Supplement to
"Risk Sharing and Strategic Choice"
(For Online Publication)

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This supplement contains extended formal results for Daley and Sadowski (2025) in three directions. First, we consider the respective special cases of the model wherein the risk sharing rule adheres to one of the three prominent cooperative bargaining solutions of Nash (1950); Kalai and Smorodinsky (1975); Kalai (1977). For each solution, we provide additional behavioral axioms that, together with the general axioms from Section 3.2, characterize the specific model. Second, we relax No Waste (Axiom 5) and characterize the two-stage model of proportional risk sharing with frictions from Section 6. Third, we provide an example of how our decision-theoretic study can serve as a foundation for further analysis in a simultaneous-move game.

S.1 Specific Sharing Rules

The following representation theorem serves to organize the next three subsections, which provide the axioms for the different solutions from Section 4.1 in turn.

Theorem S.1 (Prominent Solutions) *If preferences* $\left\{\succsim_g^i\right\}_{g\in\mathcal{F}}$ *for* $i \in \{1,2\}$ *can be explained by a two-stage model of proportional risk sharing (Axioms 1-6), then*

1. *they satisfy Axioms 7 and 8 if and only if* α *corresponds to the NBS.*
2. *they satisfy Axiom 9 if and only if* α *corresponds to the KSS.*
3. *they satisfy Axiom 10 if and only if* α *corresponds to the EBS.*

The axioms we provide differ from the well known normative desiderata on which these bargaining solutions are usually founded (see Table 1 from Section 4.1) for three related reasons. First, our theory is positive, and hence our axioms aim to capture testable restrictions on behavior, rather than properties of the distribution of surplus. In this view, the normative appeal of each bargaining solution matters only insofar as it leads agents to adopt it in practice. Second, properties that are intuitive on the domain of utility surpluses may not have an appealing counterpart on our domain of first-stage preferences.

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Third, we look only at bargaining problems that correspond to a risk sharing situation in the context of our model, precluding the vast majority of possible bargaining problems as considered in the cooperative bargaining literature. In particular, the feasible set of utility surpluses in our model is pinned down by η, $u(f+g)$, $u(f)$ and $u(g)$. The normative desiderata discussed in Section 4.1 therefore have less bite on our restricted domain. For instance, it can be shown that all three bargaining solutions satisfy Monotonicity on our domain. Because of these differences, novel axioms are needed.

We will make use of the following notation. Since $\alpha^1_{f,g} + \alpha^2_{f,g} = 1$, we drop the superscript and let $\alpha^1_{f,g} = \alpha_{f,g}$ and $\alpha^2_{f,g} = 1 - \alpha_{f,g}$. Given (f, g), recall from Section 4.1 that $\bar{a}_{f,g}$ and $\bar{b}_{f,g}$ are the best allocations for Agents 1 and 2, respectively, consistent with Voluntary Participation (Axiom 3). Because Pareto efficient sharing arrangements are proportional in our representation, there exists unique $\underline{\alpha}_{f,g}$, $\overline{\alpha}_{f,g}$ such that $\bar{a}_{f,g} = \overline{\alpha}_{f,g}(f+g)$ and $\bar{b}_{f,g} = (1 - \underline{\alpha}_{f,g})(f + g)$. Stated in preference terms,

$$\underline{\alpha}_{f,g} := \sup\{\alpha | \langle f, 0\rangle \succ^1 \langle \alpha(f+g), 0\rangle\} \quad \text{and} \quad \overline{\alpha}_{f,g} := \inf\{\alpha | \langle g, 0\rangle \succ^2 \langle (1-\alpha)(f+g), 0\rangle\}.$$

We will also rely on a notion of randomization over acts. Under the typical assumption that agents are indifferent to the source of randomization, the fact that uncertainty is objective in our model implies this randomization can be captured within our formal domain. To do so, we denote by $f\mathbf{p}f'$ an arbitrary, but fixed, act that generates the same distribution over outcomes as the randomization that yields act f with probability $\mathbf{p}$ and f' otherwise. Furthermore, when $f\mathbf{p}f'$ and $g\mathbf{p}g'$ appear together, they denote two acts such that the distribution over pairs of outcomes is the same as that generated by Agents 1 and 2 simultaneously getting f and g with probability $\mathbf{p}$ and simultaneously getting f' and g' otherwise.[1]

To distinguish situations where at least one agent values the ability to share from those situations where neither agent does, let $\mathcal{B} := \{(f, g) | \langle f, 1\rangle \succ^1_g \langle f, 0\rangle \text{ or } \langle g, 1\rangle \succ^2_f \langle g, 0\rangle\}$, and denote by $\mathcal{B}^C$ its complement in $\mathcal{F}^2$.

Remark 1 *In the context of the two-stage model with frictions (Section 6), the KSS and EBS can again be tightly characterized by requiring Axiom 9 or 10, respectively. Characterizing the NBS in the presence of frictions involves appropriate modifications to Axioms 7 and 8.*

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[1] Formally, let $f^{-1}(x)$ denote the event on which $f \in \mathcal{F}$ has outcome x. Then $f\mathbf{p}f'$ and $g\mathbf{p}g'$ denote fixed acts with $\mu\left((f\mathbf{p}f')^{-1}(x) \cap (g\mathbf{p}g')^{-1}(y)\right) = p\mu\left(f^{-1}(x) \cap g^{-1}(y)\right) + (1-p)\mu\left(f'^{-1}(x) \cap g'^{-1}(y)\right)$. The existence of such acts relies on the infinite divisibility of the probability space $(\Omega, \mathcal{A}, \mu)$.

S.1.1 Sharing according to the Nash Bargaining Solution

Given our general model, the use of the NBS is tightly characterized by two additional axioms. First, while a sharing arrangement may divide the gains unevenly in general, it seems plausible that when the stakes are very small, gains would be split about evenly. To see why this feature arises when sharing is based on the NBS, recall a critical step in its familiar construction based on the properties in Table 1. That construction begins by showing that for linear Pareto surplus frontiers, Symmetry and Scale Invariance imply that agents' surpluses must be proportional to the frontier's slope. In our context, surplus frontiers are never linear. However, the possible utility frontiers in our model are all smooth, and hence become approximately linear as the surplus from sharing shrinks sufficiently. In terms of first-stage preferences, our first additional axiom therefore requires that for sufficiently small achievable gains from sharing (f, g), each agent likes sharing as much as a final allocation that is sufficiently close to the reallocation that lies "halfway" between their best/worst proportional shares. To formalize this notion, let

$$\widetilde{\alpha}_{f,g} = \frac{\underline{\alpha}_{f,g} + \overline{\alpha}_{f,g}}{2}.$$

The axiom states that if the gains at stake shrink sufficiently, then for any $\varepsilon > 0$ Agent 1 prefers to get share $\widetilde{\alpha}_{f,g}$ of $f + g$ over (getting the autarky equivalent of) sharing f with g with high probability but having to consume f in autarky with low probability $\varepsilon > 0$, and analogously for Agent 2.

Axiom 7 (Small Stakes Symmetry)
Fix any $(f, g) \in \mathcal{B}^C$, and sequences of acts $\{f^n\}_{n=1}^{\infty} \to f$ uniformly and $\{g^n\}_{n=1}^{\infty} \to g$ uniformly. If, for $(a^n, b^n) \in A(f^n, g^n)$, $\langle f^n, 1\rangle \sim^1_{g^n} \langle a^n, 0\rangle$ and $\langle g^n, 1\rangle \sim^2_{f^n} \langle b^n, 0\rangle$, then for all $\varepsilon > 0$, there is $N > 0$ such that for all $n > N$

$$\begin{aligned} \langle \widetilde{\alpha}_{f^n,g^n}(f^n + g^n), 0\rangle &\succsim^1_{g^n} \langle f^n \varepsilon a^n, 0\rangle \\ \langle (1 - \widetilde{\alpha}_{f^n,g^n})(f^n + g^n), 0\rangle &\succsim^2_{f^n} \langle g^n \varepsilon b^n, 0\rangle. \end{aligned}$$

The second new axiom is the appropriate translation to our environment of a convexity assumption in Peters and Van Damme's (1991) alternative characterization of the NBS via its dependence on disagreement values. They consider bargaining agreements made when the parameters of the bargaining problem are uncertain; for example the agents may be sharing (f, g) or they may be sharing (f', g'), but this uncertainty resolves only after bargaining has completed. In our context, this randomness is captured by the endowments being $(f\mathbf{p}f', g\mathbf{p}g')$ for some $\mathbf{p} \in [0, 1]$. Their assumption is that bargaining from the "endowment" consisting of an after-bargaining randomization of (f, g) and its corresponding bargaining solution $(a, b) = \Gamma(f, g)$ must result in the same solution: (a, b).

