrak{su}(n)) = S_{0}^{2:2}(\mathbb{C}^{n}) \]
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   Of particular interest: The cases $n = 2$ and $n = 3$ (because of the connections with string theory and nearly Kähler geometry).
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1. $\text{SU}(n) \subset \text{SO}(2n)$

$$K(\mathfrak{su}(n)) = S^{2,2}_0(\mathbb{C}^n)$$

Of particular interest: The cases $n = 2$ and $n = 3$ (because of the connections with string theory and nearly Kähler geometry).

2. $\text{G}_2 \subset \text{SO}(7)$

$$K(\mathfrak{g}_2) \simeq V^{0,2}(\mathfrak{g}_2) \simeq \mathbb{R}^{77}.$$
Cases of interest in special holonomy

1. $SU(n) \subset SO(2n)$
   \[ K(\mathfrak{su}(n)) = S_0^{2,2}(\mathbb{C}^n) \]
   Of particular interest: The cases $n = 2$ and $n = 3$ (because of the connections with string theory and nearly Kähler geometry).

2. $G_2 \subset SO(7)$
   \[ K(\mathfrak{g}_2) \simeq V^{0,2}(\mathfrak{g}_2) \simeq \mathbb{R}^{77}. \]

3. $Spin(7) \subset SO(8)$
   \[ K(\mathfrak{so}(7)) \simeq V^{0,2,0}(\mathfrak{so}(7)) \simeq \mathbb{R}^{168}. \]
Example: The structure equations for SU(2)-holonomy

\[
\begin{pmatrix}
    d\eta_0 \\
    d\eta_1 \\
    d\eta_2 \\
    d\eta_3
\end{pmatrix} = -\begin{pmatrix}
    0 & \theta_1 & \theta_2 & \theta_3 \\
    -\theta_1 & 0 & -\theta_3 & \theta_2 \\
    -\theta_2 & \theta_3 & 0 & -\theta_1 \\
    -\theta_3 & -\theta_2 & \theta_1 & 0
\end{pmatrix} \wedge \begin{pmatrix}
    \eta_0 \\
    \eta_1 \\
    \eta_2 \\
    \eta_3
\end{pmatrix}
\]

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\begin{pmatrix}
    d\theta_1 \\
    d\theta_2 \\
    d\theta_3
\end{pmatrix} = -\begin{pmatrix}
    2 \theta_2 \wedge \theta_3 \\
    2 \theta_3 \wedge \theta_1 \\
    2 \theta_1 \wedge \theta_2
\end{pmatrix} + \begin{pmatrix}
    R_{11} & R_{12} & R_{13} \\
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where \( R_{ij} = R_{ji} \) with \( R_{11} + R_{22} + R_{33} = 0 \).
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\begin{pmatrix}
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\end{pmatrix}
= -
\begin{pmatrix}
0 & \theta_1 & \theta_2 & \theta_3 \\
-\theta_1 & 0 & -\theta_3 & \theta_2 \\
-\theta_2 & \theta_3 & 0 & -\theta_1 \\
-\theta_3 & -\theta_2 & \theta_1 & 0 
\end{pmatrix}
\wedge
\begin{pmatrix}
\eta_0 \\ \eta_1 \\ \eta_2 \\ \eta_3 
\end{pmatrix}
\]  

\[
\begin{pmatrix}
\, d\theta_1 \\ d\theta_2 \\ d\theta_3 
\end{pmatrix}
= -
\begin{pmatrix}
2\theta_2\wedge\theta_3 \\ 2\theta_3\wedge\theta_1 \\ 2\theta_1\wedge\theta_2 
\end{pmatrix}
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\begin{pmatrix}
R_{11} & R_{12} & R_{13} \\
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R_{31} & R_{32} & R_{33} 
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\begin{pmatrix}
\eta_0\wedge\eta_1 - \eta_2\wedge\eta_3 \\
\eta_0\wedge\eta_2 - \eta_3\wedge\eta_1 \\
\eta_0\wedge\eta_3 - \eta_1\wedge\eta_2 
\end{pmatrix},
\]  

where \( R_{ij} = R_{ji} \) with \( R_{11} + R_{22} + R_{33} = 0 \).

