Collapse and special holonomy metrics

Mark Haskins

Imperial College London

joint with Lorenzo Foscolo and Johannes Nordström

Inaugural Meeting
Simons Collaboration on Special Holonomy in Geometry, Analysis and Physics
8 Sept 2016

Plan of talk

1. Codimension one collapse in families of hyperKähler (hK) metrics on the K3 surface after Foscolo

Plan of talk

1. Codimension one collapse in families of hyperKähler (hK) metrics on the K3 surface after Foscolo
2. Codimension one collapse in families of G_2-holonomy metrics
   - ALC G_2 manifolds
   - ALC spaces by cohomogeneity one methods
   - G_2 analogue of the Gibbons-Hawking ansatz.
   - ALC spaces by analytic methods
   - G_2-holonomy metrics on compact manifolds collapsing to Calabi-Yau 3-folds

Collapsing hK metrics on K3

Foscolo gave the following general construction of collapsing families of hK metrics on the K3 surface utilising ALF gravitational instantons.

Theorem (Foscolo 2016) Every collection of 8 ALF spaces of dihedral type $M_1, \ldots, M_8$ and n ALF spaces of cyclic type $N_1, \ldots, N_n$ satisfying

$$\sum_{j=1}^{8} \chi(M_j) + \sum_{i=1}^{n} \chi(N_i) = 24$$

appears as the collection of "bubbles" forming in a sequence of Kähler Ricci-flat metrics on the K3 surface collapsing to the flat orbifold $T^3/\mathbb{Z}_2$ with bounded curvature away from $n+8$ points.

ALF gravitational instantons

Defn: A gravitational instanton (M, g) is called **ALF** if there exists a compact set $K \subset M$, $R > 0$ and a finite group $\Gamma \subset O(3)$ acting freely on $\mathbb{S}^2$ such that $M \setminus K$ is the total space of a circle fibration over $(\mathbb{R}^3 \setminus \mathbb{B}_R)/\Gamma$ and the metric g is asymptotic to a Riemannian submersion

$$g = \pi^* g_{\mathbb{R}^3/\Gamma} + \theta^2 + O(r^{-\tau})$$

for a connection θ on π and some $\tau > 0$.

ALF gravitational instantons

Defn: A gravitational instanton (M, g) is called **ALF** if there exists a compact set $K \subset M$, $R > 0$ and a finite group $\Gamma \subset O(3)$ acting freely on $\mathbb{S}^2$ such that $M \setminus K$ is the total space of a circle fibration over $(\mathbb{R}^3 \setminus \mathbb{B}_R)/\Gamma$ and the metric g is asymptotic to a Riemannian submersion

$$g = \pi^* g_{\mathbb{R}^3/\Gamma} + \theta^2 + O(r^{-\tau})$$

for a connection θ on π and some $\tau > 0$. There are only two possibilities for Γ: if $\Gamma = (Id)$ we say M is ALF of *cyclic* type;
if $\Gamma = \mathbb{Z}_2$ we say M is ALF of *dihedral* type.

- The multi-center Taub-NUT metrics all give ALF gravitational instantons of **cyclic** type. Minerbe proved converse. So ALF gravitational instantons of cycle type from the Gibbons-Hawking ansatz.

ALF gravitational instantons

Defn: A gravitational instanton (M, g) is called **ALF** if there exists a compact set $K \subset M$, $R > 0$ and a finite group $\Gamma \subset O(3)$ acting freely on $\mathbb{S}^2$ such that $M \setminus K$ is the total space of a circle fibration over $(\mathbb{R}^3 \setminus \mathbb{B}_R)/\Gamma$ and the metric g is asymptotic to a Riemannian submersion

$$g = \pi^* g_{\mathbb{R}^3/\Gamma} + \theta^2 + O(r^{-\tau})$$

for a connection θ on π and some $\tau > 0$. There are only two possibilities for Γ: if $\Gamma = (Id)$ we say M is ALF of *cyclic* type;
if $\Gamma = \mathbb{Z}_2$ we say M is ALF of *dihedral* type.

- The multi-center Taub-NUT metrics all give ALF gravitational instantons of **cyclic** type. Minerbe proved converse. So ALF gravitational instantons of cycle type from the Gibbons-Hawking ansatz.
- The ALF spaces of **dihedral** type are more complicated to construct because they do not arise form the Gibbons-Hawking ansatz (except asymptotically). The Atiyah–Hitchin manifold D_0 and its double cover D_1 are vital to the gluing construction. They contribute effective negative 'charge'. Without them only have collapse via the Kummer construction.

Ingredients of proof

- Construct (incomplete) S^1–invariant hyperkähler metrics on circle bundles over a punctured 3–torus using Gibbons-Hawking ansatz.

Ingredients of proof

- Construct (incomplete) S^1–invariant hyperkähler metrics on circle bundles over a punctured 3–torus using Gibbons-Hawking ansatz.
- Fix involution τ with 8 fixed points on T^3, choose a $\mathbb{Z}_2$–invariant configuration of $2n+8$ punctures and construct a monopole with Dirac-type singularities at these points. Pass to a $\mathbb{Z}_2$ quotient.

Ingredients of proof

- Construct (incomplete) S^1–invariant hyperkähler metrics on circle bundles over a punctured 3–torus using Gibbons-Hawking ansatz.
- Fix involution τ with 8 fixed points on T^3, choose a $\mathbb{Z}_2$–invariant configuration of $2n+8$ punctures and construct a monopole with Dirac-type singularities at these points. Pass to a $\mathbb{Z}_2$ quotient.
- Complete the resulting hyperkähler metrics by gluing ALF spaces at the $n+8$ punctures:

  an ALF space of dihedral type at each of the 8 fixed points of τ,

  an ALF space of cyclic type at each of the other punctures.

Ingredients of proof

- Construct (incomplete) S^1–invariant hyperkähler metrics on circle bundles over a punctured 3–torus using Gibbons-Hawking ansatz.
- Fix involution τ with 8 fixed points on T^3, choose a $\mathbb{Z}_2$–invariant configuration of $2n+8$ punctures and construct a monopole with Dirac-type singularities at these points. Pass to a $\mathbb{Z}_2$ quotient.
- Complete the resulting hyperkähler metrics by gluing ALF spaces at the $n+8$ punctures:

  an ALF space of dihedral type at each of the 8 fixed points of τ,
  an ALF space of cyclic type at each of the other punctures.
- Deform the resulting approximately hyperkähler metric using the Implicit Function Theorem. The setting of *definite triples* seems the most convenient framework to use.

Key features: collapsing hK metrics on K3

The main ingredients of the construction were the following:

1. A good perturbation theory for closed definite triples that are almost hyperKähler.

Key features: collapsing hK metrics on K3

The main ingredients of the construction were the following:

1. A good perturbation theory for closed definite triples that are almost hyperKähler.
2. The Gibbons-Hawking ansatz for constructing $\mathbb{S}^1$-invariant hK metrics. It was used in several ways

Key features: collapsing hK metrics on K3

The main ingredients of the construction were the following:

1. A good perturbation theory for closed definite triples that are almost hyperKähler.
2. The Gibbons-Hawking ansatz for constructing $\mathbb{S}^1$-invariant hK metrics. It was used in several ways
   - To construct the ALF gravitational instantons of cycle type (multi-centered Taub-NUT)

Key features: collapsing hK metrics on K3

The main ingredients of the construction were the following:

1. A good perturbation theory for closed definite triples that are almost hyperKähler.
2. The Gibbons-Hawking ansatz for constructing $\mathbb{S}^1$-invariant hK metrics. It was used in several ways
   - To construct the ALF gravitational instantons of cycle type (multi-centered Taub-NUT)
   - To construct the incomplete $\mathbb{S}^1$-invariant metric on the total space of a suitable circle bundle over a punctured 3-torus.

