Comparing Algorithms for Detecting Gerrymandering: An Investigation into District Borders Through High and Low Population Density Areas in Ensemble Map Generation

Jonas Eichenlaub (Bowdoin College), Gibbons Fellowship Summer Research Project 2021
Advisors: Jonathan Mattingly (Duke), Gregory Herschlag (Duke), Jack O'Brien (Bowdoin)

Overview

Gerrymandering has been a pervasive issue throughout American history, but only over the last decade have mathematicians begun to make noticeable contributions towards its detection and avoidance. While these contributions have come from a diverse set of mathematical disciplines and perspectives, one focus has been on simulating an ensemble of potential legislative maps and then comparing a real legislative map against this distribution. By plugging in historical vote counts into both the real and simulated maps, it is then possible to gain an understanding of just how much more or less represented a given party is under the real map than under the average map from the simulations. This understanding can be supplemented by other analyzes of the ensemble, such as by looking at how large of a swing in the popular vote the minority party must attain to win a majority of the legislative seats. While it is easy for policymakers to circumvent any single gerrymandering metric, generating an entire distribution of legally possible legislative maps allows for a wholistic analysis of how partisan and fair the real map is.

A prevalent way of creating such an ensemble of simulated maps is by encoding a state as a graph, with each node representing a voting precinct and each edge representing the border between neighboring precincts. The graph is partitioned into districts to according to constraints and preferences within the algorithm make sure that the edges that are cut to create connected components (partitions or districts) with roughly equal populations, minimal municipality splitting, and whatever other legal criteria legislative maps must meet. To create a series of maps with this method, which is termed merge-split, two adjacent districts are combined, and a new panning tree is drawn across both of them (the merge stage). Then a new edge is cut to create two new connected components that meet the specified restrictions (the split stage). This new map is then added to the ensemble of maps and the process is repeated.

Duke Professors Jonathan Mattingly and Gregory Herschlag have proposed formalizing such an approach by placing a recombination algorithm (DeFord, Duchin, Solomon, 2021) into a Metropolis Hastings Markov Chain (Autry, Carter, Herschlag, Hunter, Mattingly, 2022), where the proposal of a new map is separated from the acceptance of that proposal into the ensemble. Instead of making decisions about how maps are added to the ensemble based the structure of the algorithm, Mattingly and Herschlag's work applies the existing mathematical theory on Metropolis Hastings chains to justify their approach. Each map represents a stage in the chain, and new maps are proposed with the merge-split technique described above. Maps are then added to the chain based off an acceptance probability that quantifies how well the proposed map fits within the legal criteria, which is designed to be changeable to meet the specific policy and legal parameters of different states. By making the choices within the algorithm explicit, Mattingly and Herschlag offer an approach to detecting gerrymandering that is more assessable and defensible when used as evidence in legal challenges to gerrymandered maps.

Part of their contribution is to sample maps over possible partitions of a state rather than from possible spanning trees of the graph of the state, since each partition of a state can be formed by a different number of spanning tree configurations. In other words, historically the merge-split method for randomly proposing a potential legislative map makes it equally likely for any given edge of the graph to lie on a district border. However, there are likely many different spanning tree configurations that result in the partition of the state into the same set of legislative districts; further, the number of spanning trees that produce the same partition will vary by partition. To allow the option of using either sample spaces, Mattingly and Herschlag introduce a gamma parameter to the acceptance probability of the Metropolis Hastings chain which controls whether the acceptance probability is weighted by the number of spanning trees that could produce it (when gamma is zero) or not (when gamma is one). The latter makes more theoretical sense since partitions not spanning trees have meaning in the real-world, but the former is computationally more convenient, and the way merge-split algorithms have often functioned.

My research this summer focuses on determining whether sampling from partitions generates different ensembles of maps than that from the spanning tree method. Since the application of these ensembles to detecting gerrymandering relies on the assumption that the ensemble of maps are representative samples of all possible legal legislative maps, it is important to know that technical decisions in the algorithm do not have significant impacts on characteristics of the ensemble. More specifically, I investigate whether algorithms that sample from the space of spanning trees are less likely to generate maps that draw district lines along the border between areas with high and low population density (i.e., along the borders of cities). To this end, I ran Mattingly and Herschlag's algorithm with both gamma equal to zero and one on a variety of "test states" that I constructed. I represented each test state simply as a lattice graph, with a few nodes on the lattice replaced with smaller, denser lattice graphs representing cities. The "city" nodes along the edge of the denser lattice all had edges connecting to the same neighboring "noncity" node, which meant that drawing a district line between these two areas requires cutting far more edges than drawing it between other sections of the graph (which normally would only require one cut); the motivating idea behind my inquiry is that the spanning tree algorithm views this as costlier and is less likely to select maps with district lines the run between the city and noncity nodes.