We have \( A \simeq K(\mathfrak{su}(2)) \simeq \mathbb{R}^5 \) with

\[
( s_1, s_2, s_3, s_4, s_5, s_6, s_7 ) = (0, 3, 2, 0, 0, 0, 0).
\]

and \( \dim A^{(1)} = \dim K(\mathfrak{su}(2))^{(1)} = 12 = 2s_2 + 3s_3 \), so it’s involutive.
Unfortunately, the systems we need to study are not always involutive, and one must prolong the structure equations.
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**Example:** The SU(2) structure equations in which $R : B \rightarrow S^2_0(\mathbb{R}^3)$ has a double eigenvalue everywhere are not involutive:

\[
\begin{pmatrix}
\frac{\partial \eta_0}{\partial t} \\
\frac{\partial \eta_1}{\partial t} \\
\frac{\partial \eta_2}{\partial t} \\
\frac{\partial \eta_3}{\partial t}
\end{pmatrix} = -\begin{pmatrix}
0 & \theta_1 & \theta_2 & \theta_3 \\
-\theta_1 & 0 & -\theta_3 & \theta_2 \\
-\theta_2 & \theta_3 & 0 & -\theta_1 \\
-\theta_3 & -\theta_2 & \theta_1 & 0
\end{pmatrix} \wedge \begin{pmatrix}
\eta_0 \\
\eta_1 \\
\eta_2 \\
\eta_3
\end{pmatrix}
\]

\[
\begin{pmatrix}
\frac{\partial \theta_1}{\partial t} \\
\frac{\partial \theta_2}{\partial t} \\
\frac{\partial \theta_3}{\partial t}
\end{pmatrix} = -\begin{pmatrix}
2 \theta_2 \wedge \theta_3 \\
2 \theta_3 \wedge \theta_1 \\
2 \theta_1 \wedge \theta_2
\end{pmatrix} + \begin{pmatrix}
-2r^3 & 0 & 0 \\
0 & r^3 & 0 \\
0 & 0 & r^3
\end{pmatrix} \begin{pmatrix}
\eta_0 \wedge \eta_1 - \eta_2 \wedge \eta_3 \\
\eta_0 \wedge \eta_2 - \eta_3 \wedge \eta_1 \\
\eta_0 \wedge \eta_3 - \eta_1 \wedge \eta_2
\end{pmatrix},
\]
Applying \( d^2 = 0 \) to the equations

\[
\begin{pmatrix}
    d\eta_0 \\
    d\eta_1 \\
    d\eta_2 \\
    d\eta_3
\end{pmatrix} = - \begin{pmatrix}
    0 & \theta_1 & \theta_2 & \theta_3 \\
    -\theta_1 & 0 & -\theta_3 & \theta_2 \\
    -\theta_2 & \theta_3 & 0 & -\theta_1 \\
    -\theta_3 & -\theta_2 & \theta_1 & 0
\end{pmatrix} \wedge \begin{pmatrix}
    \eta_0 \\
    \eta_1 \\
    \eta_2 \\
    \eta_3
\end{pmatrix}
\]

\[
\begin{pmatrix}
    d\theta_1 \\
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\end{pmatrix} = - \begin{pmatrix}
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    2 \theta_3 \wedge \theta_1 \\
    2 \theta_1 \wedge \theta_2
\end{pmatrix} + \begin{pmatrix}
    -2r^3 & 0 & 0 \\
    0 & r^3 & 0 \\
    0 & 0 & r^3
\end{pmatrix} \begin{pmatrix}
    \eta_0 \wedge \eta_1 - \eta_2 \wedge \eta_3 \\
    \eta_0 \wedge \eta_2 - \eta_3 \wedge \eta_1 \\
    \eta_0 \wedge \eta_3 - \eta_1 \wedge \eta_2
\end{pmatrix},
\]

with \( r \neq 0 \) implies that there exist \( u_0, u_1, u_1, u_3 \) for which

\[
dr = 4r \left( u_0 \eta_0 + u_1 \eta_1 + u_2 \eta_2 + u_3 \eta_3 \right)
\]

\[
\theta_2 = 2 \left( -u_2 \eta_0 - u_3 \eta_1 + u_0 \eta_2 + u_1 \eta_3 \right)
\]

\[
\theta_3 = 2 \left( -u_3 \eta_0 + u_2 \eta_1 - u_1 \eta_2 + u_0 \eta_3 \right)
\]
Applying $d^2 = 0$ to the equations