Axiom 8 (Disagreement Convexity)
If $f, g, a, b \in \mathcal{F}$*,* $f + g = a + b$*, and* $\mathbf{p} \in [0,1]$ *then,*

$$\left.\begin{array}{c} \langle f, 1\rangle \sim_g^1 \langle a, 0\rangle \\ \langle g, 1\rangle \sim_f^2 \langle b, 0\rangle \end{array}\right\} \implies \left\{\begin{array}{c} \langle a\mathbf{p}f, 1\rangle \sim_{b\mathbf{p}g}^1 \langle a, 0\rangle \\ \langle b\mathbf{p}g, 1\rangle \sim_{a\mathbf{p}f}^2 \langle b, 0\rangle \end{array}\right.$$

S.1.2 Sharing according to the Kalai-Smorodinsky Solution

Kalai and Smorodinsky (1975) dropped IIA in favor of Resource Monotonicity. The resulting solution (KSS) relies on each agent's "aspiration payoff": the maximum payoff an agent can get in an agreement that respects disagreement values. In our risk sharing model with first-stage acts (f, g), aspiration payoffs are $u\left(\overline{\alpha}_{f,g}(f+g)\right)$ and $u((1-\underline{\alpha}_{f,g})(f+g))$, and the KSS of the second stage is the reallocation $(a, b) \in PS(f, g)$ such that

$$\frac{u(a) - u(f)}{u\left(\overline{\alpha}_{f,g}(f+g)\right) - u(f)} = \frac{u(b) - u(g)}{u((1-\underline{\alpha}_{f,g})(f+g)) - u(g)}. \tag{S.1}$$

The KSS is tightly characterized by one additional axiom. Not surprisingly, the additional axiom captures that the gains from sharing are proportional to the respective best outcomes each agent can get without violating the participation constraint of the other.

Axiom 9 (Proportional Gains)
If $\mathbf{p} \in [0,1]$ *is such that* $\langle \overline{\alpha}_{f,g}(f+g)\mathbf{p}f, 0\rangle \sim_g^1 \langle f, 1\rangle$*, then* $\langle (1-\underline{\alpha}_{f,g})(f+g)\mathbf{p}g, 0\rangle \sim_f^2 \langle g, 1\rangle$*.*

S.1.3 Sharing according to the Egalitarian Bargaining Solution

Recall that the EBS is egalitarian with respect to the utility surplus each agent achieves in the bargaining outcome. In our model with first-stage acts (f, g), the EBS of the second stage is the reallocation $(a, b) \in PS(f, g)$ such that $u(a) - u(f) = u(b) - u(g)$.

The EBS relies on a cardinal interpretation of utilities. In the context of our axioms and general representation, η is common, but setting $u_1 = u_2$ is a normalization. Attaching cardinal meaning to this normalization may be appropriate if, for instance, the scaling of the utilities are part of a social norm that dictates how individual outcomes are evaluated for the purpose of sharing. The use of the EBS is tightly characterized by one additional axiom.[2]

Axiom 10 (Symmetric Division) $\langle f\tfrac{\mathbf{1}}{\mathbf{2}}g, 1\rangle \sim_g^1 \langle h, 0\rangle$ *if and only if* $\langle f\tfrac{\mathbf{1}}{\mathbf{2}}g, 1\rangle \sim_f^2 \langle h, 0\rangle$*.*

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[2] If, alternatively, individual utilities were not normalized, then the model representing Axioms 1-6 would entail $u_1 = k_1 u_2 + k_2$ for arbitrary constants $k_1 > 0$ and k_2. In this context, the bargaining solution characterized by Axiom 10 has the agents' surpluses being proportional to their utility scaling: $\Gamma(f, g) = (a, b) \in PS(f, g)$ such that $u_1(a) - u_1(f) = k_1(u_2(b) - u_2(g))$.

Within the representation, for acts f, g, h that satisfy the first preference statement in Axiom 10 we have: $u(h) = \frac{1}{2}u(\alpha_{f,g}(f+g)) + \frac{1}{2}u(\alpha_{g,g}2g) = \frac{1}{2}u(\alpha_{f,g}(f+g)) + \frac{1}{2}u(g)$, where the second equality is implied by Axioms 3 and 4. Writing out the analogous expression for the second preference statement yields $u(h) = \frac{1}{2}u(f) + \frac{1}{2}u((1-\alpha_{f,g})(f+g))$. Since an act h that satisfies the first preference statement can always be found, this implies $u(\alpha_{f,g}(f+g)) - u(f) = u((1-\alpha_{f,g})(f+g)) - u(g)$, which is the utility-surplus characterization of the EBS of the second stage of our model given first-stage acts (f, g).

S.2 Sharing with Frictions: Representation Theorem

This section provides an axiomatization of the model described in Section 6, wherein frictions can be summarized by a probability of "breakdown" of the sharing arrangement. We continue to rely on the notation introduced in Section S.1.

Recall that π is a pair-specific parameter such that $1-\pi$ measures the probability of a breakdown. Specifically, if $\Gamma(f,g) = (a,b)$, then agents receive the respective allocations a and b with probability π only, and retain their respective acts f and g as the final allocation otherwise. This situation generates the same distribution over outcomes as $(a\pi f, b\pi g)$. Like α and η, the new parameter π is unobservable to the analyst, and its identification is addressed in Section 6.

Obviously, if $(f,g) \in \mathcal{B}$ and $\pi < 1$ (leaving agents with their endowments with probability $1-\pi$), then No Waste (Axiom 5) will be violated: It would be possible to achieve a Pareto improvement if breakdowns could be avoided. However, if

$$A^{\pi}(f,g) := \left\{(a,b) \in \mathcal{F}^2 \,|(a,b) = (a'\pi f, b'\pi g) \text{ for some } (a',b') \in A(f,g)\right\},$$

then No Waste should hold after replacing $A(f,g)$ with $A^{\pi}(f,g)$. Though the analyst does not observe π directly, if No Waste holds for some $A^{\pi}(f,g)$, and if π is independent of the chosen endowments, then it should also hold for $A^{\pi}(\hat{f},\hat{g})$ for any act-pair $(\hat{f},\hat{g})$. The next axiom formalizes this relaxation of No Waste.

Axiom 5′ (Consistent Waste)
If, given π, the following holds for some $(f,g) \in \mathcal{B}$, then it holds for all $(f,g) \in \mathcal{F}^2$:
$\langle a,0\rangle \succsim^1_g \langle f,1\rangle$ *and* $\langle b,0\rangle \succsim^2_f \langle g,1\rangle$*, one of them strict, implies* $(a,b) \notin A^{\pi}(f,g)$.

Because Axiom 5′ is weaker than Axiom 5, to guarantee the representation, we need a very minor strengthening of Axiom 3 that rules out that the preference for sharing compared to autarky is always immeasurably slight.

Axiom 3′ (Voluntary Participation 2) $\langle f,1\rangle \succsim^i_g \langle f,0\rangle$ *for all* $(f,g) \in \mathcal{F}^2$ *and* $i \in \{1,2\}$*, and there exist* $f,g,a \in \mathcal{F}$ *and* $i \in \{1,2\}$ *such that* $\langle f,1\rangle \succsim^i_g \langle a,0\rangle \succ^i_g \langle f,0\rangle$.

Theorem S.2 (Representation with Frictions) *Preferences* $\left\{\succsim_g^i\right\}_{g\in\mathcal{F}}$ *for* $i \in \{1,2\}$ *satisfy Axioms 1, 2, 3′, 4, 5′, and 6 if and only if they can be explained by a two-stage model of proportional risk sharing with frictions* (η, α, π)*.*

S.3 Example: Equilibrium Analysis of First-stage Choice

Our representation theorem applies to best-response data. This primitive most naturally aligns with axiomatic decision theories for individual choice, and in many contexts may best correspond to available data. Alternatively, equilibrium analysis requires the analyst to be explicit about both the first-stage game-form (e.g., simultaneous vs. sequential) and also the solution concept (e.g., Nash equilibrium, rationalizability, correlated equilibrium). Moreover, multiple equilibria may satisfy the chosen solution concept—a well-known problem for empirical application. In addition to its own merits, best-response analysis is useful for virtually every solution concept and game form, while avoiding the potential multiplicity problem.

Nevertheless, equilibrium analyses of particular first-stage game-forms should yield additional insights. We now employ our representation to analyze Nash equilibria of a simultaneous-move first stage. The example consists of the model from our representation in the following sharing environment. Two neighboring farmers (agents) each control land of measure 2. There are two equally likely states of the world: R(ainy) or D(ry). There are also two available crops, r and d, each one doing better than the other in its mnemonicly corresponding state of the world. Half of Agent 1's land is already planted with r, half of Agent 2's land is already planted with d.

