Manifolds with holonomy Sp(n)Sp(1)

SC in SHGAP

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The list

$$SO(N)$$

$$U(\tfrac{N}{2}) \qquad\qquad Sp(\tfrac{N}{4})Sp(1)$$

$$SU(\tfrac{N}{2}) \qquad\qquad Sp(\tfrac{N}{4})$$

$$G_2 \quad (N=7)$$

$$Spin(7) \; (N=8)$$

All act transitively on S^{N-1} [Si]. In particular,

$$S^6 \cong \frac{G_2}{SU(3)}, \qquad S^7 \cong \frac{Spin(7)}{G_2}$$

have 'weak holonomy' reductions. The Ricci-flat cases are characterized by the existence of parallel spinors.

The quaternionic unitary case

Mathematical interest in $Sp(n)Sp(1)$ arises from
(i) its presence in the list,
(ii) its generalization of $SO(4) = Sp(1)Sp(1)$,
(iii) being the holonomy group of the projective space

$$\mathbb{HP}^n = \frac{Sp(n+1)}{Sp(n) \times Sp(1)} = \frac{Sp(n+1)/\mathbb{Z}_2}{Sp(n)Sp(1)}.$$

The reduction corresponds to splittings

$$\mathbb{H}^{n+1} = \mathbb{H}^n \oplus \mathbb{H}$$

$$\mathbb{C}^{2n+2} = \mathbb{C}^{2n} \oplus \mathbb{C}^2 = E \oplus H,$$

and

$$(T^*\mathbb{HP}^n)_c \cong E \otimes H \quad (= S_+ \otimes S_- \text{ if } n = 1).$$

Quaternion-kähler manifolds

Definition. A *QK manifold* is a Riemannian manifold of dimension $4n$, with $n \geqslant 2$, whose holonomy group H satisfies

$$Sp(n) \subsetneq H \subseteq Sp(n)Sp(1).$$

This rules out the hyperkähler case, though HK and QK manifolds share some properties. Any QK manifold has a parallel 4-form

$$\Omega = \sum_{i=1}^{3} \omega_i \wedge \omega_i,$$

where $\omega_1, \omega_2, \omega_3$ is a *local* triple of 2-forms.

Any QK curvature tensor R belongs to $S^2(\mathfrak{sp}(n) \oplus \mathfrak{sp}(1))$ and

$$R = R_{\mathrm{HK}} \oplus sR_0, \quad R_{\mathrm{HK}} \in S^4E \subset S^2\mathfrak{sp}(n).$$

So a QK manifold is 'nearly HK' and Einstein but can have $s < 0$.

The Wolf spaces

Given a compact simple Lie algebra $\mathfrak{g}$, choose a highest root subalgebra $\mathfrak{su}(2) = \mathfrak{sp}(1)$. Then

$$H = K\,Sp(1) = \{g \in G : \mathrm{Ad}(g)(\mathfrak{sp}(1)) = \mathfrak{sp}(1)\}.$$

If G is centreless, $K \subseteq Sp(n)$.

Wolf spaces of real dimension $4n$ include $\mathbb{HP}^n$ and

$$\mathbb{G}\mathrm{r}_2(\mathbb{C}^{n+2}) = \frac{SU(n+2)}{S(U(n) \times U(2))}$$
$$\mathbb{G}\mathrm{r}_4(\mathbb{R}^{n+4}) = \frac{SO(n+4)}{SO(n) \times SO(4)}.$$

There are exceptional ones with $n = 2, 7, 10, 16, 28$.

All compact QK homogeneous spaces arise in this way [A].

Twistor space

Let M be QK. The reduction to $Sp(n)Sp(1)$ equips each tangent space $T_m M$ with a 2-sphere

$$Z_m = \{aI_1 + bI_2 + cI_3 : a^2 + b^2 + c^2 = 1\}$$

of almost complex structures, a point of which corresponds to a reduction $H_m = \theta \oplus \tilde{\theta}$ and

$$(T_m^* M)_c = E \otimes (\theta \oplus \tilde{\theta}) = \Lambda^{1,0} \oplus \Lambda^{0,1}.$$

Thus, $Z \cong H/\mathbb{C}^*$ is a bundle over M with fibre $S^2 \cong \mathbb{CP}^1$.

Theorem. The tautological almost complex structure on Z determined by the horizontal (LC) distribution is integrable. Therefore Z is a complex manifold, generalizing the AHS construction in dimension 4.