$$\begin{pmatrix} d\eta_0 \\ d\eta_1 \\ d\eta_2 \\ d\eta_3 \end{pmatrix} = - \begin{pmatrix} 0 & \theta_1 & \theta_2 & \theta_3 \\ -\theta_1 & 0 & -\theta_3 & \theta_2 \\ -\theta_2 & \theta_3 & 0 & -\theta_1 \\ -\theta_3 & -\theta_2 & \theta_1 & 0 \end{pmatrix} \wedge \begin{pmatrix} \eta_0 \\ \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix}$$

$$\begin{pmatrix} d\theta_1 \\ d\theta_2 \\ d\theta_3 \end{pmatrix} = - \begin{pmatrix} 2\theta_2^\wedge \theta_3 \\ 2\theta_3^\wedge \theta_1 \\ 2\theta_1^\wedge \theta_2 \end{pmatrix} + \begin{pmatrix} -2r^3 & 0 & 0 \\ 0 & r^3 & 0 \\ 0 & 0 & r^3 \end{pmatrix} \begin{pmatrix} \eta_0^\wedge \eta_1 - \eta_2^\wedge \eta_3 \\ \eta_0^\wedge \eta_2 - \eta_3^\wedge \eta_1 \\ \eta_0^\wedge \eta_3 - \eta_1^\wedge \eta_2 \end{pmatrix},$$

with $r \neq 0$ implies that there exist $u_0, u_1, u_1, u_3$ for which

$$dr = 4r \left( u_0 \eta_0 + u_1 \eta_1 + u_2 \eta_2 + u_3 \eta_3 \right)$$

$$\theta_2 = 2 \left( -u_2 \eta_0 - u_3 \eta_1 + u_0 \eta_2 + u_1 \eta_3 \right)$$

$$\theta_3 = 2 \left( -u_3 \eta_0 + u_2 \eta_1 - u_1 \eta_2 + u_0 \eta_3 \right)$$

These are structure equations for a coframing $(\eta_0, \eta_1, \eta_2, \eta_3, \theta_1)$ with coefficients $(r, u_0, u_1, u_2, u_3)$ that still are not involutive.
Differentiating the structure equations again yields relations of the form

\[ \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} = U(r, u_0, u_1, u_2, u_3, v_1, v_2, v_3) \begin{pmatrix} \theta_1 \\ \eta_0 \\ \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} \]

where \( U(\cdot) \) is a matrix depending on three new parameters \( v_1, v_2, v_3 \).
Differentiating the structure equations again yields relations of the form

\[
d \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} = U(r, u_0, u_1, u_2, u_3, v_1, v_2, v_3) \begin{pmatrix} \theta_1 \\ \eta_0 \\ \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix}
\]

where \( U(\cdot) \) is a matrix depending on three new parameters \( v_1, v_2, v_3 \).

Differentiating these equations gives relations of the form

\[
d \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = V(r, u_0, u_1, u_2, u_3, v_1, v_2, v_3) \begin{pmatrix} \theta_1 \\ \eta_0 \\ \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix}
\]
Differentiating the structure equations again yields relations of the form

\[
d \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} = U(r, u_0, u_1, u_2, u_3, v_1, v_2, v_3) \begin{pmatrix} \theta_1 \\ \eta_0 \\ \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix}
\]

where \(U(\cdot)\) is a matrix depending on three new parameters \(v_1, v_2, v_3\). Differentiating these equations gives relations of the form

\[
d \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = V(r, u_0, u_1, u_2, u_3, v_1, v_2, v_3) \begin{pmatrix} \theta_1 \\ \eta_0 \\ \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix}
\]

Differentiating these last relations yields no more relations. Coupled with

\[
dr = 4r (u_0 \eta_0 + u_1 \eta_1 + u_2 \eta_2 + u_3 \eta_3)
\]

This gives 8 ‘independent’ coefficients in the structure equations for which \(d^2 = 0\) is an identity.
Classical Holonomy (no curvature restrictions)

1. $H = SU(m) \subset SO(2m)$: $s^2m - 1 = 2$ is the last nonzero character. (also works for the nearly Kähler case when $m = 3$)

2. $H = G_2 \subset SO(7)$: $s^6 = 6$ is the last nonzero character. (also works for 'nearly-G_2' structures on $M^7$)