Both farmers have $u(c) = \ln(c)$ (i.e., $\eta = 1$) and need to decide how much to specialize versus hedge when deciding how much of their remaining land to dedicate to one crop or the other.[3] The benefit of specialization is due to increasing returns in each crop. Specifically, let r_i (respectively, d_i) be the amount of Agent i's land that is planted with crop r (d). Then, i's production in states R and D are,

$$f(R|r_i, d_i) = r_i^z \quad \text{and} \quad f(D|r_i, d_i) = d_i^z,$$

where $z > 1$ captures the strength of the reward to specialization.[4]

The farmers simultaneously choose their crop allocations and then bargain over risk sharing. Their tie strength is $\pi > 0$ (probability of breakdown $1 - \pi$, see Section 6).

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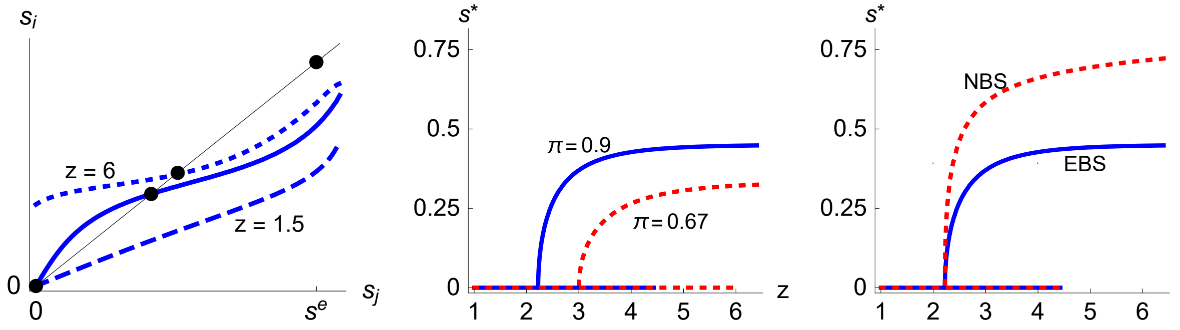
[3] The assumption that half their land is already planted allows us to focus on the degree of specialization, and bypass the coordination issue of which farmer should specialize in which crop.

[4] That $f(R|0, 2) = f(D|2, 0) = 0$ violates the specification that $f(\omega) \in \mathbb{R}_{++}$ from Section 2. However, it poses no issues for the analysis that follows and is allowed only for the sake of parsimony. Moreover, all claims regarding efficiency and/or equilibrium are the limits of the corresponding claims for the specification $f_\gamma(R|r_i, d_i) = r_i^z + \gamma d_i^z > 0$ and $f_\gamma(D|r_i, d_i) = \gamma r_i^z + d_i^z > 0$ as $\gamma > 0$ limits to zero.



(a) Best Responses ($\pi = 0.9$) (b) Change in tie-strength (c) Change to NBS ($\pi = 0.9$)

Figure 2: Equilibrium behavior behavior in the example.

Without loss, pose the game in terms of the degree of specialization each farmer chooses: $s_1 := (r_1 - 1) \in [0, 1]$ and $s_2 := (d_2 - 1) \in [0, 1]$.

Fact S.1 a) *The autarky solution is complete diversification:* $s_1 = s_2 = 0$, *so each farm is evenly split between the two crops.* b) *With risk sharing, the efficient individual specialization level is symmetric, denoted* s^e, *increasing in* z *and in* π, *and limits to* π *as* z *grows arbitrarily large.*

Suppose now that the sharing rule corresponds to the EBS (Section 4.1). Recalling Lemma 3.1, we have $\alpha^1_{f,g} = \frac{ce(f)}{ce(f)+ce(g)}$, where $ce(f)$ denotes the certainty equivalent of act f. The key tension is clear. Each farmer increases her disagreement value, captured by $ce(f)$, by individually diversifying, which increases the share of the pie she receives conditional on sharing. However, the total pie to share, $u(f + g)$ is largest when the farmers specialize (up to s^e).

The induced first-stage game is symmetric with increasing best response function. Figure 2(a) depicts the farmers' best response function for three different rewards to specialization levels, z.[5] All Nash equilibria are symmetric (lie on the 45°-line), but equilibrium multiplicity is possible. For low z (dashed curve), the reward to specialization is insufficient to overcome the incentive to self-insure, and the unique equilibrium is the autarky solution, $s_1 = s_2 = 0$. An intermediate reward to specialization (solid), gives rise to a coordination issue: the autarky solution is still an equilibrium, but now so too is a profile with a positive degree of specialization $s_1 = s_2 = s^* > 0$. Finally, for high z (dotted), the reward to specialize is large enough that each farmer wants to specialize even if the other is not, and there is a unique equilibrium.[6] Figure 2(b), illustrates the set of equilibria as it varies with the reward to specialization for two different tie strengths.

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[5] Here the efficient specialization s^e varies so little with z that it undetectable in the figure ($s^e \in (0.87, 0.9)$ for $z \geq 1.5$).

[6] For any z, if the equilibrium is unique, then it is the also the unique rationalizable outcome.

Fact S.2 *All equilibria are symmetric and at most one involves positive specialization,* $s^* > 0$.

- *The autarky solution,* $s_1 = s_2 = 0$*, is an equilibrium if and only if* $z < \frac{4}{\pi}$.
- *An equilibrium with positive specialization exists if and only if* $z > \frac{2}{\pi}$.
- *All equilibria involve inefficiently low (high) specialization (self insurance):* $s^* < s^e$.

From Fact S.2, we can see that increasing the chance of breakdown (decreasing π) increases the parameter space in which the autarky solution is an equilibrium and decreases the parameter space in which an equilibrium with positive specialization exists. Figure 2(b) illustrates the effect that the tie strength has on the equilibrium actions: both the largest and smallest equilibria increase with π. Underlying this observation is that (a) because the EBS generates a sensible sharing rule (Proposition 4.1), increasing π leads to a more efficient best response to any opponent choice of act (Theorem 6.3), and (b) the best response functions in this game are increasing and symmetric.

It is also worth observing that if instead of simultaneous moves, the first-stage choice of acts were sequential with, say, Agent 2 being able to condition her choice on Agent 1's, then the efficiency of total production would be strictly higher and both agents strictly better off compared to the most efficient equilibrium of the simultaneous-move specification. That is, sequentiality not only solves the coordination problem in the case of multiplicity, but allows the first-mover to select an even higher level of specialization, confident that the second-mover will follow suit since best-response functions are increasing. In fact, the second-mover earns a higher payoff than the first.

S.3.1 Varying the Bargaining Solution

Suppose now that instead of the EBS, the farmers shared according to the NBS. In this case, the analytic form of α is intractable. Perhaps surprisingly though, Fact S.2 remains valid verbatim. However, whenever the positive-specialization equilibrium exists, it involves more efficient specialization under the NBS than under the EBS (Figure 2(c)). That is, the NBS rewards efficient production more than the EBS does in this example.

The force underlying this ranking is more general as illustrated in the proposition below. In our context, all three prominent bargaining solutions (Section 4.1) depend only on $u(f+g)$, $u(f)$, and $u(g)$, which we can think of as efficiency and disagreement values. Because all three solutions generate sensible sharing rules (Proposition 4.1), at any best response the agent has optimally traded off increasing $u(f+g)$ versus increasing $u(f)$. Further, because the space of acts is continuous and both u and α are differentiable (under each of these bargaining solutions), we can consider the local effect from changing f slightly.

Proposition S.1 *Fix* $\pi > 0$ *and* $f \neq g$ *such that* $u(f) = u(g)$. *Then each agent's local incentive to increase efficiency,* $u(f+g)$, *at the expense of her disagreement value,* $u(f)$ *or* $u(g)$, *is*:

- *strictly greater under both the NBS and the KSS than under the EBS,*
- *strictly greater under the KSS than under the NBS if and only if* $\eta < 2$.[7]

The example and proposition point out an important facet of the two-stage environment. Even though all three of these bargaining solutions result in Pareto efficient reallocation *conditional* on (f, g) (i.e., in the second stage), they generally provide different incentives for, and engender different equilibrium levels of, efficiency in the choice of first-stage acts. One may then ask whether the strength of the incentives provided for this "global" efficiency should be important for the normative evaluation of bargaining solutions, and if any yet unheralded solutions would provide a worthwhile improvement on this dimension.

S.4 Proofs

From here until the completion of the proof of Theorem S.1, suppose that primitive $\{\succsim_g^i\}_{i=1,2}$ satisfies Axioms 1-6, so is represented by a model of proportional risk sharing denoted (η^*, α^*). Let u be the utility function from (2) with $\eta = \eta^*$, and, for $f, g \in F$, define

$$\lambda(\alpha) := -\left.\frac{\partial u\left(\hat{\alpha}(f+g)\right)/\partial\hat{\alpha}}{\partial u((1-\hat{\alpha})(f+g))/\partial\hat{\alpha}}\right|_{\hat{\alpha}=\alpha} = \left(\frac{1-\alpha}{\alpha}\right)^{\eta^*}, \tag{S.2}$$

which does not depend on (f, g).