Associated bundles

$$
\begin{array}{ccccc}
\mathscr{S}^{4n+3} & \hookrightarrow & \mathscr{U}^{4n+4} & & \\
\downarrow & & & \searrow \mu & \\
Z^{4n+2} & & \hookrightarrow & & V^{4n+3} \\
& \searrow & & \swarrow & \\
& & M^{4n},\ QK,\ s>0 & &
\end{array}
$$

The twistor space Z with fibre $\mathbb{CP}^1$ is a Kähler manifold.

V is the span of I_1, I_2, I_3, or $\omega_1, \omega_2, \omega_3$, with fibre $\mathbb{R}^3 = \mathfrak{sp}(1)$.

$\mathscr{U} = H/\mathbb{Z}_2$ has both HK and QK metrics, and $\mathscr{S}$ is 3-Sasakian.

Fano contact manifolds

When M^{4n} is a Wolf space,

$$Z = \frac{G}{K\,U(1)} \xrightarrow{\pi} \frac{G}{K\,Sp(1)} = M.$$

is an adjoint orbit in $\mathfrak{g}$, polarized by a holomorphic line bundle L. Each fibre $\pi^{-1}(m)$ is a rational curve $\mathbb{CP}^1$ with normal bundle $2n\mathcal{O}(1)$, whereas $L|_{\mathbb{CP}^1} \cong \mathcal{O}(2)$.

If M is a QK manifold, Z has a holomorphic contact structure $\theta \in H^0(Z, T^*Z \otimes L)$, so

$$0 \;\neq\; \theta \wedge (d\theta)^n \in H^0(Z, \mathcal{K} \otimes L^{n+1}),$$

and $\overline{\mathcal{K}} \cong L^{n+1}$. There is a short exact sequence

$$0 \to D \to TZ \xrightarrow{\theta} L \to 0$$

in which $D \cong L^{1/2} \otimes \pi^* E$ is horizontal, and E is an instanton.

Characterization of $\mathbf{HP}^n$

The twistor space $\mathbb{CP}^{2n+1}$ of $\mathbb{HP}^n$ has $L = \mathcal{O}(2)$, and is of course Fano of index $2n+2$.

If M is a QK manifold with $s > 0$ and the $Sp(n)Sp(n)$ structure lifts to $Sp(n) \times Sp(1)$, the same is true. It follows that Z is biholomorphic to $\mathbb{CP}^{2n+1}$ [KO].

Corollary. If $H^2(M, \mathbb{Z}_2) = 0$ then M is *isometric* to $\mathbb{HP}^n$.

In general, the twistor space Z of a QK manifold with positive scalar curvature $s > 0$ is Fano of index $n+1$. The big question is whether such a Fano contact manifold must be homogeneous. Yes, if $Z \to \mathbb{P}(H^0(Z, L)^*)$ is generically finite [Be].

The twistor dictionary

M QK, $s \neq 0$	Z complex contact
point	vertical rational curve
complex structure	holomorphic section
Killing field X	$s \in H^0(Z, L)$
Dirac operator	$\overline{\partial}$ on $\Lambda^{0,*} \otimes \mathcal{O}(-n)$
Fueter operator	$\overline{\partial}$ on $\Lambda^{0,1} \otimes \mathcal{O}(-3)$
$s > 0$	Z Kähler-Einstein
$s > 0$, compact	Z contact Fano
minimal 2-sphere	contact rational curve
$b_2(M) + 1$	$= b_2(Z)$

Rigidity

Let M^{4n} be a QK manifold with $s > 0$. The odd Betti numbers of M^{4n} are all zero.

Theorem 1 [LS,Wi]. If $b_2(M) > 0$ then M is isometric to $\mathbb{G}\mathrm{r}_2(\mathbb{C}^{n+2})$.

For, if $b_2(Z) > 1$ there exists a family of rational curves on Z transverse to the fibres, and a contraction $Z \longrightarrow \mathbb{CP}^{n+1}$ with fibres tangent to D. This forces $Z = \mathbb{P}(T^*\mathbb{CP}^{n+1})$.

Theorem 2 [LS,KMM]. For each n, there are finitely many complete QK manifolds with $s > 0$. (Three if $n = 2$.)

Theorem 3 [GS,A]. If $b_4 = 1$ and $3 \leqslant n \leqslant 6$ then $M \cong \mathbb{HP}^n$.