Key features: collapsing hK metrics on K3

The main ingredients of the construction were the following:

1. A good perturbation theory for closed definite triples that are almost hyperKähler.
2. The Gibbons-Hawking ansatz for constructing $\mathbb{S}^1$-invariant hK metrics. It was used in several ways
   - To construct the ALF gravitational instantons of cycle type (multi-centered Taub-NUT)
   - To construct the incomplete $\mathbb{S}^1$-invariant metric on the total space of a suitable circle bundle over a punctured 3-torus.
   - It provides a good approximation to the asymptotic geometry of the ALF gravitational instantons of dihedral type (up to a double cover).

Key features: collapsing hK metrics on K3

The main ingredients of the construction were the following:

1. A good perturbation theory for closed definite triples that are almost hyperKähler.
2. The Gibbons-Hawking ansatz for constructing $\mathbb{S}^1$-invariant hK metrics. It was used in several ways
   - To construct the ALF gravitational instantons of cycle type (multi-centered Taub-NUT)
   - To construct the incomplete $\mathbb{S}^1$-invariant metric on the total space of a suitable circle bundle over a punctured 3-torus.
   - It provides a good approximation to the asymptotic geometry of the ALF gravitational instantons of dihedral type (up to a double cover).
3. The existence of ALF gravitational instantons of dihedral type, not just of cyclic type, including the exceptional ones D_0 and D_1, the Atiyah–Hitchin manifold and its double cover.

Key features: collapsing hK metrics on K3

The main ingredients of the construction were the following:

1. A good perturbation theory for closed definite triples that are almost hyperKähler.
2. The Gibbons-Hawking ansatz for constructing $\mathbb{S}^1$-invariant hK metrics. It was used in several ways
   - To construct the ALF gravitational instantons of cycle type (multi-centered Taub-NUT)
   - To construct the incomplete $\mathbb{S}^1$-invariant metric on the total space of a suitable circle bundle over a punctured 3-torus.
   - It provides a good approximation to the asymptotic geometry of the ALF gravitational instantons of dihedral type (up to a double cover).
3. The existence of ALF gravitational instantons of dihedral type, not just of cyclic type, including the exceptional ones D_0 and D_1, the Atiyah–Hitchin manifold and its double cover.

We would like to develop analogues of these key features in the G_2 case.

1-dimensional collapse in the G_2 setting

Joyce already developed a very general perturbation theory for closed G_2 structures with sufficiently small torsion.

So the main tasks are to construct appropriate analogues of:

1-dimensional collapse in the G_2 setting

Joyce already developed a very general perturbation theory for closed G_2 structures with sufficiently small torsion.

So the main tasks are to construct appropriate analogues of:

1. The ALF gravitational instantons of cyclic type.

1-dimensional collapse in the G_2 setting

Joyce already developed a very general perturbation theory for closed G_2 structures with sufficiently small torsion.

So the main tasks are to construct appropriate analogues of:

1. The ALF gravitational instantons of cyclic type.
2. The ALF gravitational instantons of dihedral type.

1-dimensional collapse in the G_2 setting

Joyce already developed a very general perturbation theory for closed G_2 structures with sufficiently small torsion.

So the main tasks are to construct appropriate analogues of:

1. The ALF gravitational instantons of cyclic type.
2. The ALF gravitational instantons of dihedral type.
3. The incomplete $\mathbb{S}^1$-invariant hK metric on the circle bundle over a punctured 3-torus.

1-dimensional collapse in the G_2 setting

Joyce already developed a very general perturbation theory for closed G_2 structures with sufficiently small torsion.

So the main tasks are to construct appropriate analogues of:

1. The ALF gravitational instantons of cyclic type.
2. The ALF gravitational instantons of dihedral type.
3. The incomplete $\mathbb{S}^1$-invariant hK metric on the circle bundle over a punctured 3-torus.
   - Here we used the fact that $\mathbb{T}^3$ had a Ricci-flat metric (actually a flat metric);
   - the isometric involution τ also plays a fundamental role.

1-dimensional collapse in the G_2 setting

Joyce already developed a very general perturbation theory for closed G_2 structures with sufficiently small torsion.

So the main tasks are to construct appropriate analogues of:

1. The ALF gravitational instantons of cyclic type.
2. The ALF gravitational instantons of dihedral type.
3. The incomplete $\mathbb{S}^1$-invariant hK metric on the circle bundle over a punctured 3-torus.
    - Here we used the fact that $\mathbb{T}^3$ had a Ricci-flat metric (actually a flat metric);
    - the isometric involution τ also plays a fundamental role.

3 gives us incomplete $\mathbb{S}^1$-invariant hK metrics into which we glue spaces from 1 and 2 to give smooth but highly collapsed almost hK metrics on K3.

We therefore think of 3 as giving us a suitable singular background hK metric into which we glue two different types of hK bubbles, according to the behaviour of points under the involution τ.

1-dimensional collapse in the G_2 setting

Joyce already developed a very general perturbation theory for closed G_2 structures with sufficiently small torsion.

So the main tasks are to construct appropriate analogues of:

1. The ALF gravitational instantons of cyclic type.
2. The ALF gravitational instantons of dihedral type.
3. The incomplete $\mathbb{S}^1$-invariant hK metric on the circle bundle over a punctured 3-torus.
   - Here we used the fact that $\mathbb{T}^3$ had a Ricci-flat metric (actually a flat metric);
   - the isometric involution τ also plays a fundamental role.

3 gives us incomplete $\mathbb{S}^1$-invariant hK metrics into which we glue spaces from 1 and 2 to give smooth but highly collapsed almost hK metrics on K3.

We therefore think of 3 as giving us a suitable singular background hK metric into which we glue two different types of hK bubbles, according to the behaviour of points under the involution τ.

Q1: What's the correct G_2 analogue of an ALF gravitational instanton?

ALF in higher dims: ALC manifolds

- (Σ, g_Σ) closed (connected) Riemannian manifold of dim $n-2$.
- $\pi : N \to \Sigma$ a circle bundle. Passing to a double cover one can assume N is *principal*.
- θ a connection 1–form on $\pi : N \to \Sigma$, and a constant $\ell > 0$

Data $(\Sigma, g, \pi, \theta, \ell) \rightsquigarrow$ model metric on $M_\infty = \mathbb{R}^+ \times N$

$$g_\infty = dr^2 + r^2 \pi^* g_\Sigma + \ell^2 \theta^2$$

i.e. a circle bundle with fibres of constant length over cone $C(\Sigma)$.