Lemma S.4 *For* $f, g \in \mathcal{F}$, *the proportional allocation* $(a, b) = (\alpha^*_{f,g}(f+g), (1-\alpha^*_{f,g})(f+g))$ *is the NBS if and only if*

$$\lambda(\alpha^*_{f,g}) = \frac{u(\alpha^*_{f,g}(f+g)) - u\left(f\right)}{u((1-\alpha^*_{f,g})(f+g)) - u\left(g\right)}. \tag{S.3}$$

Proof. Given the representation, Proposition 3.1 implies that the objective for the NBS can be written as

$$\max_{\alpha\in[0,1]} \left[u\left(\alpha(f+g)\right) - u\left(f\right)\right]\left[u\left((1-\alpha)(f+g)\right) - u\left(g\right)\right].$$

Because $\eta^* > 0$, the objective is concave in α, and the solution is always interior. It is therefore characterized by its first-order condition, which rearranges to (S.3) ■

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[7] If $f = g$, so there is no scope for risk sharing, then each agent's local incentive to increase efficiency at the expense of her disagreement value is identical under all three bargaining solutions. Notice that $f = g$ is nongeneric among act-pairs with $u(f) = u(g)$.

Lemma S.5 *For $f, g \in \mathcal{F}$ and the allocation $(a, b) = (\alpha^*_{f,g}(f+g), (1-\alpha^*_{f,g})(f+g))$,*

$$\lim_{\mathbf{p}\to 1} \overline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g} = \lim_{\mathbf{p}\to 1} \underline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g} = \alpha^*_{f,g}.$$

Proof. See that

$$u\left(\underline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g}(f+g)\right) = u\left(\underline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g}(a\mathbf{p}f + b\mathbf{p}g)\right) = u(a\mathbf{p}f), \tag{S.4}$$

where the first equality is by $(f+g) = (a+b) = (a\mathbf{p}f + b\mathbf{p}g)$, and the second equality is by definition of $\underline{\alpha}$. Next,

$$\lim_{\mathbf{p}\to 1} u\left(\underline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g}(f+g)\right) = \lim_{\mathbf{p}\to 1} u(a\mathbf{p}f) = u(a) = u(\alpha^*_{f,g}(f+g)),$$

where the first equality is from (S.4), the second equality is from the continuity of u, and the third is by the assignment of a in the lemma's hypothesis. It follows from the continuity and strict monotonicity of $u(\alpha(f+g))$ in α that $\underline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g} \xrightarrow[\mathbf{p}\to 1]{} \alpha^*_{f,g}$. A symmetric argument establishes that $\overline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g} \xrightarrow[\mathbf{p}\to 1]{} \alpha^*_{f,g}$, completing the proof. ■

Proof of Theorem S.1(1).

Axioms 7-8 ⇒ NBS: Fix $f, g \in \mathcal{F}$, and to simplify notation, we drop the subscript on $\alpha^*_{f,g}$. Let $(a, b) = (\alpha^*(f+g), (1-\alpha^*)(f+g)) \in PS(f, g)$. If $(a, b) = (f, g)$, then (a, b) is the unique allocation in $A(f, g)$ that generates non-negative utility surpluses for both agents. Trivially then, (a, b) adheres to the NBS in this case. For the remainder, assume that $(a, b) \neq (f, g)$. The proof proceeds in several steps.

Step 1 *For any $\mathbf{p} \in [0, 1]$, $\alpha^*_{a\mathbf{p}f,b\mathbf{p}g} = \alpha^*$.*

▷ For any $\mathbf{p} \in [0, 1]$, Axiom 8 implies $\langle a\mathbf{p}f, 1\rangle \sim^1_{b\mathbf{p}g} \langle a, 0\rangle$ and $\langle b\mathbf{p}g, 1\rangle \sim^2_{a\mathbf{p}f} \langle b, 0\rangle$, and, in the context of the representation, this is $\alpha^*_{a\mathbf{p}f,b\mathbf{p}g} = \alpha^*$. ◁

Step 2 *For any $\mathbf{p} \in [0, 1]$, $u(\alpha^*(a\mathbf{p}f + b\mathbf{p}g)) - u(a\mathbf{p}f) = (1-\mathbf{p})(u(\alpha^*(f+g)) - u(f))$, and $u((1-\alpha^*)(a\mathbf{p}f + b\mathbf{p}g)) - u(b\mathbf{p}g) = (1-\mathbf{p})(u((1-\alpha^*)(f+g)) - u(g))$.*

▷ Because u is linear in probabilities,

$$\begin{aligned} u(\alpha^*(\underbrace{a\mathbf{p}f + b\mathbf{p}g}_{=f+g})) - u\left(a\mathbf{p}f\right) &= u\left(\alpha^*(f+g)\right) - \mathbf{p}u(\underbrace{\quad a \quad}_{=\alpha^*_{f,g}(f+g)}) - (1-\mathbf{p})u(f) \\ &= (1-\mathbf{p})\left(u\left(\alpha^*(f+g)\right) - u\left(f\right)\right). \end{aligned}$$

A symmetric argument establishes the claim's second equation. ◁

Step 3 $\lim\limits_{\mathbf{p}\to 1} \dfrac{\alpha^* - \underline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g}}{\overline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g} - \alpha^*} = 1.$

$\triangleright$ By hypothesis, $\langle a, 0\rangle \sim_b \langle a, 1\rangle$ and $\langle b, 0\rangle \sim_a \langle b, 1\rangle$, so $(a, b) \in \mathcal{B}^C$. In addition, $a\mathbf{p}f, b\mathbf{p}g$ converge uniformly to a, b, respectively, as $\mathbf{p} \to 1$. By Axiom 7 then, for any $\varepsilon > 0$, and $\mathbf{p} < 1$ large enough,

$$\langle \widetilde{\alpha}_{a\mathbf{p}f,b\mathbf{p}g}(a\mathbf{p}f + b\mathbf{p}g), 0\rangle \succsim^1_{b\mathbf{p}g} \langle (a\mathbf{p}f)\varepsilon a, 0\rangle.$$

In the representation, that is

$$u(\widetilde{\alpha}_{a\mathbf{p}f,b\mathbf{p}g}(\underbrace{a\mathbf{p}f + b\mathbf{p}g}_{=f+g})) \geq \varepsilon u(a\mathbf{p}f) + (1-\varepsilon)u(\underbrace{\quad a \quad}_{=\alpha^*(f+g)})$$

which rearranges to

$$\frac{u(\alpha^*(f+g)) - u(\widetilde{\alpha}_{a\mathbf{p}f,b\mathbf{p}g}(f+g))}{u(\alpha^*(f+g)) - u(a\mathbf{p}f)} \leq \varepsilon,$$

and then, using: $u(a\mathbf{p}f) = \mathbf{p}u(a) + (1-\mathbf{p})u(f) = \mathbf{p}u(\alpha^*(f+g)) + (1-\mathbf{p})u(\underline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g}(f+g))$, to

$$\frac{u(\alpha^*(f+g)) - u(\widetilde{\alpha}_{a\mathbf{p}f,b\mathbf{p}g}(f+g))}{(1-\mathbf{p})(u(\alpha^*(f+g)) - u(\underline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g}(f+g)))} \leq \varepsilon. \tag{S.5}$$

Because $(a, b) \neq (f, g)$ and $\mathbf{p} \in (0, 1)$, the denominator of the LHS of (S.5) is positive and hence

$$\frac{u(\alpha^*(f+g)) - u(\widetilde{\alpha}_{a\mathbf{p}f,b\mathbf{p}g}(f+g))}{u(\alpha^*(f+g)) - u(\underline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g}(f+g))} \leq \varepsilon.$$

Because $\alpha^*, \widetilde{\alpha}_{a\mathbf{p}f,b\mathbf{p}g}$ are both in $(\underline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g}, \overline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g})$, Lemma S.5 implies all four α-terms are converging to one another as $\mathbf{p} \to 1$. Because u is continuous, for any $\gamma > 0$ and $\mathbf{p} < 1$ large enough, we therefore have that

$$\frac{\alpha^* - \frac{1}{2}(\overline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g} + \underline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g})}{\alpha^* - \underline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g}} \leq \frac{\gamma}{2},$$

and hence,

$$\frac{\overline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g} - \alpha^*}{\alpha^* - \underline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g}} \geq 1 - \gamma. \tag{S.6}$$

A symmetric argument using the preferences for Agent 2 implied by Axiom 7 shows that for any $\gamma' > 0$ and $\mathbf{p} < 1$ large enough we have

$$\frac{\overline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g} - \alpha^*}{\alpha^* - \underline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g}} \leq 1 + \gamma'. \tag{S.7}$$

Together, (S.6) and (S.7) imply the claim in Step 3. $\triangleleft$

Step 4 *The proportional allocation* $(\alpha^*(f+g), (1-\alpha^*)(f+g))$ *satisfies* (S.3).