Additionally, to answer this question I needed to make sure the Metropolis Hastings chain was reaching its stationary distribution. Accordingly, a large part of this research revolved around understanding ways to check for the convergence of Markov chains and then applying those ideas to the ensembles of maps.

Ultimately, I found that the algorithm that samples uniformly from spanning trees is less likely to make districts that have edges along the border of cities, but only under certain situations. The two main factors that determine whether such bias appears in an ensemble are the district population size and the city population size. When the city population fits perfectly within a single or multiple districts, the spanning tree algorithm is significantly more likely to create maps that do not break up cities compared to the partition-space algorithm; hence, a hidden bias in the original recombination approach is evident. However, if the city population does not fit evenly into one or more districts, both algorithms are forced to break up the city and there is not a notable difference between the ensembles they generate. Numerical results that illustrate this behavior are shown below in the Example Investigation and Appendix A sections.

Methodology

To explore the behavior of the two different approaches to map ensemble generation, I created a series of fake states to run Mattingly and Herschlag's code on. In order to isolate the type of bias I was looking for, I kept the test states as simple as possible, laying out the precincts as a lattice graph, with each precinct having equal population and being part of the same municipality. The part of the code that handles avoiding municipality splitting was thus excluded, as was complications due to varied precinct sizes. To represent highly populated areas, I then added “cities” on this grid by replacing a single node with another, tighter grid of nodes, all of which still had the same population and were part of the same municipality as the rest of the state. As was mentioned above, the nodes that are right next to a city have edges that connect to all the city- nodes on the side of the city closest to them; thus, drawing a district border between the city and noncity node requires cutting far more edges than drawing a district border. Any algorithm that views a lower number of cut edges as correlating to a more compact map is less likely to create maps with district borders along population density borders. In my investigation, I also varied the placement, size, and number of these “cities” to gain insight into how the two versions of the algorithm handle the edges that transition from high density to low density areas of the state.

For each test state configuration, I generated an ensemble of maps with gamma set to zero and then a separate ensemble with a gamma of one. Before generating the gamma equals one ensemble, I first tuned the weight on the measure of compactness in the energy function of the Metropolis Hastings chain such that the mean district compactness in this ensemble matched that of the mean district ensemble from the gamma of zero run. This is important to ensure that the two ensembles are as close to each other as possible, which strengthens the validity of any difference my comparison of them illustrates. For each of the two ensembles, I then count the number of times that edges of the graph which connect a node in a city to a node not in a city fell on a legislative district border across all maps in that ensemble; this number is then divided by a tally of the total number of edges that fell on district borders. This statistic is the *fraction of edges that cross district borders that go from city to noncity*. I calculated a similar statistic termed *fraction of edges that cross district borders that go from city to anywhere* that looks at the proportion of edges that are cut which connect one city node to either a noncity node or another city node. While these statistics are quite similar, the later includes city-to-city edges in addition to city-to-noncity, and reporting both of them offers a more thorough description of the ways the algorithms sample city edges. Finally, I compared these statistics as the gamma value changes to assessing if there is a discrepancy between the type of maps ensemble each algorithm generates and, if so, what characteristics of the test state created that discrepancy.

A secondary aspect of my research involved learning how to determine that a given run of the Metropolis Hastings Markov Chain on a test state has converged. Each state of the chain is an individual legislative map, and thus, to determine convergence, I created several summary statistics that describe the characteristics of a map as single number which can then be analyzed and visualized. I plotted each statistic over time to see if it settled into a consistent range of values as the chain progressed. Some of these convergence statistics come directly from the algorithm, such as the mean and variance of compactness of the districts in each map, or the Gibbs energy score that is used in the acceptance probability by the chain. Other convergence statistics that I explored are more abstracted, such as the percentage of a specified region of the state that exists within the district the makes up the largest portion of it. While exactly how any given region is divided up into districts varies map by map, the portion of it that is encapsulated in the largest district should eventually converge to a consistent range. The same is true for the portion of a vertical line that cuts through the middle of the map that falls within the majority district, which is a variant of this convergence metric that I considered.

For each of these statistics, I assessed its distribution for each of the four quarters of the ensemble of maps (i.e. the first 25% of maps generated, the second 25% of maps, etc) and compare the quarter distributions to each other. In almost all cases, I found that there was significantly more variance in the first quarter of maps, whereas the later three quarters all behaved largely the same when I ran the Metropolis Hastings chain with one million steps. To reflect this, my analyzes exclude the first 25% of maps generated, treating this as a burn-in period. I also repeated every run of the chain with different starting states and compared each statistic between runs as a further test of convergence.