ALF in higher dims: ALC manifolds

- (Σ, g_Σ) closed (connected) Riemannian manifold of dim $n-2$.
- $\pi : N \to \Sigma$ a circle bundle. Passing to a double cover one can assume N is *principal*.
- θ a connection 1–form on $\pi : N \to \Sigma$, and a constant $\ell > 0$

Data $(\Sigma, g, \pi, \theta, \ell) \rightsquigarrow$ model metric on $M_\infty = \mathbb{R}^+ \times N$

$$g_\infty = dr^2 + r^2 \pi^* g_\Sigma + \ell^2 \theta^2$$

i.e. a circle bundle with fibres of constant length over cone $C(\Sigma)$.

Definition A complete Riemannian manifold (M^n, g) with only one end is an **ALC manifold** asymptotic to M_∞ with rate $\nu < 0$ if there exists a compact set $K \subset M$, a positive number $R > 0$ and (up to a double cover) a diffeomorphism $\phi : M_\infty \cap \{r > R\} \to M \setminus K$ such that for all $j \geq 0$

$$|\nabla^j_{g_\infty}(\phi^* g - g_\infty)|_{g_\infty} = O(r^{\nu - j}).$$

ALF in higher dims: ALC manifolds

- (Σ, g_Σ) closed (connected) Riemannian manifold of dim $n-2$.
- $\pi : N \to \Sigma$ a circle bundle. Passing to a double cover one can assume N is *principal*.
- θ a connection 1–form on $\pi : N \to \Sigma$, and a constant $\ell > 0$

Data $(\Sigma, g, \pi, \theta, \ell) \rightsquigarrow$ model metric on $M_\infty = \mathbb{R}^+ \times N$

$$g_\infty = dr^2 + r^2 \pi^* g_\Sigma + \ell^2 \theta^2$$

i.e. a circle bundle with fibres of constant length over cone $C(\Sigma)$.

Definition A complete Riemannian manifold (M^n, g) with only one end is an **ALC manifold** asymptotic to M_∞ with rate $\nu < 0$ if there exists a compact set $K \subset M$, a positive number $R > 0$ and (up to a double cover) a diffeomorphism $\phi : M_\infty \cap \{r > R\} \to M \setminus K$ such that for all $j \geq 0$

$$|\nabla^j_{g_\infty}(\phi^* g - g_\infty)|_{g_\infty} = O(r^{\nu - j}).$$

Remark: ALC reduces to ALF when $n = 4$, $N = \mathbb{S}^3/\Gamma$ where Γ is a cyclic or binary dihedral group, and $\Sigma = \mathbb{S}^2$ or $\mathbb{RP}^2$ respectively.

ALC G_2 manifolds

G_2 holonomy: specified by a closed and coclosed 3–form φ.

If the ALC manifold M has G_2 holonomy the data determining the model M_∞ should satisfy further constraints; should include a suitable model 3-form φ_∞:

ALC G_2 manifolds

G_2 holonomy: specified by a closed and coclosed 3–form φ.

If the ALC manifold M has G_2 holonomy the data determining the model M_∞ should satisfy further constraints; should include a suitable model 3-form φ_∞:

- Σ is a Sasaki–Einstein 5–manifold $\Leftrightarrow$ the cone $C(\Sigma)$ is Calabi–Yau (CY).
- (ω_C, Ω_C) conical CY structure + Hermitian–Yang–Mills connection θ on a circle bundle $M_\infty \to C(\Sigma)$ $\rightsquigarrow$ model 3–form that is coclosed but not closed

$$\varphi'_\infty = \theta \wedge \omega_C + \mathrm{Re}\,\Omega_C$$

ALC G_2 manifolds

G_2 holonomy: specified by a closed and coclosed 3–form φ.

If the ALC manifold M has G_2 holonomy the data determining the model M_∞ should satisfy further constraints; should include a suitable model 3-form φ_∞:

- Σ is a Sasaki–Einstein 5–manifold $\Leftrightarrow$ the cone $C(\Sigma)$ is Calabi–Yau (CY).
- (ω_C, Ω_C) conical CY structure + Hermitian–Yang–Mills connection θ on a circle bundle $M_\infty \to C(\Sigma) \rightsquigarrow$ model 3–form that is coclosed but not closed

$$\varphi'_\infty = \theta \wedge \omega_C + \mathrm{Re}\,\Omega_C$$

But we can add a term to make φ_∞ closed (but no longer coclosed).

$$\varphi_\infty = \theta \wedge \omega_C + \mathrm{Re}\,\Omega_C$$

ALC G_2 manifolds

G_2 holonomy: specified by a closed and coclosed 3–form φ.

If the ALC manifold M has G_2 holonomy the data determining the model M_∞ should satisfy further constraints; should include a suitable model 3-form φ_∞:

- Σ is a Sasaki–Einstein 5–manifold $\Leftrightarrow$ the cone $C(\Sigma)$ is Calabi–Yau (CY).
- (ω_C, Ω_C) conical CY structure + Hermitian–Yang–Mills connection θ on a circle bundle $M_\infty \to C(\Sigma) \rightsquigarrow$ model 3–form that is coclosed but not closed

$$\varphi'_\infty = \theta \wedge \omega_C + \mathrm{Re}\,\Omega_C$$

But we can add a term to make φ_∞ closed (but no longer coclosed).

$$\varphi_\infty = \theta \wedge \omega_C + \mathrm{Re}\,\Omega_C - \tfrac{1}{2} r^2 \eta \wedge d\theta$$

where η is the contact 1-form on Σ.

Method I to construct ALC G_2 manifolds

Impose extra symmetry: reduce to a cohomogeneity one problem and try to deal with the resulting ODEs directly, e.g. as in Johannes' talk.

Method I to construct ALC G_2 manifolds

Impose extra symmetry: reduce to a cohomogeneity one problem and try to deal with the resulting ODEs directly, e.g. as in Johannes' talk.

- Approach taken by various physicists in series of papers circa 2001-2005; the group acting has $G_0 = SU(2) \times SU(2)$ or $SU(2) \times SU(2) \times U(1)$.
- One explicit ALC G_2 metric found by Brandhuber-Gomis-Gubner-Gukov.
- Numerical evidence for a 2-parameter family of ALC G_2 metrics containing the BGGG example: physicists called this family $\mathbb{B}_7$.

Method I to construct ALC G_2 manifolds

Impose extra symmetry: reduce to a cohomogeneity one problem and try to deal with the resulting ODEs directly, e.g. as in Johannes' talk.

- Approach taken by various physicists in series of papers circa 2001-2005; the group acting has $G_0 = SU(2) \times SU(2)$ or $SU(2) \times SU(2) \times U(1)$.
- One explicit ALC G_2 metric found by Brandhuber-Gomis-Gubner-Gukov.
- Numerical evidence for a 2-parameter family of ALC G_2 metrics containing the BGGG example: physicists called this family $\mathbb{B}_7$.
- Bogoyavlenskaya (2013) rigorously proved existence of the family $\mathbb{B}_7$ by qualitative analysis of the ODEs. Crucially uses the extra $U(1)$-symmetry to reduce number of variables in the ODE system.

Method I to construct ALC G_2 manifolds

Impose extra symmetry: reduce to a cohomogeneity one problem and try to deal with the resulting ODEs directly, e.g. as in Johannes' talk.