$\triangleright$ For any acts $f', g' \in \mathcal{F}$ and triple $\overline{\alpha} > \alpha' > \underline{\alpha}$, the concavity of u implies

$$\left.\frac{\partial u\left(\alpha(f'+g')\right)}{\partial \alpha}\right|_{\alpha=\underline{\alpha}} > \frac{u\left(\alpha'(f'+g')\right) - u(f')}{\alpha' - \underline{\alpha}} > \left.\frac{\partial u\left(\alpha(f'+g')\right)}{\partial \alpha}\right|_{\alpha=\overline{\alpha}}$$

and

$$-\left.\frac{\partial u\left((1-\alpha)(f'+g')\right)}{\partial \alpha}\right|_{\alpha=\overline{\alpha}} > \frac{u\left((1-\alpha')(f'+g')\right) - u(g')}{(1-\alpha') - (1-\overline{\alpha})} > -\left.\frac{\partial u\left((1-\alpha)(f'+g')\right)}{\partial \alpha}\right|_{\alpha=\underline{\alpha}}.$$

Hence,

$$\lambda(\overline{\alpha}) < \theta(\alpha', \underline{\alpha}, \overline{\alpha} | f', g') := \frac{\frac{u(\alpha'(f'+g')) - u(f')}{\alpha' - \underline{\alpha}}}{\frac{u((1-\alpha')(f'+g')) - u(g')}{(1-\alpha') - (1-\overline{\alpha})}} < \lambda(\underline{\alpha}). \tag{S.8}$$

For any $\mathbf{p} < 1$, we have $\alpha^*_{a\mathbf{p}f,b\mathbf{p}g} \in (\underline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g}, \overline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g})$ and

$$\begin{aligned}
\theta(\alpha^*_{a\mathbf{p}f,b\mathbf{p}g}, \underline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g}, \overline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g} | a\mathbf{p}f, b\mathbf{p}g) &= \left(\frac{u\left(\alpha^*(a\mathbf{p}f + b\mathbf{p}g)\right) - u(a\mathbf{p}f)}{u\left((1-\alpha^*)(a\mathbf{p}f + b\mathbf{p}g)\right) - u(b\mathbf{p}g)}\right)\left(\frac{\overline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g} - \alpha^*}{\alpha^* - \underline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g}}\right) \\
&= \left(\frac{u\left(\alpha^*(f+g)\right) - u(f)}{u\left((1-\alpha^*)(f+g)\right) - u(g)}\right)\left(\frac{\overline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g} - \alpha^*}{\alpha^* - \underline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g}}\right),
\end{aligned} \tag{S.9}$$

where the first equality is from Step 1 and the second equality is from Step 2.

By Lemma S.5, $\underline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g}$ and $\overline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g}$ both limit to α^* as $\mathbf{p} \to 1$. Using the continuity of λ, (S.8) then implies

$$\begin{aligned}
\lambda(\alpha^*) = \lim_{\mathbf{p}\to 1} \lambda(\overline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g}) = \lim_{\mathbf{p}\to 1} \lambda(\underline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g}) &= \lim_{\mathbf{p}\to 1} \theta(\alpha^*_{a\mathbf{p}f,b\mathbf{p}g}, \underline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g}, \overline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g} | a\mathbf{p}f, b\mathbf{p}g) \\
\text{(using (S.9))} \quad &= \lim_{\mathbf{p}\to 1} \left(\frac{u\left(\alpha^*(f+g)\right) - u(f)}{u\left((1-\alpha^*)(f+g)\right) - u(g)}\right)\left(\frac{\overline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g} - \alpha^*}{\alpha^* - \underline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g}}\right) \\
&= \frac{u\left(\alpha^*(f+g)\right) - u(f)}{u\left((1-\alpha^*)(f+g)\right) - u(g)} \left(\lim_{\mathbf{p}\to 1} \frac{\overline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g} - \alpha^*}{\alpha^* - \underline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g}}\right) \\
\text{(using Step 3)} \quad &= \frac{u\left(\alpha^*(f+g)\right) - u(f)}{u\left((1-\alpha^*)(f+g)\right) - u(g)},
\end{aligned}$$

meaning α^* satisfies (S.3). $\triangleleft$

It follow from Step 4 and Lemma S.4 that $(a, b) = (\alpha^*_{f,g}(f+g), (1-\alpha^*_{f,g})(f+g))$ adheres to the NBS.

NBS $\Rightarrow$ Axioms 7-8: Consider the model (η, α^*) where α^* satisfies the NBS, and $f, g, \{(f^n, g^n)\}_{n=1}^{\infty}$ all satisfying the hypotheses of Axiom 7. If $\underline{\alpha}_{f^n,g^n} = \overline{\alpha}_{f^n,g^n}$, then since the NBS respects disagreement values, we have $\alpha^*_{f^n,g^n} = \widetilde{\alpha}_{f^n,g^n} = \underline{\alpha}_{f^n,g^n} = \overline{\alpha}_{f^n,g^n}$, and the implication of Axiom 7 is immediately satisfied. For the remainder, consider any subsequence for which $\underline{\alpha}_{f^n,g^n} < \overline{\alpha}_{f^n,g^n}$ for all n. Using (S.2), define

$$\rho := \lim_{n\to\infty} \lambda(\alpha^*_{f^n,g^n}|f^n, g^n) = \lim_{n\to\infty} \left(\frac{1-\alpha^*_{f^n,g^n}}{\alpha^*_{f^n,g^n}}\right)^{\eta} = \left(\frac{1-\alpha^*_{f,g}}{\alpha^*_{f,g}}\right)^{\eta}.$$

From Lemma S.4,

$$\begin{aligned}\rho &= \lim_{n\to\infty} \frac{u(\alpha^*_{f^n,g^n}(f^n+g^n)) - u\left(f^n\right)}{u((1-\alpha^*_{f^n,g^n})(f^n+g^n)) - u\left(g^n\right)} \\ &= \lim_{n\to\infty} \frac{u(\alpha^*_{f^n,g^n}(f^n+g^n)) - u\left(\underline{\alpha}_{f^n,g^n}(f^n+g^n)\right)}{u((1-\alpha^*_{f^n,g^n})(f^n+g^n)) - u\left((1-\overline{\alpha}_{f^n,g^n})(f^n+g^n)\right)},\end{aligned}$$

where the second equality is by definition of $\underline{\alpha}, \overline{\alpha}$. By the Product Rule of Limits, this can be rewritten as

$$\begin{aligned}\rho = \lim_{n\to\infty} &\frac{u(\alpha^*_{f^n,g^n}(f^n+g^n)) - u\left(\underline{\alpha}_{f^n,g^n}(f^n+g^n)\right)}{\alpha^*_{f^n,g^n} - \underline{\alpha}_{f^n,g^n}} \\ &\times \lim_{n\to\infty} \frac{(1-\alpha^*_{f^n,g^n}) - (1-\overline{\alpha}_{f^n,g^n})}{u((1-\alpha^*_{f^n,g^n})(f^n+g^n)) - u\left((1-\overline{\alpha}_{f^n,g^n})(f^n+g^n)\right)} \\ &\times \lim_{n\to\infty} \frac{\alpha^*_{f^n,g^n} - \underline{\alpha}_{f^n,g^n}}{\overline{\alpha}_{f^n,g^n} - \alpha^*_{f^n,g^n}}. \qquad \text{(S.10)}\end{aligned}$$

Evaluating the limit of the first factor, note that for any convergent subsequence $\{(f^n, g^n)\}_n$ in a sufficiently small neighborhood of the strictly positive acts (f, g), $\underline{\alpha}_{f^n,g^n}$ is bounded away from 0 because $u(\underline{\alpha}_{f^n,g^n}(f^n+g^n)) = u(f^n)$. Let $\alpha^- := \inf_n(\underline{\alpha}_{f^n,g^n})$. According to the NBS, $u(\alpha^*_{f^n,g^n}(f^n+g^n)) > u(f^n) = u(\underline{\alpha}_{f^n,g^n}(f^n+g^n))$, it follows that $\alpha^- \leq \underline{\alpha}_{f^n,g^n} < \alpha^*_{f^n,g^n} < \overline{\alpha}_{f^n,g^n}$, where the last inequality comes from the parallel oberservation for Agent 2. Hence, for the concave CRRA utility u and any $\alpha' > \alpha'' \geq \alpha^-$, we have that

$$0 \leq \frac{u(\alpha'(f^n+g^n)) - u\left(\alpha''(f^n+g^n)\right)}{\alpha' - \alpha''} \leq \inf_n \left\{ \left.\frac{\partial u(\alpha(f^n+g^n))}{\partial \alpha}\right|_{\alpha=\alpha^-} \right\}.$$

Therefore, $\frac{u(\alpha'(f^n+g^n)) - u(\alpha''(f^n+g^n))}{\alpha'-\alpha''}$ converges to $\frac{u(\alpha'(f+g)) - u(\alpha''(f+g))}{\alpha'-\alpha''}$ uniformly for $\alpha' > \alpha'' \geq \alpha^-$.