3. $H = \text{Spin}(7) \subset SO(8)$: $s^7 = 12$ is the last nonzero character.
Classical Holonomy (no curvature restrictions)

1. $H = \text{SU}(m) \subset \text{SO}(2m)$: $s_{2m-1} = 2$ is last nonzero character.
   (also works for the nearly Kähler case when $m = 3$)
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**Classical Holonomy (no curvature restrictions)**

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3. $H = \text{Spin}(7) \subset \text{SO}(8)$: $s_7 = 12$ is last nonzero character.
Curvature restrictions in the $\text{SU}(2) \subset \text{SO}(4)$ case

The $\text{SU}(2)$-invariants on $K(\mathfrak{su}(2)) \simeq S_0^2(\mathbb{R}^3) \simeq \mathbb{R}^5$ are generated by $\sigma_2, \sigma_3 : S_0^2(\mathbb{R}^3) \to \mathbb{R}$, satisfying

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   Not involutive. Prolongation show that solutions only exist in the trivial case $R = 0$.

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   This is the 'double eigenvalue case', with nontrivial stabilizer $S^1 \subset \text{SU}(2)$.
   Not involutive, but prolongation yields a 2-parameter family of solutions, not all of which are complete, but some are.
Curvatures in $K(\mathfrak{h})$ with nontrivial $H$-stabilizers

Classifying the general $H$-invariant $A \subset K(\mathfrak{h})$ for which the corresponding $H$-structures have nontrivial solutions is probably intractable.
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This is the project that I have been engaged in.
**Table:** Stabilized curvatures for subgroups of SU(3) and Generality

<table>
<thead>
<tr>
<th>$G$</th>
<th>$\dim(K(\mathfrak{su}(3)))^G$</th>
<th>$G$-splitting of $\mathbb{C}^3$</th>
<th>Generality</th>
</tr>
</thead>
<tbody>
<tr>
<td>U(2)</td>
<td>1</td>
<td>$\mathbb{C} \oplus \mathbb{C}^2$</td>
<td>1 const. (known)</td>
</tr>
<tr>
<td>SU(2)</td>
<td>1</td>
<td>$\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{C}^2$</td>
<td>1 const. (known)</td>
</tr>
<tr>
<td>SO(3)</td>
<td>1</td>
<td>$\mathbb{R}^3 \oplus \mathbb{R}^3$</td>
<td>does not exist</td>
</tr>
<tr>
<td>$\mathbb{T}^2$</td>
<td>3</td>
<td>$\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$</td>
<td>8 constants</td>
</tr>
<tr>
<td>$S^1(p/q)\dagger$</td>
<td>3</td>
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<td>8 constants</td>
</tr>
<tr>
<td>$S^1(0)$</td>
<td>5</td>
<td>$\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{C} \oplus \mathbb{C}$</td>
<td>$s_1 = 2$ ??</td>
</tr>
<tr>
<td>$S^1(1)$</td>
<td>7</td>
<td>$\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$</td>
<td>$s_1 = 4$ ??</td>
</tr>
</tbody>
</table>

$\dagger$ $p/q \neq 0, 1$, where $S^1(p/q)$ is the circle of diagonal matrices $	ext{diag}(e^{ipt}, e^{iqt}, e^{-i(p+q)t})$. 
### Table: Stabilized curvatures of subgroups of $G_2$

<table>
<thead>
<tr>
<th>$G$</th>
<th>$\dim(K(g_2))^G$</th>
<th>$G$-splitting of $\mathbb{R}^7$</th>
<th>Generality</th>
</tr>
</thead>
<tbody>
<tr>
<td>SU(3)</td>
<td>0</td>
<td>$\mathbb{R}^1 \oplus \mathbb{C}^3$</td>
<td>only flat</td>
</tr>
<tr>
<td>SO(4)</td>
<td>1</td>
<td>$\mathbb{R}^3 \oplus \mathbb{R}^4$</td>
<td>only $\Lambda^2_+ (S^4)$</td>
</tr>
<tr>
<td>$U(2)_1$</td>
<td>2</td>
<td>$\mathbb{R}^3 \oplus \mathbb{R}^4$</td>
<td>only $\Lambda^2_+ (\mathbb{C}P^2)$</td>
</tr>
<tr>
<td>$U(2)_2$</td>
<td>2</td>
<td>$\mathbb{R}^1 \oplus \mathbb{R}^2 \oplus \mathbb{R}^4$</td>
<td>DNE</td>
</tr>
<tr>
<td>$\mathbb{T}^2$</td>
<td>5</td>
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<td>'only' consts.</td>
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<td>SU(2)$_1$</td>
<td>3</td>
<td>$\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{C}^2$</td>
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</tr>
<tr>
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<td>6</td>
<td>$\mathbb{R}^3 \oplus \mathbb{R}^4$</td>
<td>???</td>
</tr>
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<tr>
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<td>9</td>
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<td>$s_1 = 4$</td>
</tr>
<tr>
<td>$S^1(0)$</td>
<td>13</td>
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<td>???</td>
</tr>
</tbody>
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$\dagger$ $p/q \neq 0, \frac{1}{2}, 1$
Part II: Second order associative 3-folds