- Approach taken by various physicists in series of papers circa 2001-2005; the group acting has $G_0 = SU(2) \times SU(2)$ or $SU(2) \times SU(2) \times U(1)$.
- One explicit ALC G_2 metric found by Brandhuber-Gomis-Gubner-Gukov.
- Numerical evidence for a 2-parameter family of ALC G_2 metrics containing the BGGG example: physicists called this family $\mathbb{B}_7$.
- Bogoyavlenskaya (2013) rigorously proved existence of the family $\mathbb{B}_7$ by qualitative analysis of the ODEs. Crucially uses the extra $U(1)$-symmetry to reduce number of variables in the ODE system.
- Geometrically the two parameters control the size of the exceptional orbit (a round 3-sphere) and the length of the asymptotic circle.
- Geometric degenerations occur at special points in the parameter space.
  - The AC limit and the conically singular ALC space discussed by Johannes.
  - A **collapsed limit** where the ALC G_2 manifold Gromov-Hausdorff converges to a noncompact Calabi-Yau 3-fold.

Method II to construct ALC G_2 manifolds

Basic idea: Try to understand ALC G_2 manifolds close to the collapsed Calabi–Yau limit.

Example: In the $\mathbb{B}_7$ family expect to see collapse to the AC CY metric (Stenzel/Candelas-de la Ossa) on the smoothing of the conifold.

Method II to construct ALC G_2 manifolds

Basic idea: Try to understand ALC G_2 manifolds close to the collapsed Calabi–Yau limit.

Example: In the $\mathbb{B}_7$ family expect to see collapse to the AC CY metric (Stenzel/Candelas-de la Ossa) on the smoothing of the conifold.

General expectation from theory of collapse: in our setting a highly collapsed space should be composed of pieces that are approximately invariant under an isometric circle action.

Method II to construct ALC G_2 manifolds

Basic idea: Try to understand ALC G_2 manifolds close to the collapsed Calabi–Yau limit.

Example: In the $\mathbb{B}_7$ family expect to see collapse to the AC CY metric (Stenzel/Candelas-de la Ossa) on the smoothing of the conifold.

General expectation from theory of collapse: in our setting a highly collapsed space should be composed of pieces that are approximately invariant under an isometric circle action.

Recall **Gibbons–Hawking Ansatz**: the $U(1)$-invariant metric on P

$$g := h\,\pi^* g_{\mathbb{R}^3} + h^{-1}\theta^2$$

is hyperKähler iff (h, θ) is an abelian **monopole**: $*dh = d\theta$.

- U open subset of $\mathbb{R}^3$; h positive function on U
- $\pi : P \to U$ a principal $U(1)$–bundle with a connection θ

Method II to construct ALC G_2 manifolds

Basic idea: Try to understand ALC G_2 manifolds close to the collapsed Calabi–Yau limit.

Example: In the $\mathbb{B}_7$ family expect to see collapse to the AC CY metric (Stenzel/Candelas-de la Ossa) on the smoothing of the conifold.

General expectation from theory of collapse: in our setting a highly collapsed space should be composed of pieces that are approximately invariant under an isometric circle action.

Recall **Gibbons–Hawking Ansatz**: the $U(1)$-invariant metric on P

$$g := h\,\pi^* g_{\mathbb{R}^3} + h^{-1}\theta^2$$

is hyperKähler iff (h, θ) is an abelian **monopole**: $*dh = d\theta$.

- U open subset of $\mathbb{R}^3$; h positive function on U
- $\pi : P \to U$ a principal $U(1)$–bundle with a connection θ

Essentially reduces us to solving a **linear** PDE $\Delta h = 0$ plus some topological constraints.

G_2 analogue of Gibbons-Hawking

- **Apostolov–Salamon** (2004): Let $M^7 \to Y^6$ be a principal circle bundle with connection θ and $(\omega, h^{-\frac{3}{4}}\Omega)$ an $SU(3)$–structure on Y with h some positive function on Y. We define a G_2-structure on P by

$$\varphi = \theta \wedge \omega + \mathrm{Re}\,\Omega.$$

The induced metric is

$$g = \sqrt{h}\, g_M + h^{-1}\theta^2$$

and the 4-form is given by

$$*_\varphi \varphi = -h^{-1/2}\theta \wedge \mathrm{Im}\Omega + \frac{1}{2} h\omega^2.$$

φ is an $\mathbb{S}^1$-invariant torsion-free G_2-structure iff

$$d\omega = 0, \qquad d\mathrm{Re}\,\Omega = -d\theta \wedge \omega,$$
$$d\left(h^{-\frac{1}{2}}\mathrm{Im}\,\Omega\right) = 0 \qquad \tfrac{1}{2} dh \wedge \omega^2 = h^{-\frac{1}{2}} d\theta \wedge \mathrm{Im}\,\Omega.$$

- Introduce a parameter ϵ:

$$\varphi = \epsilon\,\theta \wedge \omega + \mathrm{Re}\,\Omega, \qquad g = \sqrt{h}\,g_Y + \epsilon^2 h^{-1}\theta^2,$$
$$d\omega = 0, \qquad d\mathrm{Re}\,\Omega = -\epsilon\,d\theta \wedge \omega,$$
$$d\left(h^{-\frac{1}{2}}\mathrm{Im}\,\Omega\right) = 0, \qquad \tfrac{1}{2}dh \wedge \omega^2 = \epsilon\,h^{-\frac{1}{2}}d\theta \wedge \mathrm{Im}\,\Omega.$$

The collapsed limit of Apostolov–Salamon eqns

- Pass to the limit $\epsilon \to 0$: $h_0 \equiv 1$ and (ω_0, Ω_0) is a CY structure on Y.

The collapsed limit of Apostolov–Salamon eqns

- Pass to the limit $\epsilon \to 0$: $h_0 \equiv 1$ and (ω_0, Ω_0) is a CY structure on Y.
- *Linearised* equations over the collapsed limit: a **CY monopole** on Y

$$*dh = d\theta \wedge \mathrm{Re}\,\Omega_0, \qquad d\theta \wedge \omega_0^2 = 0$$

together with a 3–form ρ satisfying a coupled linear equation

$$d\rho = -d\theta \wedge \omega_0, \qquad d\hat{\rho} = \tfrac{1}{2} dh \wedge \mathrm{Im}\,\Omega_0.$$

The collapsed limit of Apostolov–Salamon eqns

- Pass to the limit $\epsilon \to 0$: $h_0 \equiv 1$ and (ω_0, Ω_0) is a CY structure on Y.
- *Linearised* equations over the collapsed limit: a **CY monopole** on Y

$$*dh = d\theta \wedge \mathrm{Re}\,\Omega_0, \qquad d\theta \wedge \omega_0^2 = 0$$

together with a 3–form ρ satisfying a coupled linear equation

$$d\rho = -d\theta \wedge \omega_0, \qquad d\hat{\rho} = \tfrac{1}{2} dh \wedge \mathrm{Im}\,\Omega_0.$$

CY monopole equation arises as a dimensional reduction of G_2 instanton equation just as ordinary monopole equation does in the 4d instanton case.

The collapsed limit of Apostolov–Salamon eqns

- Pass to the limit $\epsilon \to 0$: $h_0 \equiv 1$ and (ω_0, Ω_0) is a CY structure on Y.
- *Linearised* equations over the collapsed limit: a **CY monopole** on Y

$$*dh = d\theta \wedge \mathrm{Re}\,\Omega_0, \qquad d\theta \wedge \omega_0^2 = 0$$

together with a 3–form ρ satisfying a coupled linear equation

$$d\rho = -d\theta \wedge \omega_0, \qquad d\hat{\rho} = \tfrac{1}{2} dh \wedge \mathrm{Im}\,\Omega_0.$$

CY monopole equation arises as a dimensional reduction of G_2 instanton equation just as ordinary monopole equation does in the 4d instanton case.