Example Investigation

Over the course of the summer, I investigated a variety of different test state configurations, including ones that were asymmetrical and had multiple cities of different sizes. Ultimately, the placement and number of cities did not seem to have an impact on the results, so the test state I use in this analysis is simple in structure: a 9x9 lattice grid graph with the middle node replaced by a 4x4 lattice grid graph "city." Each precinct is given a population of 1000 people, making the total test state population 96000 and the city population 16000.

Since the most important factor in my findings is the relationship between the ideal district population size and the city population size, I (somewhat arbitrarily) opt to vary the number and size of districts instead of varying the map itself. I run the Metropolis Hastings algorithm with gamma equals zero and gamma equals one on this test state with 3 districts with ideal district population 32000; 4 districts with ideal district population 24000; 6 districts with ideal district population 16000; 8 districts with an ideal district population of 12000; and 12 districts with an ideal district population of 8000 people.

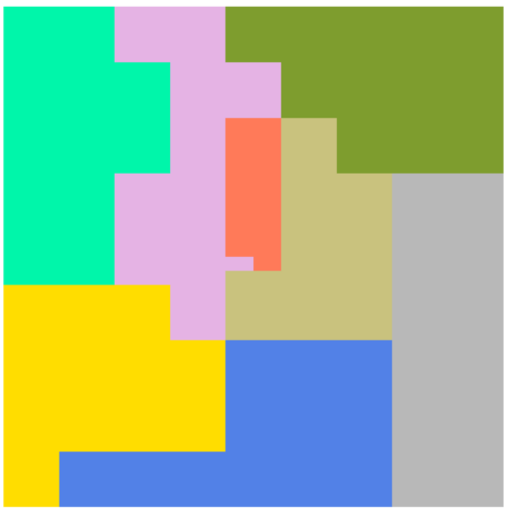


Image: Visualization of a sample map of the test state from an 8 district, gamma equals one run

After generating the gamma equals zero ensembles for each number of districts, I calculate the distribution of compactness scores for each ensemble and set them as targets. I then tune the weight on the compactness score in the energy function such that the distributions of compactness scores generated when gamma equals one are close to these targets. To ensure convergence, I run two trials of the Markov Chain with 1 million steps for each combination of gamma value and number. After discarding the first quarter of the ensemble as a burn in period, I then compare the results of all of the convergence statistics from the two trials against one another. The results from the compactness weight tuning and compactness convergence statistic for each scenario are reported in Appendix B of this document.

Finally, I calculate the *fraction of edges that cross district borders that go from city to noncity* and the *fraction of edges that cross district borders that go from city to anywhere* for each scenario and each gamma value. These findings are summarized in the table below and shown in full in Appendix A, along with visualizations of the frequency that any given edge in the test state falls on a district border. As the bottom two rows of the below table indicate, the gamma equals one algorithm is more likely than the gamma equals zero algorithm to draw maps that have district borders cutting city edges for every district configuration. However, the magnitude of this difference varies significantly, from ~0.287 when there are 4 districts to ~0.029 when there are 12 districts. When the city and surrounding nodes can be included in one district, as is perfectly the case when there are 4 districts and largely the case when there are 3 districts, the uniform over spanning trees (gamma equals zero) algorithm is much less likely to break up that region between districts. When most of the city and surrounding area can be fit into one or two districts, as is the case for the 6 and 8 district scenarios, this phenomenon still occurs, but to a lesser extent. When there are so many districts that the high population density area must always be broken up, as is the case in the 12 district scenario, there is almost no difference between the ensembles generated by each algorithm.

Number of Districts	3	4	6	8	12
Ideal District Pop	32000	24000	16000	12000	8000
Δ City-to-noncity	0.0535122	0.0991	0.03579	0.0338853	0.019087
Δ City-to-anywhere	0.13672	0.28727	0.06528	0.06211	0.02969

Table: Summary of the difference between the gamma equals zero and gamma equals one ensemble for the *fraction of edges that cross district borders that go from city to noncity* and the *fraction of edges that cross district borders that go from city to anywhere* statistics for each scenario. Note that a positive value indicates that the statistic for gamma equals one ensemble has a larger value.

Conclusion

The results from the sample experiment above support my broader findings that the algorithm that samples uniformly from partitions is more likely to draw maps that break up cities than the algorithm that samples uniformly from spanning trees if the city and surrounding nodes fit into a single district. While there are many situations in which both versions of the algorithm produce similar ensembles, it is important to be aware that there are some instances where sampling uniformly from spanning tree creates an ensemble that rarely breaks apart cities. These results reenforce why it is important to use Mattingly and Herschlag's approach, as their algorithm can be centered on partitions does not automatically keep municipalities within the same district in a hidden and uncontrollable manner. Instead, it allows for policymakers to explicitly decide how much they want to preserve cities by incorporating this preference into the acceptance probability. Given that the redistricting process from the 2020 Census is currently underway, this insight could find application in the upcoming legal challenges to the inevitably gerrymandered maps that politicians propose.