Next, because the NBS is Pareto efficient, $(f,g) \in \mathcal{B}^C$ implies that $\underline{\alpha}_{f,g} = \overline{\alpha}_{f,g}$. Because u is continuous, we have $\lim_{n\to\infty}[\overline{\alpha}_{f^n,g^n} - \underline{\alpha}_{f^n,g^n}] = 0$. Furthermore, since u is continuously differentiable, and since $\alpha^*_{f^n,g^n} \in (\underline{\alpha}_{f^n,g^n}, \overline{\alpha}_{f^n,g^n})$,

$$\lim_{n\to\infty} \frac{u(\alpha^*_{f^n,g^n}(f+g)) - u\left(\underline{\alpha}_{f^n,g^n}(f+g)\right)}{\alpha^*_{f^n,g^n} - \underline{\alpha}_{f^n,g^n}} = \left.\frac{\partial u\left(\alpha(f+g)\right)}{\partial\alpha}\right|_{\alpha=\alpha^*_{f,g}}. \tag{S.11}$$

An analogous argument yields that

$$\lim_{n\to\infty} \frac{(1-\alpha^*_{f^n,g^n}) - (1-\overline{\alpha}_{f^n,g^n})}{u((1-\alpha^*_{f^n,g^n})(f+g)) - u\left((1-\overline{\alpha}_{f^n,g^n})(f+g)\right)} = -\left(\left.\frac{\partial u\left((1-\alpha)(f+g)\right)}{\partial\alpha}\right|_{\alpha=\alpha^*_{f,g}}\right)^{-1}. \tag{S.12}$$

The product of (S.11) and (S.12) is

$$\lambda(\alpha^*_{f,g}) = \left(\frac{1-\alpha^*_{f,g}}{\alpha^*_{f,g}}\right)^{\eta} = \rho.$$

Apply the Order of Limits Theorem (the Moore-Osgood Theorem) to (S.10) and see that

$$\rho = \rho \lim_{n\to\infty} \frac{\alpha^*_{f^n,g^n} - \underline{\alpha}_{f^n,g^n}}{\overline{\alpha}_{f^n,g^n} - \alpha^*_{f^n,g^n}}.$$

Therefore, we have

$$\lim_{n\to\infty} \frac{\alpha^*_{f^n,g^n} - \underline{\alpha}_{f^n,g^n}}{\overline{\alpha}_{f^n,g^n} - \alpha^*_{f^n,g^n}} = 1 = \lim_{n\to\infty} \frac{\overline{\alpha}_{f^n,g^n} - \alpha^*_{f^n,g^n}}{\alpha^*_{f^n,g^n} - \underline{\alpha}_{f^n,g^n}}.$$

Hence,

$$\begin{aligned} 0 = \lim_{n\to\infty}\left(\frac{\overline{\alpha}_{f^n,g^n} - \alpha^*_{f^n,g^n}}{\alpha^*_{f^n,g^n} - \underline{\alpha}_{f^n,g^n}} - \frac{\alpha^*_{f^n,g^n} - \underline{\alpha}_{f^n,g^n}}{\alpha^*_{f^n,g^n} - \underline{\alpha}_{f^n,g^n}}\right) &= \lim_{n\to\infty} \frac{(\overline{\alpha}_{f^n,g^n} + \underline{\alpha}_{f^n,g^n}) - 2\alpha^*_{f^n,g^n}}{\alpha^*_{f^n,g^n} - \underline{\alpha}_{f^n,g^n}} \\ &= \lim_{n\to\infty} \frac{2\widetilde{\alpha}_{f^n,g^n} - 2\alpha^*_{f^n,g^n}}{\alpha^*_{f^n,g^n} - \underline{\alpha}_{f^n,g^n}}. \end{aligned}$$

Dividing both sides by (-2) yields,

$$\lim_{n\to\infty} \frac{\alpha^*_{f^n,g^n} - \widetilde{\alpha}_{f^n,g^n}}{\alpha^*_{f^n,g^n} - \underline{\alpha}_{f^n,g^n}} = 0.$$

Using again the Order of Limits Theorem, as well as Taylor approximation given that each of the α-terms have the same limit, yields

$$0 = \lim_{n\to\infty} \frac{u(\alpha^*_{f^n,g^n}(f^n+g^n)) - u(\widetilde{\alpha}_{f^n,g^n}(f^n+g^n))}{u(\alpha^*_{f^n,g^n}(f^n+g^n)) - u(\underline{\alpha}_{f^n,g^n}(f^n+g^n))} = \lim_{n\to\infty} \frac{u(a^n) - u(\widetilde{\alpha}_{f^n,g^n}(f^n+g^n))}{u(a^n) - u(f^n)}. \tag{S.13}$$

Because $u(a^n) > u(f^n)$, (S.13) implies that for any fixed $\varepsilon > 0$ there exists N such that for all $n > N$,

$$\frac{u(a^n) - u(\widetilde{\alpha}_{f^n,g^n}(f^n+g^n))}{u(a^n) - u(f^n)} \leq \varepsilon,$$

which rearranges to $u(\widetilde{\alpha}_{f^n,g^n}(f^n+g^n)) \geq \varepsilon u(f^n) + (1-\varepsilon)u(a^n)$. This is identical to the preference statement, $\langle \widetilde{\alpha}_{f^n,g^n}(f^n+g^n), 0\rangle \succsim^1_{g^n} \langle f^n \varepsilon a^n, 0\rangle$ for all n large enough. An analogous argument establishes the counterpart for Agent 2: $\langle (1-\widetilde{\alpha}_{f^n,g^n})(f^n+g^n), 0\rangle \succsim^2_{f^n} \langle g^n \varepsilon b^n, 0\rangle$ for all n large enough. This is Axiom 7.

To establish Axiom 8, recall that, given endowment (f, g), under the NBS the objective function for α is the maximization of

$$(u(\alpha(f+g)) - u(f))\,(u((1-\alpha)(f+g)) - u(g))$$

Due to the linearity of the CRRA utility u, the objective function for endowment $(a\mathbf{p}f, b\mathbf{p}g)$ with $a\mathbf{p}f + b\mathbf{p}g = f + g$ is

$$\begin{aligned}(u(\alpha(f+g)) - u(a\mathbf{p}f))\,(u((1-\alpha)(f+g)) - u(b\mathbf{p}g)) = \\ (1-p)^2\,(u(\alpha(f+g)) - u(f))\,(u((1-\alpha)(f+g)) - u(g)).\end{aligned}$$

Since the objective functions for the two endowments are identical up to a scalar, the solutions must be the same. This implies Axiom 8. ■

Proof of Theorem S.1(2).

Axiom 9 ⇒ KSS: Fix $f, g \in \mathcal{F}$. By definition of $\overline{\alpha}_{f,g}$, we have that $u\left(\alpha_{f,g}(f+g)\right) \in \left[u\left(f\right), u\left(\overline{\alpha}_{f,g}(f+g)\right)\right]$. Hence, there exist $\mathbf{p} \in [0,1]$ such that

$$\mathbf{p}\, u\left(\overline{\alpha}_{f,g}(f+g)\right) + (1-\mathbf{p})u\left(f\right) = u\left(\alpha_{f,g}(f+g)\right).$$

The equivalent preference statement is $\langle \overline{\alpha}_{f,g}(f+g)\mathbf{p}f, 0\rangle \sim^1_g \langle f, 1\rangle$. It then follows from Axiom 9 that we have $\langle (1-\underline{\alpha}_{f,g})(f+g)\mathbf{p}g, 0\rangle \sim^2_f \langle g, 1\rangle$. The equivalent utility statement is $\mathbf{p}u\left((1-\underline{\alpha}_{f,g})(f+g)\right) + (1-\mathbf{p})u\left(g\right) = u\left((1-\alpha_{f,g})(f+g)\right)$. Solving each utility statement for $\mathbf{p}$ gives

$$\mathbf{p} = \frac{u\left(\alpha_{f,g}(f+g)\right) - u\left(f\right)}{u\left(\overline{\alpha}_{f,g}(f+g)\right) - u\left(f\right)} = \frac{u\left((1-\alpha_{f,g})(f+g)\right) - u\left(g\right)}{u\left((1-\underline{\alpha}_{f,g})(f+g)\right) - u\left(g\right)}.$$

Since $\alpha_{f,g}$ is also Pareto efficient given u (Proposition 3.1), it corresponds to the KSS.