Associative submanifolds $M^3 \subset \mathbb{R}^n$ can be defined by the condition that their tangent spaces belong to the 8-dimensional associative Grassmannian

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one can speak of $\Sigma$-manifolds $M^m \subset \mathbb{R}^n$, whose tangent spaces belong to $\Sigma$. The Gauss map $\gamma_M : M \to \Sigma$ given by $\gamma_M(x) = T_x M \in \Sigma$ has a derivative

$$\gamma'_M(x) : T_x M \to T_{\gamma_M(x)} \Sigma \subset N_{\gamma_M(x)} \otimes T^*_x M \cong \mathbb{R}^{n-m} \otimes (\mathbb{R}^m)^*$$

that satisfies (because of symmetry of second partials),

$$\mathbb{I}_x = \gamma'_M(x) \in T_{\gamma_M(x)} \Sigma \otimes T^*_x M \cap (N_{\gamma_M(x)} \otimes S^2(T^*_x M)).$$
All of the spaces $\mathbb{R}^m$, $(\mathbb{R}^m)^\perp = \mathbb{R}^{n-m}$, and $T_{\mathbb{R}^m}\Sigma \simeq \mathfrak{g}/\mathfrak{h}$ are $H$-modules, and so is the space

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**Example:** Special Lagrangian submanifolds. $G = SU(m)$, $H = SO(m)$. Then

$$\Pi(g, h) = \left( S^2_0(\mathbb{R}^m) \otimes \mathbb{R}^m \right) \cap \left( \mathbb{R}^m \otimes S^2(\mathbb{R}^m) \right) = S^3_0(\mathbb{R}^m) \simeq \mathbb{R}^7.$$
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In 2000, I analyzed the case $m = 3$ of special Lagrangian 3-folds whose second fundamental forms (harmonic cubic forms) had a nontrivial symmetry, and found many integrable cases. (Second order families of special Lagrangian 3-folds, arXiv:math/0007128.)
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My student, Marianty Ionel, did a similar analysis of the case $m = 4$ in 2002 and also found many integrable cases.
**Associative 3-folds:** $G = G_2, H = SO(4)$. ($\text{Spin}(4) = \text{Sp}(1) \times \text{Sp}(1).$) 

\[ g/\mathfrak{h} \cong V^\mathbb{R}_{3,1} = S^3(V_{1,0}) \otimes_{\mathbb{H}} V_{0,1} \cong \mathbb{R}^8, \]
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Thus, one can think of the second fundamental form of an associative 3-fold as a **quintic polynomial** in two complex variables

\[ p(z_1, z_2) = a_5 z_1^5 + 5a_4 z_1^4 z_2 + 10a_3 z_1^3 z_2^2 + 10a_2 z_1^2 z_2^3 + 5a_1 z_1 z_2^4 + a_0 z_2^5. \]
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One can interpret a second fundamental form of an associative 3-manifold (up to real scalar multiples) as a degree \( \leq 5 \) rational mapping

\[ P : \mathbb{CP}^1 \to \mathbb{CP}^1, \quad P(z_1, z_2) = [p(z_1, z_2), p(-\overline{z_2}, \overline{z_1})]. \]

up to (independent) isometric rotations in the domain and range 2-spheres.
By comparison, for co-associative submanifolds $M^4 \subset \mathbb{R}^7$, the coassociative Grassmannian is also $G_2/\text{SO}(4) \subset \text{Gr}^+_4(\mathbb{R}^7)$, so again,