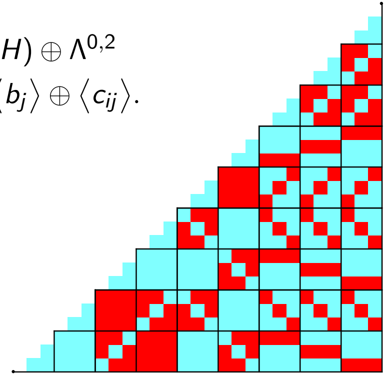
A curiosity: $E_6/SU(6)Sp(1)$

The configuration of 27 lines on a smooth cubic surface in $\mathbb{CP}^3$ can be described using

$$\mathfrak{e}_6 = \mathfrak{su}(6) \oplus \mathfrak{sp}(1) \oplus (E \otimes H), \quad E = \Lambda^{3,0}.$$

and choosing a basis of

$$\begin{aligned} \mathbb{C}^{27} &= (\Lambda^{1,0} \otimes H) \oplus \Lambda^{0,2} \\ &= \langle a_i \rangle \oplus \langle b_j \rangle \oplus \langle c_{ij} \rangle. \end{aligned}$$



QK reduction

Suppose that M^{4n} is a QK manifold with an isometric $U(1)$ action and Killing field X such that $\mathscr{L}_X\Omega \equiv 0$. Define a 2-form

$$\eta = \pi(dX^\flat) = \sum_{i=1}^{3} \eta_i\omega_i \in \Gamma(M, V)$$

and set $f = \frac{1}{2}\|\eta\|^2$. Then, up to constants,

$$df = X \lrcorner\, \eta, \qquad d\eta = X \lrcorner\, \Omega,$$

and η determines a holomorphic section $\widehat{\eta} \in H^0(Z, L)$.

If $s > 0$, $M \setminus \{f=0\}$ has an associated Kähler metric [Ha].

Stony Brook Theorem [GL, HKLR]. If $U(1)$ acts freely on $f^{-1}(0)$ then $f^{-1}(0)/U(1)$ has a natural QK structure.

An M-theory example

The diagonal action of $U(1) \subset Sp(3)$ on $\mathbb{H}^3$ has fixed point set $\mathbb{CP}^2$, and gives rise to an $SU(3)$-equivariant picture

$$\begin{array}{ccc} f^{-1}(0) = S^5 & \subset & \mathbb{HP}^2 \setminus \mathbb{CP}^2 \\ \downarrow & & \downarrow \\ \mathbb{CP}^2 & \longleftarrow & \Lambda^2_- T^*\mathbb{CP}^2 = \mathbb{X}, \end{array}$$

and $\mathbb{HP}^2/U(1) \approx S^7$ [AW].

Lemma [CMS]. The 3-form on $\mathbb{X}$ defining the BS metric is

$$\varphi = \mu(X \lrcorner \Omega) + \frac{\mu}{1048576}(\cot 2t)^2 \nabla X^\flat \wedge \alpha_1 + \lambda \alpha_{123},$$

$$\lambda = \frac{f^{3/2}\left(3\sqrt{(f-2)^2\left(64+(f-2)^2 f^2\right)} + f(f(3f-10)+4)+8\right)}{32(2-f)^{5/2}}, \quad \mu = -\frac{2048\sqrt{6}\left(\sqrt{(f-2)^2\left(64+(f-2)^2 f^2\right)} + f(f-2)^2\right)}{(2-f)^{3/2}}.$$

Nilpotency

Theorem [Sw]. For G compact simple,

$$f(\langle e_1, e_2, e_3\rangle) = B(e_1, [e_2, e_3])$$

is a Morse-Bott function on $\mathbb{G}\mathrm{r}_3(\mathfrak{g})$. Critical points are subalgebras. The unstable manifold determined by an $\mathfrak{su}(2)$ is QK with twistor space $\mathscr{U}/\mathbb{C}^*$, where $\mathscr{U}$ is the associated complex nilpotent orbit.

Example. $\mathfrak{su}(3)$ has two TDA's up to conjugacy: $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$.
(i) $\mathfrak{su}(2)$ gives rise to the Wolf space $\mathbb{CP}^2 = SU(3)/U(1)SU(2)$.
(ii) $\mathfrak{so}(3)$ gives an 8-dimensional incomplete QK manifold, covered by the total space of the rank 3 vector bundle

$$\mathbb{V} \longrightarrow \mathbb{L} = SU(3)/SO(3),$$

which can be identified with an open subset of $G_2/SO(4)$.