KSS $\Rightarrow$ Axiom 9: Straightforward by running the argument above in reverse. ■

Proof of Theorem S.1(3).
Axiom 10 $\Rightarrow$ EBS: Fix $f, g \in \mathcal{F}$. By the continuity of u, there exists $h \in \mathcal{F}$ with $u\left(h\right) = \frac{1}{2}u\left(\alpha_{f,g}(f+g)\right) + \frac{1}{2}u\left(g\right) = \frac{1}{2}u(\alpha_{f,g}(f+g)) + \frac{1}{2}u(\alpha_{g,g}2g)$, where the second equality is implied by Axioms 3 and 4. The equivalent preference statement is $\langle f\boldsymbol{\tfrac{1}{2}}g, 1\rangle \sim_g^1 \langle h, 0\rangle$. By Axiom 10, we then have that $\langle f\boldsymbol{\tfrac{1}{2}}g, 1\rangle \sim_f^2 \langle h, 0\rangle$ and, equivalently,

$$u\left(h\right) = \frac{1}{2}u((1-\alpha_{f,g})(f+g)) + \frac{1}{2}u(\alpha_{f,f}2f) = \frac{1}{2}u\left((1-\alpha_{f,g})(f+g)\right) + \frac{1}{2}u\left(f\right),$$

where again the second equality is implied by Axioms 3 and 4. That $u\left(\alpha_{f,g}(f+g)\right) - u\left(f\right) = u\left((1-\alpha_{f,g})(f+g)\right) - u\left(g\right)$ follows from algebraic rearrangement of the two utility statements above. Since $\alpha_{f,g}$ is also Pareto efficient given u (Proposition 3.1), it corresponds to the EBS.

EBS $\Rightarrow$ Axiom 10: Straightforward by running the argument above in reverse. ■

Proof of Theorem S.2.

Definition S.1 *Let $T(f, g, \pi)$ to be the statement: $\langle a, 0\rangle \succsim_g^1 \langle f, 1\rangle$ and $\langle b, 0\rangle \succsim_f^2 \langle g, 1\rangle$, one of them strict, implies $(a, b) \notin A^\pi(f, g)$. So Axiom 5′ is: If $T(f, g, \pi)$ is true for $(f, g) \in \mathcal{B}$, then $T(f', g', \pi)$ is true for all $(f', g') \in \mathcal{F}^2$.*

Model $\Rightarrow$ Axioms: Let $(\eta^*, \alpha^*, \pi^*)$ be a particular model. Immediately, we have $\langle f, 1\rangle \sim_g^1 \langle \alpha^*_{f,g}(f+g)\pi^* f, 0\rangle$ and $\langle g, 1\rangle \sim_f^2 \langle (1-\alpha^*_{f,g})(f+g)\pi^* g, 0\rangle$. Establishing Axioms 1, 2, 3′, 4, and 6 is analogous to the proof of Theorem 3.1. For Axiom 5′, it is straightforward that $T(f, g, \pi)$ is true for all $(f, g) \in \mathcal{F}^2$ when $\pi \leq \pi^*$. Now let $\pi > \pi^*$. For any (f, g),

$$\begin{aligned}
&u(f) \leq \pi^* u(\alpha^*_{f,g}(f+g)) + (1-\pi^*)u(f) \leq \pi u(\alpha^*_{f,g}(f+g)) + (1-\pi)u(f), \\
&u(g) \leq \pi^* u((1-\alpha^*_{f,g})(f+g)) + (1-\pi^*)u(g) \leq \pi u((1-\alpha^*_{f,g})(f+g)) + (1-\pi)u(g),
\end{aligned}$$

where the first inequality in each line follows from Axiom 3′, and the second via $\pi > \pi^*$. Moreover, $(f, g) \in \mathcal{B}$ implies the inequalities are strict for at least one agent. Translating back to preference statements, and since $\left(\alpha^*_{f,g}(f+g)\pi f, (1-\alpha^*_{f,g})(f+g))\pi g\right) \in A^\pi(f, g)$, we have that $T(f, g, \pi)$ is false for all $(f, g) \in \mathcal{B}$ when $\pi > \pi^*$, which completes the proof.

Axioms ⇒ Model: The result is established in several steps. Throughout, assume that the primitive, $\left\{\succsim^i_g\right\}_{g\in\mathcal{F}}$ for $i \in \{1, 2\}$, satisfies Axioms 1, 2, 3′, 4, 5′, and 6.

Step 1 *There exists $(f, g) \in \mathcal{B}$ and $\pi > 0$ such that $T(f, g, \pi)$ is true.*

▷ From Axiom 3′ there exist $f, g, a \in \mathcal{F}$ and $i \in \{1, 2\}$ such that $\langle f, 1\rangle \succsim^i_g \langle a, 0\rangle \succ^i_g \langle f, 0\rangle$. Because $\langle g, 1\rangle \succsim^j_f \langle g, 0\rangle$ (also by Axiom 3′), we have $(f, g) \in \mathcal{B}$. Without loss, let $i = 1$ and $j = 2$. For the purpose of contradiction, suppose that $T(f, g, \pi)$ is false for all $\pi > 0$. So, for all $\pi > 0$, there exists $(a_\pi, b_\pi) \in A^\pi(f, g)$ such that $\langle a_\pi, 0\rangle \succsim^1_g \langle f, 1\rangle \succsim^1_g \langle a, 0\rangle \succ^1_g \langle f, 0\rangle$. But, as $\pi \to 0$, we have $a_\pi \to f$, and by the continuity of autarky preferences (Axiom 2): $\langle f, 0\rangle \succsim^1_g \langle a, 0\rangle \succ^1_g \langle f, 0\rangle$ which is an obvious contradiction. ◁

Step 2 *If $(f, g) \in \mathcal{B}^C$, then $(f, g) \in PS(f, g)$.*

▷ Let u_1, u_2 be the expected utility functions representing each agent's autarky preferences, which exist by Axiom 2. Fix $(f, g) \in \mathcal{B}^C$. Then Axiom 3′ implies $\langle f, 1\rangle \sim^1_g \langle f, 0\rangle$ and $\langle g, 1\rangle \sim^2_f \langle g, 0\rangle$. Therefore, $T(f, g, \pi)$ can be written: $u_1(a) \geq u_1(f)$ and $u_2(b) \geq u_2(g)$, one of them strict, implies $(a, b) \notin A^\pi(f, g)$.

For the purpose of contradiction, suppose $(f, g) \notin PS(f, g)$: there exists $(a^*, b^*) \in A(f, g)$ such that $\langle a^*, 0\rangle \succ^1 \langle f, 0\rangle$ and $\langle b^*, 0\rangle \succsim^2 \langle g, 0\rangle$, or equivalently, $u_1(a^*) > u_1(f)$ and $u_2(b^*) \geq u_2(g)$ (where assigning the strict preference to Agent 1 is without loss). Then, for any $\pi > 0$, we have $(a^*\pi f, b^*\pi g) \in A^\pi(f, g)$ and $u_1(a^*\pi f) = \pi u_1(a^*) + (1-\pi)u_1(f) > u_1(f)$ and analogously for Agent 2 with the inequality weak. Hence, $T(f, g, \pi)$ is false for all $\pi > 0$. However, by Step 1, there exists $(f', g') \in \mathcal{B}$ and $\pi' > 0$ such that $T(f', g', \pi')$ is true. Axiom 5′ then implies that $T(f, g, \pi')$ is true, which is a contradiction. ◁

Step 3 *If $(f, g) \in \mathcal{B}^C$, then $T(f, g, \pi)$ is true for all π.*

▷ Fix $(f, g) \in \mathcal{B}^C$ and (a, b) such that $\langle a, 0\rangle \succ^1_g \langle f, 1\rangle \sim^1_g \langle f, 0\rangle$ and $\langle b, 0\rangle \succsim^2_f \langle g, 1\rangle \sim^2_f \langle g, 0\rangle$, where the assignment of the strict preference to Agent 1 is without loss and the indifferences follow from $(f, g) \in \mathcal{B}^C$ and Axiom 3′. Transitivity (Axiom 1) and $(f, g) \in PS(f, g)$ by Step 2 therefore imply $(a, b) \notin A(f, g)$. So, $T(f, g, 1)$ is true. Further, $A^\pi(f, g) \subseteq A(f, g)$ for any π, implying that $T(f, g, \pi)$ is also true. ◁

Step 4 *There exists $\overline{\pi} \in (0, 1]$ such that the following analogy of Axiom 5 holds if and only if $\pi \leq \overline{\pi}$: For all $(f, g) \in \mathcal{F}^2$, the statement $T(f, g, \pi)$ is true.*

▷ By Step 1, there exists $(f, g) \in \mathcal{B}$ and $\pi > 0$ such that $T(f, g, \pi)$ is true. By Axiom 5′, then, $T(f', g', \pi)$ is true for all $(f', g') \in \mathcal{F}^2$.