Special case: $h \equiv 1$ and θ a HYM connection.

The collapsed limit of Apostolov–Salamon eqns

- Pass to the limit $\epsilon \to 0$: $h_0 \equiv 1$ and (ω_0, Ω_0) is a CY structure on Y.
- *Linearised* equations over the collapsed limit: a **CY monopole** on Y

$$*dh = d\theta \wedge \mathrm{Re}\,\Omega_0, \qquad d\theta \wedge \omega_0^2 = 0$$

together with a 3–form ρ satisfying a coupled linear equation

$$d\rho = -d\theta \wedge \omega_0, \qquad d\hat{\rho} = \tfrac{1}{2} dh \wedge \mathrm{Im}\,\Omega_0.$$

CY monopole equation arises as a dimensional reduction of G_2 instanton equation just as ordinary monopole equation does in the 4d instanton case.

Special case: $h \equiv 1$ and θ a HYM connection.

In general CY monopoles may have Dirac-type singularities along special Lagrangian submanifolds.

The collapsed limit of Apostolov–Salamon eqns

- Pass to the limit $\epsilon \to 0$: $h_0 \equiv 1$ and (ω_0, Ω_0) is a CY structure on Y.
- *Linearised* equations over the collapsed limit: a **CY monopole** on Y

$$*dh = d\theta \wedge \mathrm{Re}\,\Omega_0, \qquad d\theta \wedge \omega_0^2 = 0$$

together with a 3–form ρ satisfying a coupled linear equation

$$d\rho = -d\theta \wedge \omega_0, \qquad d\hat{\rho} = \tfrac{1}{2} dh \wedge \mathrm{Im}\,\Omega_0.$$

CY monopole equation arises as a dimensional reduction of G_2 instanton equation just as ordinary monopole equation does in the 4d instanton case.

Special case: $h \equiv 1$ and θ a HYM connection.

In general CY monopoles may have Dirac-type singularities along special Lagrangian submanifolds.

Key fact: A solution of these equations yields a **closed** ALC G_2 structure with small torsion on M.

Collapsed ALC G_2 manifolds from AC CY 3–folds

- On $Y = K_{\mathbb{P}^1\times\mathbb{P}^1}$ or the small resolution of the conifold construct a HYM connection; On $Y = T^*S^3$ construct a CY monopole with Dirac-type singularities along the special Lagrangian S^3
  $\rightsquigarrow$ closed G_2-structures with small torsion on circle bundles over Y.

Collapsed ALC G_2 manifolds from AC CY 3–folds

- On $Y = K_{\mathbb{P}^1 \times \mathbb{P}^1}$ or the small resolution of the conifold construct a HYM connection; On $Y = T^*S^3$ construct a CY monopole with Dirac-type singularities along the special Lagrangian S^3
  $\rightsquigarrow$ closed G_2-structures with small torsion on circle bundles over Y.
- Perturb these approximate solutions near infinity by hand to improve the decay of the torsion.

Collapsed ALC G_2 manifolds from AC CY 3–folds

- On $Y = K_{\mathbb{P}^1 \times \mathbb{P}^1}$ or the small resolution of the conifold construct a HYM connection; On $Y = T^*S^3$ construct a CY monopole with Dirac-type singularities along the special Lagrangian S^3
  $\rightsquigarrow$ closed G_2-structures with small torsion on circle bundles over Y.
- Perturb these approximate solutions near infinity by hand to improve the decay of the torsion.
- Deform to exact ALC G_2 metrics using Joyce's deformation result.

Collapsed ALC G_2 manifolds from AC CY 3–folds

- On $Y = K_{\mathbb{P}^1 \times \mathbb{P}^1}$ or the small resolution of the conifold construct a HYM connection; On $Y = T^*S^3$ construct a CY monopole with Dirac-type singularities along the special Lagrangian S^3
  $\rightsquigarrow$ closed G_2-structures with small torsion on circle bundles over Y.
- Perturb these approximate solutions near infinity by hand to improve the decay of the torsion.
- Deform to exact ALC G_2 metrics using Joyce's deformation result.
- Use deformation theory to calculate the dimension of the moduli space $\mathcal{M}$ of ALC G_2 metrics modulo scalings.
- Both these use detailed Fredholm theory and understanding of indicial roots on ALC manifolds.

Collapsed ALC G_2 manifolds from AC CY 3–folds

- On $Y = K_{\mathbb{P}^1\times\mathbb{P}^1}$ or the small resolution of the conifold construct a HYM connection; On $Y = T^*S^3$ construct a CY monopole with Dirac-type singularities along the special Lagrangian S^3
  $\rightsquigarrow$ closed G_2-structures with small torsion on circle bundles over Y.
- Perturb these approximate solutions near infinity by hand to improve the decay of the torsion.
- Deform to exact ALC G_2 metrics using Joyce's deformation result.
- Use deformation theory to calculate the dimension of the moduli space $\mathcal{M}$ of ALC G_2 metrics modulo scalings.
- Both these use detailed Fredholm theory and understanding of indicial roots on ALC manifolds.

Theorem For $\epsilon > 0$ sufficiently small there exist ALC G_2 manifolds with an S^1 action with orbits of asymptotic length $2\pi\epsilon$:

ALC G_2	$\mathbb{B}_7$	$\mathbb{C}_7$	$\mathbb{D}_7$
$\epsilon \to 0$	T^*S^3	$K_{\mathbb{P}^1\times\mathbb{P}^1}$	small resolution
$\dim \mathcal{M}$	1 (rate -3)	1 (rate -3)	1 (rate -2)

Collapsing G_2-metrics on compact spaces

Want to use our highly collapsed ALC G_2-spaces $\mathbb{B}_7$ as bubbles in a gluing construction. $\mathbb{B}^7$ has a global isometric circle action that fixes the exceptional orbit, which is a round $\mathbb{S}^3$. In the limit $\mathbb{B}^7$ converges to the Stenzel metric on the smoothing $T^*\mathbb{S}^3$ of the conifold.

Collapsing G_2-metrics on compact spaces

Want to use our highly collapsed ALC G_2-spaces $\mathbb{B}_7$ as bubbles in a gluing construction. $\mathbb{B}^7$ has a global isometric circle action that fixes the exceptional orbit, which is a round $\mathbb{S}^3$. In the limit $\mathbb{B}^7$ converges to the Stenzel metric on the smoothing $T^*\mathbb{S}^3$ of the conifold.

Q: What should replace the singular limit space $\mathbb{T}^3/\mathbb{Z}_2$ in the hK case?

Collapsing G_2-metrics on compact spaces

Want to use our highly collapsed ALC G_2-spaces $\mathbb{B}_7$ as bubbles in a gluing construction. $\mathbb{B}^7$ has a global isometric circle action that fixes the exceptional orbit, which is a round $\mathbb{S}^3$. In the limit $\mathbb{B}^7$ converges to the Stenzel metric on the smoothing $T^*\mathbb{S}^3$ of the conifold.