8-manifolds of cohomogeneity one

There are 3 compact Wolf spaces in dimension 8. Remarkably, all admit an action by $SU(3)$, with principal orbits $SU(3)/U(1)_{1,-1}$ and two ends chosen from

$$S^5, \qquad \mathbb{CP}^2, \qquad \mathbb{L} = \frac{SU(3)}{SO(3)}.$$

$\mathbb{G}\mathrm{r}_2(\mathbb{C}^4)$	$SU(4)/U(2)Sp(1)$	$\mathbb{CP}^2,\ \mathbb{CP}^2$
$\mathbb{HP}^2$	$Sp(3)/Sp(2)Sp(1)$	$\mathbb{CP}^2,\ S^5$
$G_2/SO(4)$	$G_2/SU(2)Sp(1)$	$\mathbb{CP}^2,\ \mathbb{L}$
$SU(3)$	$SU(3)^2/\Delta SU(3)$	$S^5,\ \mathbb{L}$

In the last case, the action is $P \mapsto \overline{Q}PQ^{-1}$. Mapping P to $\overline{P}P$ converts this to conjugation.

Corollary. The vector bundle $\mathbb{V}$ over $\mathbb{L}$ admits three distinct $SU(3)$-invariant parallel 4-forms with stabilizer $Sp(2)Sp(1)$.

The one corresponding to $G_2/SO(4)$ is

$$\frac{3\sin^2(t)\cos^2(t)}{t^2}\mathbf{bb}\boldsymbol{\beta} + \frac{\sqrt{3}\sin(2t)}{t}\mathbf{b}\tilde{\boldsymbol{\beta}} + \frac{\sin^2(t)\cos^2(t)}{t^2}\mathbf{a}\tilde{\boldsymbol{\beta}}\boldsymbol{\epsilon} - \frac{-5\sin(2t)+\sin(6t)+4t\cos(2t)}{128\sqrt{3}t^3}\boldsymbol{\gamma\epsilon\epsilon}$$
$$+\frac{\sin^4(t)(\cos(2t)+\cos(4t)+1)}{2\sqrt{3}t^4}\mathbf{bbb}\,\mathbf{a}\boldsymbol{\epsilon} + \frac{\sqrt{3}(2t\cos(2t)-\sin(2t))}{8t^3}\mathbf{b}\boldsymbol{\beta}\,\mathbf{a}\boldsymbol{\epsilon}$$
$$+\frac{3(2t\sin(4t)+\cos(4t)-1)}{4t^4}\mathbf{ab}\,\mathbf{ab}\boldsymbol{\beta} + \frac{\sin^2(t)(5t-6\sin(2t)-3\sin(4t)+t(13\cos(2t)+5\cos(4t)+\cos(6t)))}{96\sqrt{3}t^5}\mathbf{ab}\,\boldsymbol{\epsilon\epsilon\epsilon}$$
$$+\frac{\sin^3(2t)(\sin(2t)-2t\cos(2t))}{32t^6}\mathbf{abb}\,\mathbf{a}\boldsymbol{\epsilon\epsilon} - \frac{\sin^3(2t)\cos(2t)}{8t^3}\mathbf{a}\boldsymbol{\gamma}\,\mathbf{a}\boldsymbol{\gamma}$$

Closed 4-forms

If M has an $Sp(n)Sp(1)$-structure then

$$\begin{aligned} \nabla\Omega \in T^* \otimes (\mathfrak{sp}(n)+\mathfrak{sp}(1))^\perp &\cong E\,H \otimes \Lambda^2_0 E\, S^2 H \\ &\cong (E \oplus E' \oplus \Lambda^3_0 E) \otimes (H \oplus S^3 H) \end{aligned}$$

has 6 irreducible components if $n \geqslant 3$.

Lemma [Sw]. If $n \geqslant 3$ the condition $d\Omega = 0$ implies that $\nabla\Omega = 0$ and so the holonomy reduces.

If $n = 2$, one can have $d\Omega = 0$ without M being Einstein:

Theorem [CMS]. $G_2/SO(4)$ admits $SU(3)$ invariant closed non-parallel 4-forms with stabilizer $Sp(2)Sp(1)$.

Proof

$G_2/SO(4)$ does not admit a $U(1)$ commuting with $SU(3)$, but there is still a circle action and vector field X tangent to the orbits.