Appendix A: City to Non-City Edge Count Results

<u>3 Districts</u>

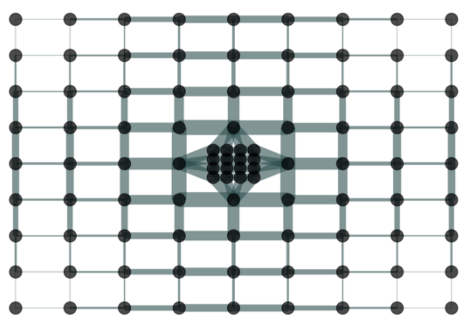
Gamma 0:

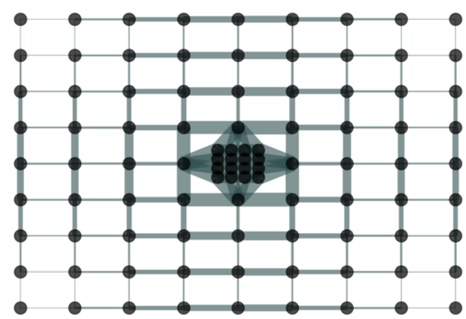
Total number of graph edges that cross district borders: 3898403
Number of city graph edges that cross district borders: 1313635
Number of city to noncity graph edges that cross district borders: 582675
Fraction of edges that cross district borders that go from city to anywhere: 0.336967471038781
Fraction of edges that cross district borders that go from city to noncity: 0.1494650501756745

Gamma 1:

Total number of graph edges that cross district borders: 2663934
Number of city graph edges that cross district borders: 1261873
Number of city to noncity graph edges that cross district borders: 540718
Fraction of edges that cross district borders that go from city to anywhere: 0.473687786559276
Fraction of edges that cross district borders that go from city to noncity: 0.20297725093789862

Difference in fraction of edges that cross district borders that go from city to anywhere: 0.13672
Difference in fraction of edges that cross district borders that go from city to noncity: 0.0535122





Images: Visualization of the graph of the test state with edge thickness corresponding to the frequency that that edge lies on a district board in the ensemble of maps when gamma is 0 (left) and gamma is 1 (right)

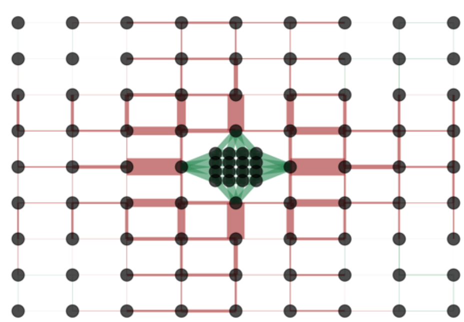
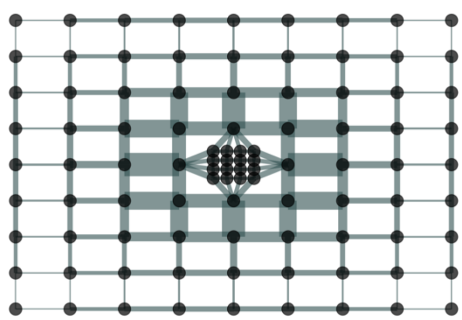
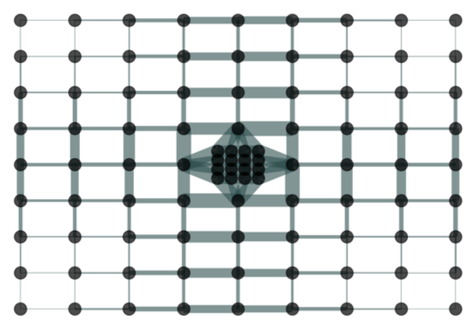


Image: Visualization of the difference between the previous two images. A green edge indicates that the gamma 1 ensemble cuts the edge more often and red indicates the gamma 0 ensemble cuts the edge more often; the edge thickness corresponds to the magnitude of the difference.