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However, the second fundamental form space is different:

$$\mathbb{I}(\mathfrak{g}, \mathfrak{h}) = V^\mathbb{R}_{4,2} = S^4(V_{1,0})^\mathbb{R} \otimes_{\mathbb{R}} S^2(V_{0,1})^\mathbb{R} = \mathbb{R}^5 \otimes \mathbb{R}^3 \simeq \mathbb{R}^{15},$$
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For Cayley submanifolds $M^4 \subset \mathbb{R}^8$, the Cayley Grassmannian is $\text{Spin}(7)/H \subset \text{Gr}^+(\mathbb{R}^8)$, where $H = (\text{Sp}(1) \times \text{Sp}(1) \times \text{Sp}(1))/\mathbb{Z}_2$. Then

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while the second fundamental form space turns out to be

$$\Pi(g, \mathfrak{h}) = V_{2,3,1}^\mathbb{R} \simeq \mathbb{R}^{24},$$
The classification of the stabilizer types in the associative case can now be worked out.

Proposition 0:
If \( p(z_1, z_2) \) represents an associative second fundamental form with nontrivial stabilizer in \( H = \text{SO}(4) \), then \( p \) is in the orbit of one of the following types (where \( a, b, u, v \) are real)

1. \( p = a z_1^5 \), \( \text{Stab}(p) \cong \text{SO}(2) \)
2. \( p = 5a z_1^4 z_2 \), \( \text{Stab}(p) \cong \text{SO}(2) \)
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4. \( p = a z_1^5 + 5b z_1 z_2^4 \), \( \text{Stab}(p) \cong \mathbb{Z}_4 \)
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(N.B. There are some inequalities among \( a, b, u, v \) in the above cases in order to ensure no larger symmetry. Also, the three circles in Cases 1–3 are not conjugate in \( \text{SO}(4) \).)
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The classification of the stabilizer types in the associative case can now be worked out.

**Proposition 0:** If \( p(z_1, z_2) \) represents an associative second fundamental form with nontrivial stabilizer in \( H = \text{SO}(4) \), then \( p \) is in the orbit of one of the following types (where \( a, b, u, v \) are real)

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(N.B. There are some inequalities among \( a, b, u, v \) in the above cases in order to ensure no larger symmetry. Also, the three circles in Cases 1–3 are not conjugate in \( \text{SO}(4) \).)
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2. \( p = 5 a z_1^4 z_2 \), \( \text{Stab}(p) \simeq \text{SO}(2) \)
3. \( p = 10 a z_1^3 z_2^2 \), \( \text{Stab}(p) \simeq \text{SO}(2) \)
4. \( p = a z_1^5 + 5 b z_1 z_2^4 \), \( \text{Stab}(p) \simeq \mathbb{Z}_4 \)

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2. $p = 5az_1^4z_2$, $\text{Stab}(p) \simeq \text{SO}(2)$
3. $p = 10az_1^3z_2^2$, $\text{Stab}(p) \simeq \text{SO}(2)$
4. $p = az_1^5 + 5bz_1^2z_2^4$, $\text{Stab}(p) \simeq \mathbb{Z}_4$
5. $p = az_1^5 + 5ibz_1^2z_2^3$, $\text{Stab}(p) \simeq \mathbb{Z}_3$
6. $p = az_1^5 + 5bz_1^3z_2^3 + 5(u+iv)z_1z_2^4$, $\text{Stab}(p) \simeq \mathbb{Z}_2$
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Proposition 1: The associative 3-folds in $\mathbb{R}^7$ whose second fundamental forms have type $p = a z_1^5$

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**Remark:** The ruled associative 3-folds in $\mathbb{R}^7$ can be regarded as surfaces in $\Lambda(\mathbb{R}^7) \cong TS^6$, the space of lines in $\mathbb{R}^7$. There is a unique almost complex structure on $\Lambda(\mathbb{R}^7)$ such that these surfaces are the pseudoholomorphic curves in $\Lambda(\mathbb{R}^7)$. Thus, they locally depend on $s_1 = 12$ functions of 1 variable.
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Proposition 4: ($\mathbb{Z}_4$) The only associative 3-folds in $\mathbb{R}^7$ whose second fundamental forms have type $p = a z_1^5 + 5b z_1 z_2^4$ must actually have either $a = 0$ or $b = 0$ (and so have continuous symmetry).
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**Proposition 6:** ($\mathbb{Z}_2$) In progress.