Q: What should replace the singular limit space $\mathbb{T}^3/\mathbb{Z}_2$ in the hK case?

A: Quotient of a Calabi-Yau 3-fold by an anti-holomorphic involution τ. Fixed point set of τ is a totally geodesic special Lagrangian 3-fold L

Collapsing G_2-metrics on compact spaces

Want to use our highly collapsed ALC G_2-spaces $\mathbb{B}_7$ as bubbles in a gluing construction. $\mathbb{B}^7$ has a global isometric circle action that fixes the exceptional orbit, which is a round $\mathbb{S}^3$. In the limit $\mathbb{B}^7$ converges to the Stenzel metric on the smoothing $T^*\mathbb{S}^3$ of the conifold.

Q: What should replace the singular limit space $\mathbb{T}^3/\mathbb{Z}_2$ in the hK case?

A: Quotient of a Calabi-Yau 3-fold by an anti-holomorphic involution τ. Fixed point set of τ is a totally geodesic special Lagrangian 3-fold L

Obvious problem: In Stenzel have a round $\mathbb{S}^3$. How do we get round SL 3-spheres in CY 3-folds?

Collapsing G_2-metrics on compact spaces

Want to use our highly collapsed ALC G_2-spaces $\mathbb{B}_7$ as bubbles in a gluing construction. $\mathbb{B}^7$ has a global isometric circle action that fixes the exceptional orbit, which is a round $\mathbb{S}^3$. In the limit $\mathbb{B}^7$ converges to the Stenzel metric on the smoothing $T^*\mathbb{S}^3$ of the conifold.

Q: What should replace the singular limit space $\mathbb{T}^3/\mathbb{Z}_2$ in the hK case?

A: Quotient of a Calabi-Yau 3-fold by an anti-holomorphic involution τ. Fixed point set of τ is a totally geodesic special Lagrangian 3-fold L

Obvious problem: In Stenzel have a round $\mathbb{S}^3$. How do we get round SL 3-spheres in CY 3-folds?

Answer: Start with a CY 3-fold X_0 that has only ordinary double point (conifold) singularities and assume that X is smoothable.

Collapsing G_2-metrics on compact spaces

Want to use our highly collapsed ALC G_2-spaces $\mathbb{B}_7$ as bubbles in a gluing construction. $\mathbb{B}^7$ has a global isometric circle action that fixes the exceptional orbit, which is a round $\mathbb{S}^3$. In the limit $\mathbb{B}^7$ converges to the Stenzel metric on the smoothing $T^*\mathbb{S}^3$ of the conifold.

Q: What should replace the singular limit space $\mathbb{T}^3/\mathbb{Z}_2$ in the hK case?

A: Quotient of a Calabi-Yau 3-fold by an anti-holomorphic involution τ. Fixed point set of τ is a totally geodesic special Lagrangian 3-fold L

Obvious problem: In Stenzel have a round $\mathbb{S}^3$. How do we get round SL 3-spheres in CY 3-folds?

Answer: Start with a CY 3-fold X_0 that has only ordinary double point (conifold) singularities and assume that X is smoothable. By very recent work of Hein and Sun we know that X admits an incomplete CY metric that is asymptotic to the standard KRF cone metric on the conifold at each ODP. (This is a great theorem!)

Collapsing G_2-metrics on compact spaces

Want to use our highly collapsed ALC G_2-spaces $\mathbb{B}_7$ as bubbles in a gluing construction. $\mathbb{B}^7$ has a global isometric circle action that fixes the exceptional orbit, which is a round $\mathbb{S}^3$. In the limit $\mathbb{B}^7$ converges to the Stenzel metric on the smoothing $T^*\mathbb{S}^3$ of the conifold.

Q: What should replace the singular limit space $\mathbb{T}^3/\mathbb{Z}_2$ in the hK case?

A: Quotient of a Calabi-Yau 3-fold by an anti-holomorphic involution τ. Fixed point set of τ is a totally geodesic special Lagrangian 3-fold L

Obvious problem: In Stenzel have a round $\mathbb{S}^3$. How do we get round SL 3-spheres in CY 3-folds?

Answer: Start with a CY 3-fold X_0 that has only ordinary double point (conifold) singularities and assume that X is smoothable. By very recent work of Hein and Sun we know that X admits an incomplete CY metric that is asymptotic to the standard KRF cone metric on the conifold at each ODP. (This is a great theorem!) Can now construct nearby smooth CY 3-folds Y_t by gluing in Stenzel metrics at each ODP. This way we get SL 3-spheres as close as required to round.

Collapsing G_2-metrics on compact spaces II

We have to require that the nodal CY 3-fold X admits an anti-holomorphic involution whose fixed set is a subset of the nodes. This appears to be very restrictive.

Collapsing G_2-metrics on compact spaces II

We have to require that the nodal CY 3-fold X admits an anti-holomorphic involution whose fixed set is a subset of the nodes. This appears to be very restrictive.

Because we need to find a circle bundle over our 6-dimensional space (with the correct topological properties) and we need to lift the involution to the circle bundle we also need the existence of a divisor passing through the nodes of X in a specified way,

Collapsing G_2-metrics on compact spaces II

We have to require that the nodal CY 3-fold X admits an anti-holomorphic involution whose fixed set is a subset of the nodes. This appears to be very restrictive.

Because we need to find a circle bundle over our 6-dimensional space (with the correct topological properties) and we need to lift the involution to the circle bundle we also need the existence of a divisor passing through the nodes of X in a specified way,

We do know at least one such nodal CY 3-fold satisfying our wish list!

On the total space of the circle bundle over Y' we also need to construct a highly collapsed circle-invariant approximate G_2-metric. Here again we use the linearised version of Apostolov-Salamon.

Collapsing G_2-metrics on compact spaces II

We have to require that the nodal CY 3-fold X admits an anti-holomorphic involution whose fixed set is a subset of the nodes. This appears to be very restrictive.

Because we need to find a circle bundle over our 6-dimensional space (with the correct topological properties) and we need to lift the involution to the circle bundle we also need the existence of a divisor passing through the nodes of X in a specified way,

We do know at least one such nodal CY 3-fold satisfying our wish list!

On the total space of the circle bundle over Y' we also need to construct a highly collapsed circle-invariant approximate G_2-metric. Here again we use the linearised version of Apostolov-Salamon.

We observe that Hitchin's work on gerbes and SL 3-folds can be used to solve this system on the complement of a homologically trivial collection of smooth compact disjoint SL 3-folds. Argument uses Hodge theory for currents plus clever arguments using type decomposition repeatedly.

ALC G_2 analogue of Atiyah–Hitchin

We are still missing one type of bubble to glue in, the analogues of ALF dihdredal gravitational instantons. Why needed?

ALC G_2 analogue of Atiyah–Hitchin

We are still missing one type of bubble to glue in, the analogues of ALF dihdredal gravitational instantons. Why needed? We need a family of ALC G_2 manifolds that collapses to Stenzel metric on $T^*\mathbb{S}^3$ quotiented by its standard antiholomorphic involution to glue at fixed points of τ.

ALC G_2 analogue of Atiyah–Hitchin

We are still missing one type of bubble to glue in, the analogues of ALF dihdredal gravitational instantons. Why needed? We need a family of ALC G_2 manifolds that collapses to Stenzel metric on $T^*\mathbb{S}^3$ quotiented by its standard antiholomorphic involution to glue at fixed points of τ.