<u>4 Districts</u>
Gamma 0:
Total number of graph edges that cross district borders: 8034949
Number of city graph edges that cross district borders: 1198847
Number of city to noncity graph edges that cross district borders: 613529
Fraction of edges that cross district borders that go from city to anywhere: 0.14920405842028
Fraction of edges that cross district borders that go from city to noncity: 0.07635754750901344

Gamma 1:
Total number of graph edges that cross district borders: 7744024
Number of city graph edges that cross district borders: 3380111
Number of city to noncity graph edges that cross district borders: 1358937
Fraction of edges that cross district borders that go from city to anywhere: 0.436479923099411
Fraction of edges that cross district borders that go from city to noncity: 0.17548202329951457

Difference in fraction of edges that cross district borders that go from city to anywhere: 0.28727
Difference in fraction of edges that cross district borders that go from city to noncity: 0.0991

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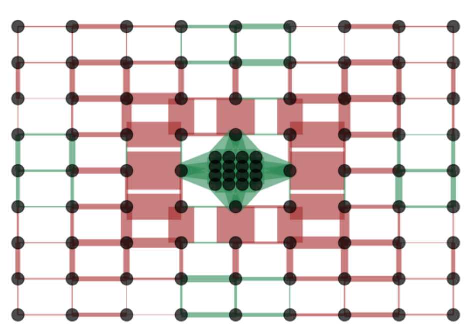

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6 Districts

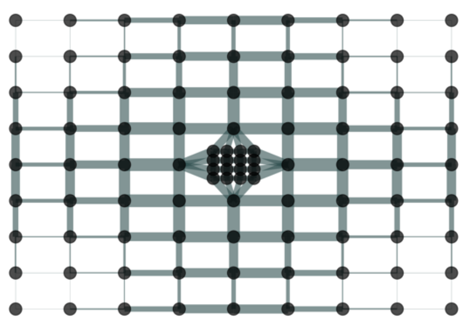
Gamma 0:

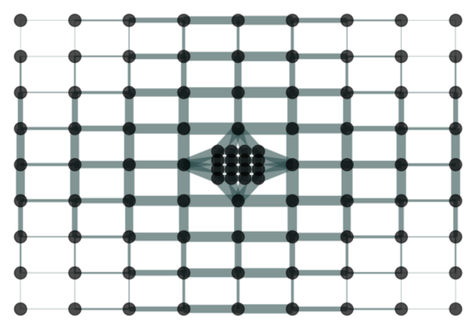
Total number of graph edges that cross district borders: 15750415
Number of city graph edges that cross district borders: 3642556
Number of city to noncity graph edges that cross district borders: 1627540
Fraction of edges that cross district borders that go from city to anywhere: 0.231267303115505
Fraction of edges that cross district borders that go from city to noncity: 0.10333315026937386

Gamma 1:

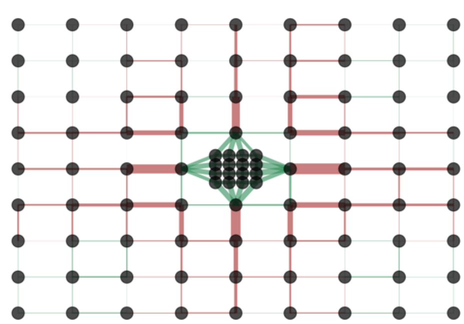
Total number of graph edges that cross district borders: 14044939
Number of city graph edges that cross district borders: 4165026
Number of city to noncity graph edges that cross district borders: 1954033
Fraction of edges that cross district borders that go from city to anywhere: 0.296549952975943
Fraction of edges that cross district borders that go from city to noncity: 0.13912719734845413

Difference in fraction of edges that cross district borders that go from city to anywhere: 0.06528
Difference in fraction of edges that cross district borders that go from city to noncity: 0.03579





Images: Visualization of the graph of the test state with edge thickness corresponding to the frequency that that edge lies on a district board in the ensemble of maps when gamma is 0 (left) and gamma is 1 (right)



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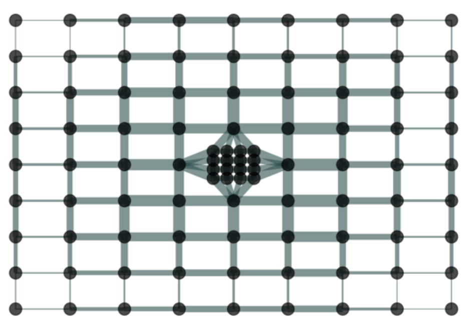
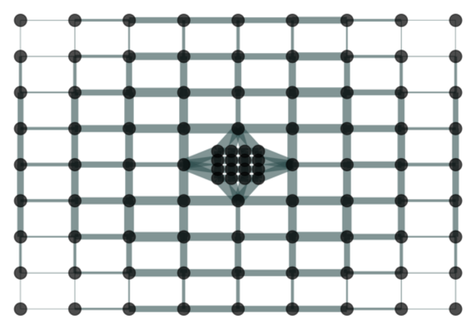
<u>8 districts</u>
Gamma 0:
Total number of graph edges that cross district borders: 23161606
Number of city graph edges that cross district borders: 5439565
Number of city to noncity graph edges that cross district borders: 2391035
Fraction of edges that cross district borders that go from city to anywhere: 0.234852669542863
Fraction of edges that cross district borders that go from city to noncity: 0.10323269465856556