The stabilizer of an exterior form is always preserved by a *linear* deformation

$$\Omega \ \rightsquigarrow \ \Omega + N\cdot\Omega,$$

where $N \in \mathfrak{gl}(8,\mathbb{R})$ satisfies $N\cdot(N\cdot\Omega) = 0$.

In our case, we can take $N = f(t)dt \otimes X$ to preserve closure, where f is smooth and odd around the end points $t = 0, \frac{\pi}{4}$, so

$$\Omega \ \rightsquigarrow \ \Omega + f(t)dt \wedge (X \lrcorner\, \Omega), \quad 0 \leqslant t \leqslant \tfrac{\pi}{4}.$$

If $f \neq 0$, this gives non-Einstein metrics with scalar curvature

$$s = 64 - \tfrac{4}{3}(\tan 2t)^2 f(t)^2.$$

The EH formalism: Spinors

Let $\Delta_+ + \Delta_-$ be the spin representation of $Spin(4n)$. Then

$$\Delta_+ - \Delta_- = \Lambda_0^n(E - H) = \bigoplus_{p+q=n} (-1)^p R^{p,q},$$

where $R^{p,q} = \Lambda_0^p E \otimes S^q H$.

Corollary. M^{4n} is spin if n is even, so if $s > 0$ then $\widehat{A}_n = 0$.

The Dirac operator

$$\Gamma(M, \Delta_+ \otimes R^{p,q}) \longrightarrow \Gamma(M, \Delta_- \otimes R^{p,q})$$

has index $\int_M \mathrm{ch}(R^{p,q})\widehat{A}(M)$.

If $s > 0$ this vanishes if $p + q < n$. Used by [GMS]. . .

One also gets estimates on the dimension of the isometry group involving $h = c_2(H) \in H^4(M, \frac{1}{4}\mathbb{Z})$. (E.g. $d = 5 + 16h^2$ for $n = 2$.)

Example. Over $\mathbb{HP}^2$,

$$\Delta_+ - \Delta_- = \Lambda^2_0(E - H) = \Lambda^2_0 E - E\,H + S^2 H.$$

There are analogies with $Spin(7)$ structures $(1 - 8 + 7)$.

$E - H$ can't be a vector bundle because an inclusion $H \hookrightarrow E$ would define a nowhere zero section of $E \otimes H \cong T\mathbb{HP}^2$. Indeed, $E - H$ has rank 2, but

$$c(E - H) = c(\underline{\mathbb{C}}^6 - 2H) = (1 - h)^{-2} = 1 + 2h + 3h^2,$$

and $c_4 \neq 0$. By contrast,

$$c(\Lambda^2_0 E - H) = 1 + 3h.$$

This time the difference *is* a genuine vector bundle.

Horrocks' bundle

Theorem. There exists a rank 3 complex vector bundle V over $\mathbb{HP}^2$ with $c_2 = 3h$, and an $SU(3)$-connection with $F \in \Gamma(\mathfrak{sp}(2))$.

Recall that $E = \ker(p_1\colon\ \underline{\mathbb{C}}^6 \to H)$. Similarly,

$$\Lambda^2_0 E = \ker\left(\Lambda^2_0(\underline{\mathbb{C}}^6) \longrightarrow \underline{\mathbb{C}}^6 \wedge H\right).$$

Fix a reduction of $Sp(3)$ to $SU(3)$, giving $\mathbb{C}^6 = \Lambda^{1,0} \oplus \Lambda^{0,1}$ and

$$p_2\colon\ \Lambda^2_0(\underline{\mathbb{C}}^6) \longrightarrow \underline{\mathbb{C}}^6.$$

Then $p_1 \circ p_2$ has rank 2 everywhere.

The instanton connection on $V = \ker(p_1 \circ p_2)$ is induced from that on $\Lambda^2_0 E$ and ultimately E, using ADHM techniques, and there is a moduli space $SL(3,\mathbb{H})/SU(3) \to SL(3,\mathbb{H})/Sp(3)$.

A curiosity: $E_8/E_7Sp(1)$

The signature of an ADE Wolf space equals its rank: $b_{2n}^{+} = b_{2n} = r$.
M^{112} has 8 primitive cohomology classes $\sigma_k \in H^{4k}(M, \mathbb{R})$, and

$$H^{56}(M, \mathbb{R}) = \langle \sigma_k \cup h^{14-k} : k = 0, 3, 5, 6, 8, 9, 11, 14 \rangle,$$

exhibiting 'secondary Poincaré duality' about $k = 7$: