# RICCI CURVATURE: SOME RECENT PROGRESS AND OPEN QUESTIONS

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# Introduction.

We will survey some recent (and less recent) progress on Ricci curvature and mention some open problems.

We will give statements but little or no indication of proofs.

Apart from the Kähler (i.e. $U(\frac{n}{2})$) case, manifolds with special holonomy are necessarily Einstein.

The structure theory for general manifolds with lower or 2-sided Ricci bounds has been of some use in their study.

Various open questions in the general case are open even in the context of special holonomy.

# Gromov-Hausdorff limits.

We consider Gromov-Hausdorff limit spaces,

$$(M_i^n, p_i) \xrightarrow{d_{GH}} (X^n, \underline{x}) \,, \tag{1}$$

with or without special holonomy, satisfying

$$|\mathrm{Ric}_{M_i^n}| \leq n-1 \,. \tag{2}$$

First, we make the noncollapsing assumption

$$\mathrm{Vol}(B_1(p_i)) \geq \mathrm{v} > 0 \,. \tag{3}$$

We may just assume $\mathrm{Ric}_{M_i^n} \geq -(n-1)$; later we drop (3).

# Tangent cones.

So begin by assuming

$$\text{(4)} \qquad \mathrm{Ric}_{M_i^n} \geq -(n-1)\,.$$

Recall that a *tangent cone* $X_x$ at $x \in X$ is a pointed rescaled Gromov-Hausdorff limit as $r_i \to 0$,

$$(X, x, r_i^{-1} d^X) \xrightarrow{d_{GH}} (X_x, \underline{x}, d^\infty)\,.$$

If (4) holds, Gromov's compactness theorem tangent cones exist for all $x$ but need not be unique.

In the noncollapsed case, every tangent cone $X_x$ is a *metric cone* $C(Z)$ where $Z$ is a length space and $\mathrm{diam}(Z) \leq \pi$; [ChColding,1997].

# The regular set.

The *regular set* $\mathcal{R}$ is the set for which some tangent cone is isometric to $\mathbb{R}^n$.

In this case, every tangner cone is isometric to $\mathbb{R}$.

If we just assume (4), then $\mathcal{R}$ need *not* be open.

However, a sufficiently small neighborhood, $\mathcal{R}_\epsilon \supset \mathcal{R}$, is bi-Hölder to a smooth riemannian manifold.

# The singular set.

The singular set, $\mathcal{S} := X \setminus \mathcal{R}$, has a filtration

$$\mathcal{S}_0 \subset \cdots \subset \mathcal{S}_{n-1} = \mathcal{S}\,.$$

Here, $\mathcal{S}_k$ consists of those points for which no tangent cone splits off $\mathbb{R}^{k+1}$ isometrically.

By [ChCo,1997], in the sense of Hausdorff dimension,

$$\dim \mathcal{S}_k \leq k\,.$$

**Remark.** For minimal submanifolds and harmonic maps, such a theory was pioneered by De Giorgi, Federer, Fleming, Almgren and Schoen-Uhlenbech.

# 2-sided bounds.

If the 2-sided Ricci bound (3) holds, then $\mathcal{S}$ is *closed* and $\mathcal{R}$ is a $C^{1,\alpha}$ smooth riemannian manifold in harmonic coordinates.

If in addition, the $M_i^n$ are Einstein, then $C^{1,\alpha}$ can be replaced by $C^\infty$.

The above depends on an $\epsilon$-regularity theorem of Mike Anderson stating that if $|\mathrm{Ric}_{M^n}| \leq \delta(n)$ and

$$\frac{\mathrm{Vol}(B_1(p))}{\mathrm{Vol}(B_1(0))} \geq 1 - \delta(n)\,, \qquad \text{(where } 0 \in \mathbb{R}^n\text{)},$$

then the metric on $B_{1/2}(p)$ is smooth.

# Special holonomy and Ricci curvature.

Except in the $U(\frac{n}{2})$ case, special holonomy implies *Einstein*.

There is a distinguished parallel $(n-4)$-form $\Omega$ for which $R_0$, the trace free part of the curvature is *anti-self-dual*:

$$*R_0 = -\Omega \wedge R_0\,.$$

Let $p_1$ denote the first Pontrjagin class.

The $L_2$-norm of the curvature is bounded by the topological invariant

$$\int_{M^n} |Rm|^2 \leq -(p_1 \cup [\Omega])(M^n)\,.$$

For this, see [Salomon,1987,1996] and [ChTi,2005].

# Rectifiability of singular sets.

**Theorem.** ([ChTi,2005]) Let $X$ denote a noncollapsed pointed Gromov-Hausdorff limit of manifolds with bounded Ricci and special holonomy.

1) The singular set $\mathcal{S} \subset X$ has Hausdorff codimension $\geq 4$.

2) Bounded subsets of $\mathcal{S} \cap B_1(x)$ are $\mathcal{H}^{n-4}$-rectifiable with a definite bound on $\mathcal{H}^{n-4}(\mathcal{S} \cap B_1(x))$.

3) Tangent cones at $\mathcal{H}^{n-4}$ a.e. $x \in \mathcal{S}$ are unique and of orbifold type.

**Remark.** For a strong generalization, see below.

# Definition of rectifiability.

**Definition.** A metric space $B$ is *k-rectifiable* with respect to a measure $\mu$ if there exist measurable subsets $A_i$ with:

1) $\mu(B \setminus \bigcup_i A_i) = 0$.

2) Each $A_i$ is bi-Lipschitz to a subset of $\mathbb{R}^k$; $f_i : A_i \to \mathbb{R}^k$.

3) $\mu \,|\, A_i$ is absolutely continuous with respect to the push-forward measure $(f_i^{-1})_*(\mathcal{L}_k)$.

**Remark.** Assuming just the *lower* Ricci bound (4) and an $L_p$ curvature bound, rectifiability of the (nonexceptional part) of the singular set was proved in [Ch,2003].

In the Kähler case, the full singular set is nonexceptional.

# The Kähler–Einstein case.

In the Kähler-Einstein case, [Donaldson-Sun,2015] showed that the tangent cone is unique at *all* $x \in X$.

They prove much more; we won't be able to go into it here.

# Bounded Ricci implies $\mathcal{S}$ has codimension $\geq 4$.

**Theorem** ([ChNaber, 2015]) If (2), (3), the 2-sided Ricci bound and noncollapsing condition hold, then

$$\dim \mathcal{S} \leq n - 4\,.$$

# Diffeomorphism finiteness in dimension 4.

**Theorem.** ([ChNa,2015]) For $n = 4$, $|\mathrm{Ric}_{M^4}| \leq 3$ the are at most $N(d, \mathrm{v})$ diffeomorphism types of manifolds admitting a metric such that

$$|\mathrm{Ric}_{M^4}| \leq 3\,,$$

$$\mathrm{diam}(M^n) \leq d\,,$$

$$\mathrm{Vol}(M^n) \geq \mathrm{v} > 0\,.$$

Moreover, there is an a priori $L_2$ curvature bound,

$$\int_{B_1(p)} |Rm|^2 \leq c(n, \mathrm{v})\,.$$

At singular points of the corresponding limit spaces, the metric is of orbifold type.

# Historical remarks on dimension 4.

The dimension 4 results were conjectured by Anderson; [An,1993].

For $n = 4$, for assuming (2), (3) *and* an $L_2$ bound on $|Rm|$, the (isolated) singularities were known since the 1990's to be of *orbifold type*.

In dimension 4, the Chern-Gauss-Bonnet form bounds the $L_2$ norm of $|Rm(x)|$.

Thus, it was understood that a bound on the Euler characteristic would imply an a priori $L_2$ on $|Rm|$.

# A priori $L_2$-curvature bound in all dimensions.

**Theorem** (JiangNa,2016) If (2), (3) hold, then:

1)$\mathcal{H}^{n-4}(\mathcal{S} \cap B_1(x)) \leq c(n, \mathrm{v})$.

2) $\mathcal{S} \cap B_1(x)$ is rectifiable.

3) For $\mathcal{H}^{n-4}$ a.e. $x \in \mathcal{S}$, the tangent cone is unique and of orbifold type.

(5) $$\int_{\mathcal{S} \cap B_1(x)} |Rm|^2 \leq c(n, \mathrm{v})\,.$$

5) There are results corresponding to 2), 3) for lower strata.

**Note.** (1), (4) were conjectured in [ChCo,1997], [CN,2015].

# Lower bounded Ricci.

With Naber, we are checking a proof that if the 2-sided bound on Ricci is weakened to (4),

$$\mathrm{Ric}_{M_i^n} \geq -(n-1)\,,$$

then 1) and 2) continue to hold i.e.

1)$\mathcal{H}^{n-4}(\mathcal{S}\cap B_1(x)) \leq c(n,\mathrm{v})$.

2) $\mathcal{S}\cap B_1(x)$ is rectifiable.

3) There are corresponding result for lower strata.

# Orbifold structure of the singular set.

In case the Kähler-Einstein case, it is known (TiLi,2015]) that on the top stratum of $\mathcal{S}$, the metric is of *orbifold type*.

This could also follow from [DonaldsonSun,2015].

**Open Problem 1.** As conjectured in [ChTi,2005], for special holonomy, are singularities of orbifold type off a subset of codimension 6?

For general noncollapsed limit spaces with bounded Ricci curvature, does this hold $\mathcal{H}^{n-4}$-a.e.?

# $\epsilon$-regularity; integral curvature bounds.

Given (2), (3), [Anderson,1989], [Nakajima,1990], showed that

$$\fint_{B_1(p)} |Rm|^{n/2} \leq \delta(n)\,, \tag{6}$$

implies

$$\sup_{x\in B_{1/2}(p)} |Rm(x)| \leq c(n)\,. \tag{7}$$

**Remark.** Note the integral in (6) is normalized by volume.

# Generalizations.

Let $0 \in \mathbb{R}^{n-4}$ and let $\underline{z}$ denote the vertex of $C(S^3/\Gamma)$.

If $n > 4$, for *most* $\Gamma$, the curvature bound,

$$\sup_{x \in B_{1/2}(p)} |Rm(x)| \le c(n)\,,$$

was obtained in [ChCoTi,2002], assuming

$$d_{GH}(B_1(p), B_1((0,\underline{p}))\,,$$

$$⨍_{B_1(p)} |Rm|^2 < \delta(n)\,.$$

In the Kähler case there were no exceptions and more generally ([ChTi,2005]), none for special holonomy.

# $L_p$ bounds; higher codimension

By a quite different argument, assuming the 2-sided Ricci bound, [ChenDonaldson,2011] proved $\epsilon$-regularity in the above case with no exceptional cases.

Assuming just the *lower* Ricci bound (4), the previous $\epsilon$-regularity theorem was generalized to $L_p$ bounds and rectifiability of the singular set was shown; [Ch,2003].

However, there were still some exceptional cases, though not in the Kähler case.

# $\epsilon$-regularity with no exceptions.

**Problem 2.** In the noncollapsed case, prove an $\epsilon$-regularity theorem for integral curvature and lower Ricci bounds, with no exceptional cases.

**Remark.** Though this is problem is rather technical, the argument, if such exists, might be interesting.

# Structure theory in the collapsed case.

**Theorem.** ([ChCo,2000]) Let $X$ satisfy (4), $\mathrm{Ric}_{M_i^n} \geq -(n-1)$.

1) There is a renormalized limit measure $\nu$ whose measure class depends only on the metric structure of $X$.

2) The regular set $\mathcal{R}$ has full measure and the singular set $\mathcal{S}$ has codimension 1 with respect to $\nu$.

3) $X$ is rectifiable with respect to $\nu$.

**Theorem.** ([CoNa,2011]) The dimension of $\mathcal{R}$ is constant and the isometry group of $X$ is a Lie group (as conjectured in [ChCo,2000]).

# Open problems in the collapsed case.

For lower or 2-sided Ricci bounds:

**Problem 1.** Show the existence of (a dense set of) manifold points.

**Problem 2.** Show that the Hausdorff dimension of the singular set is less than that of the regular set.

**Remark.** A lot is known about collapsed limits in the Calabi-Yau case:

[Gross-Wilson,2000], [Tosatti,2009,2010], [GrossTosattiZhang,2013], [TosattiWeinkovYang,2014], [HeinTosatti2015].

# $\epsilon$-regularity: collapse and topology.

**Theorem.** ([NaZhang]) Suppose $M^n$ is Einstein with

$$|\mathrm{Ric}_{m^n}| \le n-1\,,$$

$$d_{GH}(B_1(p), B_1(0^k)) < \delta(n) \qquad (0^k \in \mathbb{R}^k)\,.$$

Then $i_*(\pi_1(B_{\delta(n)}(p))) \subset \pi_1(B_1(p))$ is nilpotent of rank $\le n-k$.

If equality holds, then

$$\sup_{x\in B_{1/2}(p)} |Rm(x)| \le c(n)\,.$$

**Remark.** This builds in part on [KapovitchWilking,2011] and is consistent with e.g. [GrossWilson,2000].

# $\epsilon$-regularity with possible collapsing, $n = 4$.

**Theorem.** ([ChTi,2006]) There exists $\delta > 0$ such that if $M^4$ is Einstein,

$$|\mathrm{Ric}_{M^4}| \leq n - 1\,,$$

$$\int_{B_1(p))} |Rm|^2 \leq \delta\,, \tag{8}$$

then

$$\sup_{x \in B_{1/2}(p)} |Rm(x)| \leq c\,.$$

**Note.** The integral in (8) is *not* normalized by volume.

# Some ingredients in the proof.

Theory of collapse with locally bounded curvature.

Vanishing of the Euler characteristic of the collapsed region.

Equivariant good chopping.

For $n = 4$, the Chern-Gauss-Bonnet form bounds $|Rm(x)|^2$.

An iteration argument.

The almost volume cone implies almost metric cone, implemented in the possibly collapsed case.

# Open problems in the collapsed case.

**Problem 3.** To the extent possible, completely classify Einstein manifolds in dimension 4.

There is important progress related to the ALF case; see [Minerbe,2010,2011], [ChenChen,2015,2016].

**Problem 4.** Is there a possibility of generalizing the $\epsilon$-regularity theorem of [ChTi,2006] to higher dimensions.