Gamma 1:
Total number of graph edges that cross district borders: 22573096
Number of city graph edges that cross district borders: 6703373
Number of city to noncity graph edges that cross district borders: 3095179
Fraction of edges that cross district borders that go from city to anywhere: 0.296962942079367
Fraction of edges that cross district borders that go from city to noncity: 0.1371180541650113

Difference in fraction of edges that cross district borders that go from city to anywhere: 0.06211
Difference in fraction of edges that cross district borders that go from city to noncity: 0.0338853

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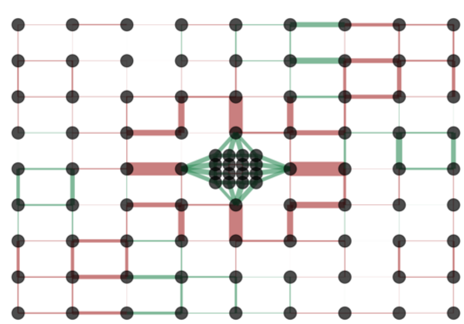


Image: Visualization of the difference between the previous two images. A green edge indicates that the gamma 1 ensemble cuts the edge more often and red indicates the gamma 0 ensemble cuts the edge more often; the edge thickness corresponds to the magnitude of the difference.

12 Districts

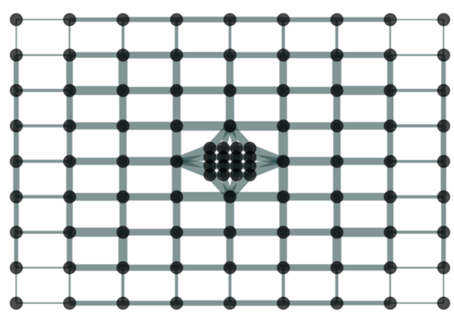
Gamma 0:

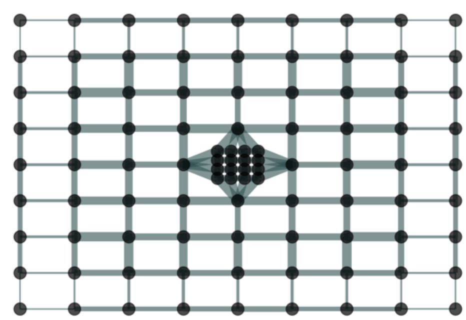
Total number of graph edges that cross district borders: 36803772
Number of city graph edges that cross district borders: 8758773
Number of city to noncity graph edges that cross district borders: 3826070
Fraction of edges that cross district borders that go from city to anywhere: 0.237985742330976
Fraction of edges that cross district borders that go from city to noncity: 0.10395863771789479

Gamma 1:

Total number of graph edges that cross district borders: 36096671
Number of city graph edges that cross district borders: 9662459
Number of city to noncity graph edges that cross district borders: 4441562
Fraction of edges that cross district borders that go from city to anywhere: 0.267682828701849
Fraction of edges that cross district borders that go from city to noncity: 0.12304630529502292

Difference in fraction of edges that cross district borders that go from city to anywhere: 0.02969
Difference in fraction of edges that cross district borders that go from city to noncity: 0.019087





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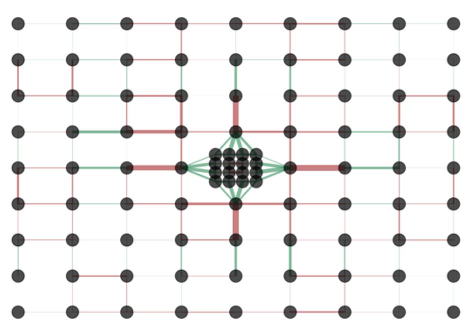

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Appendix B: Compactness Weight Tuning and Convergence Checks Results by State

3 Districts: population 32000, compactness weight .5396

Gamma 0 Compactness Summary by Quarter of Ensemble:
Run #1:
Means: 71.57338655322349 71.6040104480509 71.82091632655622 71.70195193433987
Standard Deviations: 7.613753290351 7.5380596882495 7.7717735695194 7.5962462720196
Ranges: 66.4971236236279 61.08422928558396 62.774950462636994 57.44145339635263
Run #2:
Means: 71.53558798745712 71.51543384880365 71.76653054153229 71.65329302905855
Standard Deviations: 7.5879449887502 7.647850535337 7.746647747328 7.6428397010778
Ranges: 65.7760214004857 64.51948888059623 60.41257693180678 62.196196660482386