Physicists suggested that there should be an "M-theory lift" of the Atiyah–Hitchin manifold. Hori et al proposed a cohomogeneity one construction of such a family $\mathbb{A}_7$ of G_2 manifolds and studied the ODEs numerically. There is a $SU(2) \times SU(2)$ symmetry but the extra $U(1)$ symmetry is absent. Makes the ODE system much less tractable directly.

Instead we proof existence of the $\mathbb{A}_7$ family in the highly collapsed limit by perturbation methods. This is also more involved than in the previous cases.

ALC G_2 analogue of Atiyah–Hitchin

We are still missing one type of bubble to glue in, the analogues of ALF dihdredal gravitational instantons. Why needed? We need a family of ALC G_2 manifolds that collapses to Stenzel metric on $T^*\mathbb{S}^3$ quotiented by its standard antiholomorphic involution to glue at fixed points of τ.

Physicists suggested that there should be an "M-theory lift" of the Atiyah–Hitchin manifold. Hori et al proposed a cohomogeneity one construction of such a family $\mathbb{A}_7$ of G_2 manifolds and studied the ODEs numerically. There is a $SU(2) \times SU(2)$ symmetry but the extra $U(1)$ symmetry is absent. Makes the ODE system much less tractable directly.

Instead we proof existence of the $\mathbb{A}_7$ family in the highly collapsed limit by perturbation methods. This is also more involved than in the previous cases. We needed to construct a better approximation to the geometry in the neighbourhood of the singular orbit.

ALC G_2 analogue of Atiyah–Hitchin

We are still missing one type of bubble to glue in, the analogues of ALF dihdredal gravitational instantons. Why needed? We need a family of ALC G_2 manifolds that collapses to Stenzel metric on $T^*\mathbb{S}^3$ quotiented by its standard antiholomorphic involution to glue at fixed points of τ.

Physicists suggested that there should be an “M-theory lift” of the Atiyah–Hitchin manifold. Hori et al proposed a cohomogeneity one construction of such a family $\mathbb{A}_7$ of G_2 manifolds and studied the ODEs numerically. There is a $SU(2) \times SU(2)$ symmetry but the extra $U(1)$ symmetry is absent. Makes the ODE system much less tractable directly.

Instead we proof existence of the $\mathbb{A}_7$ family in the highly collapsed limit by perturbation methods. This is also more involved than in the previous cases. We needed to construct a better approximation to the geometry in the neighbourhood of the singular orbit. We do this by rescaling in the normal directions and adapting ideas from Simon Donaldson’s adiabatic limits of coassociative fibrations. Gives a rigorous way to interpret the physics statement that these manifolds are families of “AH metrics fibred over $\mathbb{S}^3$”.

END OF TALK

Donaldson’s adiabatic limit equations

BGGG: “the similarity of our solution with the Taub-NUT metric”.

Donaldson's adiabatic limit equations

BGGG: "the similarity of our solution with the Taub-NUT metric".

Donaldson (2016): adiabatic limit of coassociative fibrations.

Donaldson's adiabatic limit equations

BGGG: "the similarity of our solution with the Taub-NUT metric".

Donaldson (2016): adiabatic limit of coassociative fibrations.

- (Y^3, g) an oriented Riemannian 3–manifold with spin bundle $\mathcal{S} \to Y$.

  $\underline{\boldsymbol{\theta}}$ tautological $\mathbb{R}^3$–valued 1–form and $\underline{\boldsymbol{\eta}}$ the LC connection on $\mathcal{S}$.

Donaldson's adiabatic limit equations

BGGG: "the similarity of our solution with the Taub-NUT metric".

Donaldson (2016): adiabatic limit of coassociative fibrations.

- (Y^3, g) an oriented Riemannian 3–manifold with spin bundle $\mathcal{S} \to Y$.

  $\underline{\boldsymbol{\theta}}$ tautological $\mathbb{R}^3$–valued 1–form and $\underline{\boldsymbol{\eta}}$ the LC connection on $\mathcal{S}$.
- $(F^4, \underline{\boldsymbol{\omega}})$ HK 4–manifold with isometric $SU(2)$–action which rotates the complex structures (flat $\mathbb{R}^4$, Taub–NUT, Atiyah–Hitchin).
  (Up to a finite cover) dense open set $\mathbb{R}^+ \times SU(2) \subset F$

Donaldson's adiabatic limit equations

BGGG: "the similarity of our solution with the Taub-NUT metric".

Donaldson (2016): adiabatic limit of coassociative fibrations.

- (Y^3, g) an oriented Riemannian 3–manifold with spin bundle $\mathcal{S} \to Y$.

  $\underline{\boldsymbol{\theta}}$ tautological $\mathbb{R}^3$–valued 1–form and $\underline{\boldsymbol{\eta}}$ the LC connection on $\mathcal{S}$.
- $(F^4, \underline{\boldsymbol{\omega}})$ HK 4–manifold with isometric $SU(2)$–action which rotates the complex structures (flat $\mathbb{R}^4$, Taub–NUT, Atiyah–Hitchin).
  (Up to a finite cover) dense open set $\mathbb{R}^+ \times SU(2) \subset F$

$SU(2)$–invariant G_2 structure on $\pi : M = \mathcal{S} \times_{SU(2)} F \to Y$ with coassociative fibres:

$$\varphi_\epsilon = V\theta_1 \wedge \theta_2 \wedge \theta_3 + \epsilon^2 \left(\theta_1 \wedge \omega_1 + \theta_2 \wedge \omega_2 + \theta_3 \wedge \omega_3\right).$$

where $\omega_i = a_i dt \wedge \eta_i + b_i \, \eta_j \wedge \eta_k$ with $a_i b_i > 0$, i.e.

$\underline{\boldsymbol{\omega}} := u\left(\omega_1, \omega_2, \omega_3\right) u^{-1}$ at $(t, u) \in \mathbb{R}^+ \times SU(2)$ is a definite triple rotated by the $SU(2)$ action.

Induced metric $g_\epsilon = V^{\frac{2}{3}} g_Y + \epsilon^2 V^{-\frac{1}{3}} g_{\underline{\boldsymbol{\omega}}}$

Donaldson's adiabatic limit equations

$$\varphi_\epsilon = V\theta_1 \wedge \theta_2 \wedge \theta_3 + \epsilon^2 \left(\theta_1 \wedge \omega_1 + \theta_2 \wedge \omega_2 + \theta_3 \wedge \omega_3\right)$$

$$\Omega^{p,q}(M) := \pi^*\Omega^p(Y) \otimes \Gamma(M; \Lambda^q \mathcal{V}^*) \qquad d = d_F + d_H + R$$

Donaldson's adiabatic limit equations

$$\varphi_\epsilon = V\theta_1 \wedge \theta_2 \wedge \theta_3 + \epsilon^2 \left(\theta_1 \wedge \omega_1 + \theta_2 \wedge \omega_2 + \theta_3 \wedge \omega_3\right)$$

$$\Omega^{p,q}(M) := \pi^* \Omega^p(Y) \otimes \Gamma(M; \Lambda^q \mathcal{V}^*) \qquad d = d_F + d_H + R$$

$$d\varphi_\epsilon = 0 \iff d_F \left(\sum \theta_i \wedge \omega_i\right) = 0 = d_H \left(\sum \theta_i \wedge \omega_i\right)$$

$$d_F \left(V\theta_1 \wedge \theta_2 \wedge \theta_3\right) + \epsilon^2 R \cdot \left(\sum \theta_i \wedge \omega_i\right) = 0$$