Gamma 0 Compactness Summary Post Burn-In:
Run #1:
Means: 71.708959569649
Standard Deviations: 7.636521098099622
Ranges: 62.774950462636994
Run #2:
Means: 71.6450858814365
Standard Deviations: 7.6799474403666235
Ranges: 64.51948888059623

Gamma 1 Compactness Summary by Quarter of Ensemble:
Run #1:
Means: 69.70751961626624 69.65135947922388 69.5823823838181 69.47049021362706
Standard Deviations: 6.3510553189747 6.344377430157 6.3257073476118 6.263496221221
Ranges: 55.977306701102336 45.46514751457646 49.21836211811362 46.446868402533426
Run #2:
Means: 69.59858874716187 69.5596625955256 69.5395818554374 69.65189381325207
Standard Deviations: 6.286361367958 6.296495305112 6.3036267263457 6.302359743030
Ranges: 43.242271488249585 51.38383275744819 50.19840594210082 46.43221804770785

Gamma 1 Compactness Summary Post Burn-In:
Run #1:
Means: 69.5680773588897
Standard Deviations: 6.311728425014293
Ranges: 49.21836211811362
Run #2:
Means: 69.58371275473836
Standard Deviations: 6.301017800486444
Ranges: 51.38383275744819

<u>4 Districts:</u> population 24000, compactness weight .348

Gamma 0 Compactness by Quarter of Ensemble:
<u>Run #1:</u>
Means: 96.66765568673294 96.87597185485266 96.5382619163998 96.5820868747472
Standard Deviations: 9.318165553786 9.1900811826952 9.209704549311 9.296390709
Ranges: 77.2578580865681 81.76433944865231 87.82331602086956 87.49767830378049
<u>Run #2:</u>
Means: 96.3425734700747 96.59928434247783 96.69501243353994 96.4684727437191
Standard Deviations: 9.257259806793625 9.242361000825 9.351284434298 9.16223078119
Ranges: 85.45276320669207 84.55410836523959 83.0244801793709 83.82324855816547

Gamma 0 Compactness Post Burn-In:
<u>Run #1:</u>
Means: 96.66544021533323
Standard Deviations: 9.233391863826958
Ranges: 87.82331602086956
<u>Run #2:</u>
Means: 96.58758983991228
Standard Deviations: 9.252749061516196
Ranges: 84.55410836523959

Gamma 1 Compactness by Quarter of Ensemble:
<u>Run #1:</u>
Means: 97.68381267199362 97.72575811562625 97.72731362027591 97.95473235470469
Standard Deviations: 11.79338668845379 11.51251670010 11.9163385004294 11.7467337801
Ranges: 91.11393147303446 92.36082970183591 95.08121321183918 88.65147355612636
<u>Run #2:</u>
Means: 97.45999113139756 97.76382716049311 97.74997615587667 98.01079216227375
Standard Deviations: 11.548304556675252 11.87834479556 11.5563198235215 11.719349889
Ranges: 85.4723985995405 93.29637175407527 84.59717279052981 95.02251752893241

Gamma 1 Compactness Post Burn-In:
<u>Run #1:</u>
Means: 97.80260238058315
Standard Deviations: 11.726858705787533
Ranges: 95.08121321183918
<u>Run #2:</u>
Means: 97.84153296325063
Standard Deviations: 11.719354858029131
Ranges: 95.02251752893241

<u>6 Districts:</u> population 16000, compactness weight .421

Gamma 0 Compactness by Quarter of Ensemble:
<u>Run #1:</u>
Means: 136.72483371387736 137.90431135547738 137.74375396961867 137.82234793575313
Standard Deviations: 11.347076097690 11.804755290285 11.7342712779 11.539596497514
Ranges: 99.6626042096766 100.48147153598282 97.58230508523002 86.04411764705884
<u>Run #2:</u>
Means: 137.244744972088 137.91669654943365 137.9016454780686 137.28514833320733
Standard Deviations: 11.709994354587 12.495952976577 11.9898558544 11.632201663096
Ranges: 88.9486513743341 100.03697917356777 92.29411764705884 81.54472841992055

Gamma 1 Compactness Post Burn-In:
<u>Run #1:</u>
Means: 137.82347136181076
Standard Deviations: 11.693595662024535
Ranges: 100.48147153598282
<u>Run #2:</u>
Means: 137.7011612211542
Standard Deviations: 12.048141766731733
Ranges: 101.95370830014966

Gamma 1 Compactness by Quarter of Ensemble:
<u>Run #1:</u>
Means: 135.24933980401823 136.23970427655686 136.39398417628422 135.69054434491923
Standard Deviations: 10.738139269747 10.672389891110 10.57964385306 10.565055011664
Ranges: 100.31116363344611 89.01085546763926 72.47064295413426 85.53740085069302
<u>Run #2:</u>
Means: 135.9470312744596 136.1163752875754 135.82004052825744 136.1702325440887
Standard Deviations: 10.902643419788 10.69634585374 10.467957367887 10.522844740865
Ranges: 77.26470588235296 82.65939515207322 85.67099675796189 86.03848457670853