Donaldson's adiabatic limit equations

$$\varphi_\epsilon = V\theta_1 \wedge \theta_2 \wedge \theta_3 + \epsilon^2 \left(\theta_1 \wedge \omega_1 + \theta_2 \wedge \omega_2 + \theta_3 \wedge \omega_3\right)$$

$$\Omega^{p,q}(M) := \pi^* \Omega^p(Y) \otimes \Gamma(M; \Lambda^q \mathcal{V}^*) \qquad d = d_F + d_H + R$$

$$d\varphi_\epsilon = 0 \iff d_F\left(\sum \theta_i \wedge \omega_i\right) = 0 = d_H\left(\sum \theta_i \wedge \omega_i\right)$$

$$d_F\left(V\theta_1 \wedge \theta_2 \wedge \theta_3\right) + \epsilon^2 R \cdot \left(\sum \theta_i \wedge \omega_i\right) = 0$$

In the limit $\epsilon \to 0$ $V \equiv 1$ and $\underline{\boldsymbol{\omega}}$ is a hyperkähler triple:

$$\omega_i = a_i dt \wedge \eta_i + a_j a_k \, \eta_j \wedge \eta_k, \qquad (a_j a_k)' = a_i - a_j - a_k.$$

Donaldson's adiabatic limit equations

$$\varphi_\epsilon = V\theta_1 \wedge \theta_2 \wedge \theta_3 + \epsilon^2 (\theta_1 \wedge \omega_1 + \theta_2 \wedge \omega_2 + \theta_3 \wedge \omega_3)$$

$$\Omega^{p,q}(M) := \pi^* \Omega^p(Y) \otimes \Gamma(M; \Lambda^q \mathcal{V}^*) \qquad d = d_F + d_H + R$$

$$d\varphi_\epsilon = 0 \Longleftrightarrow d_F \left(\sum \theta_i \wedge \omega_i \right) = 0 = d_H \left(\sum \theta_i \wedge \omega_i \right)$$
$$d_F (V\theta_1 \wedge \theta_2 \wedge \theta_3) + \epsilon^2 R \cdot \left(\sum \theta_i \wedge \omega_i \right) = 0$$

In the limit $\epsilon \to 0$ $V \equiv 1$ and $\underline{\boldsymbol{\omega}}$ is a hyperkähler triple:

$$\omega_i = a_i dt \wedge \eta_i + a_j a_k \, \eta_j \wedge \eta_k, \qquad (a_j a_k)' = a_i - a_j - a_k.$$

Given $\underline{\boldsymbol{\omega}}$ make φ_0 closed by solving the last equation:

$$d_F \circ R \cdot \left(\sum \theta_i \wedge \omega_i \right) = 0 \Longleftrightarrow \text{ 2nd Bianchi identity.}$$

$\rightsquigarrow$ closed G_2 structure φ_ϵ with small torsion.

Donaldson's adiabatic limit equations

The first two equations are
Modify φ to make it **closed**:

$$\varphi' = V\, \theta_1 \wedge \theta_2 \wedge \theta_3 + \sum_{i=1}^{3} \theta_i \wedge (a_i dt \wedge \eta_i + a_j a_k \eta_j \wedge \eta_k)$$

for a function $V : M \to \mathbb{R}^+$ satisfying a linear equation along the fibres depending on the curvature of Y.

Donaldson's adiabatic limit equations

The first two equations are
Modify φ to make it **closed**:

$$\varphi' = V\,\theta_1 \wedge \theta_2 \wedge \theta_3 + \sum_{i=1}^{3} \theta_i \wedge \left(a_i dt \wedge \eta_i + a_j a_k \eta_j \wedge \eta_k\right)$$

for a function $V : M \to \mathbb{R}^+$ satisfying a linear equation along the fibres depending on the curvature of Y.

- If $Y = S^3$ and $F =$ flat $\mathbb{R}^4$ then φ_ϵ is Bryant–Salamon AC G_2 structure on the spinor bundle of S^3.

Donaldson's adiabatic limit equations

The first two equations are
Modify φ to make it **closed**:

$$\varphi' = V\,\theta_1 \wedge \theta_2 \wedge \theta_3 + \sum_{i=1}^{3} \theta_i \wedge (a_i dt \wedge \eta_i + a_j a_k \eta_j \wedge \eta_k)$$

for a function $V : M \to \mathbb{R}^+$ satisfying a linear equation along the fibres depending on the curvature of Y.

- If $Y = S^3$ and $F =$ flat $\mathbb{R}^4$ then φ_ϵ is Bryant–Salamon AC G_2 structure on the spinor bundle of S^3.

- φ_ϵ with $F =$ Taub–NUT as a model for G_2 structures with an S^1 symmetry close to a codimension 4 fixed point locus.

  Rescalings of the $\mathbb{B}_7$ ALC family do converge to the model φ_0 with $Y = S^3$.

  Moreover φ_ϵ coincides with the leading-order terms of the closed G_2 structure obtained from a CY monopole on $T^*S^3 \setminus S^3$.

Donaldson's adiabatic limit equations

The first two equations are
Modify φ to make it **closed**:

$$\varphi' = V\,\theta_1 \wedge \theta_2 \wedge \theta_3 + \sum_{i=1}^{3} \theta_i \wedge (a_i dt \wedge \eta_i + a_j a_k \eta_j \wedge \eta_k)$$

for a function $V : M \to \mathbb{R}^+$ satisfying a linear equation along the fibres depending on the curvature of Y.

- If $Y = S^3$ and $F =$ flat $\mathbb{R}^4$ then φ_ϵ is Bryant–Salamon AC G_2 structure on the spinor bundle of S^3.

- φ_ϵ with $F =$ Taub–NUT as a model for G_2 structures with an S^1 symmetry close to a codimension 4 fixed point locus.

  Rescalings of the $\mathbb{B}_7$ ALC family do converge to the model φ_0 with $Y = S^3$.

  Moreover φ_ϵ coincides with the leading-order terms of the closed G_2 structure obtained from a CY monopole on $T^*S^3 \setminus S^3$.

Tools we have already developed

1. Good function spaces and Fredholm theory for elliptic operators on ALC spaces, including index change formulae and sufficiently good understanding of indicial roots for various operators.
2. ALC analogue of the AC G_2 deformation theory.
3. How to solve the linearised Apostolov-Salamon equations on the complement of a union of smooth compact disjoint special Lagrangian 3-folds in a compact CY 3-fold.

Further work needed

- Understand more generally which AC Calabi-Yau 3-folds arise as the collapsed limit of a family of ALC (*full*)G_2 holonomy metrics.
- Construct families of collapsing smooth compact G_2 manifolds collapsing to a Calabi-Yau 3-fold with isolated conical singularities.
- Construct families of compact G_2 spaces with *isolated conical singularities* modelled on the cone over the homogeneous nearly Kähler structure on $S^3 \times S^3$ collapsing to a Calabi-Yau 3-fold with isolated conical singularities.