Gamma 1 Compactness Post Burn-In:
<u>Run #1:</u>
Means: 136.10807759925342
Standard Deviations: 10.610098328426085
Ranges: 89.53740085069302
<u>Run #2:</u>
Means: 136.03555033413264
Standard Deviations: 10.56395361836241
Ranges: 85.65939515207322

<u>8 Districts:</u> population 12000, compactness weight .317

Gamma 0 Compactness by Quarter of Ensemble:
<u>Run #1:</u>
Means: 185.45839540213998 184.3925630034526 184.76119271253654 185.42694087015755
Standard Deviations: 16.651965428431 16.01013174857 16.548511359662 16.847597294193
Ranges: 111.853996555751 116.01725091950658 110.43489324191086 121.47392735029831
<u>Run #2:</u>
Means: 184.29309586302034 185.15789103882707 184.08065660516354 185.83795696394577
Standard Deviations: 16.411306249530 16.270914052235 16.578570166453 16.52906465634
Ranges: 121.10201836880606 121.00432900432907 132.02792445649595 141.27934810287

Gamma 0 Compactness Post Burn-In:
<u>Run #1:</u>
Means: 185.86023219538217
Standard Deviations: 16.51022661138295
Ranges: 122.22368481192018
<u>Run #2:</u>
Means: 185.35883335276037
Standard Deviations: 16.470061913973524
Ranges: 141.27934810287758

Gamma 1 Compactness by Quarter of Ensemble:
<u>Run #1:</u>
Means: 186.82376023638272 186.49006904187684 185.96082291576585 186.354250593007
Standard Deviations: 15.514977062696 15.902406649181 15.538650754801 15.536469253493
Ranges: 120.98989898989907 112.3352986882399 113.90909090909099 113.33388236544289
<u>Run #2:</u>
Means: 186.55049410003633 186.59436809682248 186.522094378543 186.19983398683704
Standard Deviations: 15.747353002255 15.93747593393 15.861114111536 15.78886171415
Ranges: 115.65381323008444 109.36241607912316 117.42924399554511 119.51441254467

Gamma 1 Compactness Post Burn-In:
<u>Run #1:</u>
Means: 186.26838112325493
Standard Deviations: 15.661727824260572
Ranges: 113.90909090909099
<u>Run #2:</u>
Means: 186.43876473314313
Standard Deviations: 15.863526893787775
Ranges: 119.51441254467892

<u>12 Districts:</u> population 8000, compactness weight .254

Gamma 0 Compactness by Quarter of Ensemble:
<u>Run #1:</u>
Means: 276.13516920719866 276.20063136895226 275.01961459102046 276.98682739100633
Standard Deviations: 19.08140450725 18.670236349850 18.811332507473 18.52437527384
Ranges: 136.87297024072 133.48628925254312 131.40174835046108 135.55381078024348
<u>Run #2:</u>
Means: 275.5869008611792 276.74331187049796 275.9139876323468 274.8806665042182
Standard Deviations: 18.438100316987 18.59634610195 19.43735763312 18.42644594682
Ranges: 156.31236851450188 137.58323778354767 144.72629783254092 149.3054300188606

Gamma 0 Compactness Post Burn-In:
<u>Run #1:</u>
Means: 276.06902445032637
Standard Deviations: 18.686513666492402
Ranges: 138.1393036468506
<u>Run #2:</u>
Means: 275.84598866902104
Standard Deviations: 18.84065223571458
Ranges: 154.5225018328361

Gamma 1 Compactness by Quarter of Ensemble:
<u>Run #1:</u>
Means: 273.7669908250047 275.9966414279346 276.348101744423 276.5792731789054
Standard Deviations: 17.859413941113 17.83192872960 18.05944477489 18.386655450841
Ranges: 134.0462258037294 158.5640993239401 132.5092003037561 132.82511189783514
<u>Run #2:</u>
Means: 275.992928029425 275.47100797095567 275.83437612295916 275.26022616428605
Standard Deviations: 18.067543202899813 18.078150934671953 17.600738 17.759119802708
Ranges: 130.56169534101838 125.53222717225074 123.40156062424967 132.8731531512499

Gamma 1 Compactness Post Burn-In:
<u>Run #1:</u>
Means: 276.308005352828
Standard Deviations: 18.095694492236696
Ranges: 160.31920136475642
<u>Run #2:</u>
Means: 275.52187008606694
Standard Deviations: 17.81538372388175
Ranges: 132.8731531512499