Next, for any $(f, g) \in \mathcal{B}$, the set $\{(a', b') | \langle f, 1\rangle \succsim^1_g \langle a', 0\rangle \wedge \langle g, 1\rangle \succsim^2_f \langle b', 0\rangle\}$ is closed, because $\left\{\succsim^i_g\right\}_{g\in\mathcal{F}}$ are transitive and complete (Axiom 1) and continuous on $\mathcal{F} \times \{0\}$ (Axiom 2). The set $A^\pi(f, g)$ is closed for all $f, g \in \mathcal{F}$ and all $\pi \in [0, 1]$. Furthermore, the mapping $H : [0,1] \to 2^{\mathcal{F}^2}$ with $H(\pi) = A^{\pi}(f,g)$ is continuous and increasing (in the sense that $A^{\pi'}(f,g) \subset A^{\pi}(f,g)$ for $\pi' < \pi$). The mapping H has a closed graph by the Closed Graph Theorem for Set Valued Functions (Aliprantis and Border, 1999, ch. 7). Hence, the intersection of the graph of H with $[0,1] \times \{(a',b') | \langle f,1\rangle \succsim_g^1 \langle a',0\rangle \wedge \langle g,1\rangle \succsim_f^2 \langle b',0\rangle\}$ is closed. Therefore, there exists a maximal $\overline{\pi}$ such that $A^{\overline{\pi}}(f,g) \subseteq \{(a',b') | \langle f,1\rangle \succsim_g^1 \langle a',0\rangle \wedge \langle g,1\rangle \succsim_f^2 \langle b',0\rangle\}$. Note that the analogy does not hold for π if and only if $A^{\pi} \not\subset \{(a',b') | \langle f,1\rangle \succsim_g^1 \langle a',0\rangle \wedge \langle g,1\rangle \succsim_f^2 \langle b',0\rangle\}$. Since $A^{\pi}(f,g) \subseteq A^{\pi'}(f,g)$ if and only if $\pi \leq \pi'$, we have that the analogy holds if and only if $\pi \leq \overline{\pi}$. $\lhd$

Step 5 *The $\overline{\pi}$ identified in Step 4 is the unique π at which the analogy of Axiom 5 and the following analogy of Axiom 6 simultaneously hold: For all $f,g \in \mathcal{F}$, $(a,b) \in PS^{\pi}(f,g)$ implies $\langle a,0\rangle \succsim_g^1 \langle f,1\rangle$ or $\langle b,0\rangle \succsim_f^2 \langle g,1\rangle$, where*

$$PS^{\pi}(f,g) := \{(a,b) \in \mathcal{F}^2 | (a,b) = (a'\pi f, b'\pi g) \textit{ for some } (a',b') \in PS(f,g)\}.$$

$\rhd$ By Step 4, the analogy of Axiom 5 does not hold for any $\pi > \overline{\pi}$. We next argue that the analogy of Axiom 6 holds at $\overline{\pi}$. There are two cases to consider. In the first case, if $\overline{\pi} = 1$, then $PS^{\overline{\pi}}(f,g) = PS(f,g)$, and Axiom 6 and its analogy are equivalent.

In the second case, $\overline{\pi} < 1$. For $(f,g) \in \mathcal{B}^C$, let $(a,b) \in PS^{\overline{\pi}}(f,g)$. Then there exists $(a',b') \in PS(f,g)$ such that $(a,b) = (a'\overline{\pi}f, b'\overline{\pi}g)$. Because $(a',b') \in PS(f,g)$, by autarky preferences being linear in probabilities (Axiom 2), we have $\langle a,0\rangle \succsim_g^1 \langle f,0\rangle$ or $\langle b,0\rangle \succsim_f^2 \langle g,0\rangle$. Moreover, $(f,g) \in \mathcal{B}^C$ implies that $\langle f,0\rangle \succsim_g^1 \langle f,1\rangle$ and $\langle g,0\rangle \succsim_f^2 \langle g,1\rangle$, giving the desired implication by transitivity (Axiom 1).

For act-pairs in $\mathcal{B}$, it is helpful to first consider $\hat{\pi} > \overline{\pi}$. By Step 4, there exists $(f,g) \in \mathcal{B}$ such that $T(f,g,\hat{\pi})$ is false. By the contrapositive of Axiom 5′, $T(f',g',\hat{\pi})$ must then be false for all $(f',g') \in \mathcal{B}$. For any $(f,g) \in \mathcal{B}$ then, for any $(a,b) \in PS^{\hat{\pi}}(f,g)$, it must be that $\langle a,0\rangle \succsim_g^1 \langle f,1\rangle$ or $\langle b,0\rangle \succsim_f^2 \langle g,1\rangle$. Since this holds for all $\hat{\pi} > \overline{\pi}$ and since, by continuity of autarky preferences (Axiom 2), the intersection of weakly better sets with $\mathcal{F} \times 0$ are closed, it follows that $\langle a',0\rangle \succsim_g^1 \langle f,1\rangle$ or $\langle b',0\rangle \succsim_f^2 \langle g,1\rangle$ for all $(a',b') \in PS^{\overline{\pi}}(f,g)$, which completes the analogy of Axiom 6 at $\overline{\pi}$.

It remains to show that the analogy of Axiom 6 does not hold for $\pi < \overline{\pi}$. Consider $(f,g) \in \mathcal{B}$, which is nonempty by Axiom 3′. Then, by Axiom 6, for $(a,b) \in PS(f,g)$, $\langle a,0\rangle \succsim_g^1 \langle f,0\rangle$ or $\langle b,0\rangle \succsim_f^1 \langle g,0\rangle$, one of them strict, and hence $(f,g) \notin PS(f,g)$. Let $(a,b) \in PS(f,g)$ be such that $\langle a,0\rangle \succ_g^1 \langle f,0\rangle$ and $\langle b,0\rangle \succ_f^2 \langle g,0\rangle$, which is possible because $(f,g) \notin PS(f,g)$ and autarky preferences are monotonic and continuous (Axiom 2). Then for arbitrary π, $a^{\pi} := a\pi f$, and $b^{\pi} := b\pi g$, we have $(a^{\pi}, b^{\pi}) \in PS^{\pi}(f,g)$. We make two observations. First, according to the analogy of Axiom 5, $\langle f,1\rangle \succsim_g^1 \langle a^{\overline{\pi}},0\rangle$ and $\langle g,1\rangle \succsim_f^2 \langle b^{\overline{\pi}},0\rangle$. Second, for $\pi < \overline{\pi}$ the linearity of autarky preferences in probabilities implies that $\langle a^{\overline{\pi}},0\rangle \succ_g^1 \langle a^{\pi},0\rangle$ and $\langle b^{\overline{\pi}},0\rangle \succ_f^2 \langle b^{\pi},0\rangle$. Hence, for $\pi < \overline{\pi}$, we established that $(a^\pi, b^\pi) \in PS^\pi(f,g)$ but $\langle f, 1\rangle \succ_g^1 \langle a^\pi, 0\rangle$ and $\langle g, 1\rangle \succ_f^2 \langle b^\pi, 0\rangle$. That is, the analogy of Axiom 6 does not hold for $\pi < \overline{\pi}$. $\lhd$

Step 6 *For the remainder of the proof, fix $\pi = \overline{\pi}$ as identified in Step 4. For any $f, g \in \mathcal{F}$ and $(a,b) \in PS^\pi(f,g)$ there exists $\alpha \in (0,1)$ such that $\langle a, 0\rangle \sim_g^1 \langle \alpha(f+g)\pi f, 0\rangle$ and $\langle b, 0\rangle \sim_f^2 \langle (1-\alpha)(f+g)\pi g, 0\rangle$.*

$\rhd$ This argument parallels the one for Step 1 in the proof of Theorem 3.1. Fix $f, g \in \mathcal{F}$ and $(a,b) \in \mathcal{A}^\pi(f,g)$. Define

$$\begin{aligned}\Theta_1 &:= \left\{\alpha \in [0,1] \,\middle|\, \langle a, 0\rangle \succ_b^1 \langle \alpha(f+g)\pi f, 0\rangle \right\}\\ \Theta_2 &:= \left\{\alpha \in [0,1] \,\middle|\, \langle b, 0\rangle \succ_a^2 \langle (1-\alpha)(f+g)\pi g, 0\rangle \right\}.\end{aligned}$$

and, for the purpose of contradiction, suppose that $\Theta_1 \cup \Theta_2 = (0,1)$. Because $a < (f+g)\pi f$, by monotonicity of autarky preferences (Axiom 2), there exists $\alpha < 1$ large enough that $\langle \alpha(f+g)\pi f, 0\rangle \succsim_b^1 \langle a, 0\rangle$ and hence $\Theta_2 \neq \emptyset$. By symmetric argument, $\Theta_1 \neq \emptyset$.

Continuity and monotonicity of autarky preferences in α (which follow immediately from Axiom 2) imply $\Theta_1 = (0, \overline{\alpha}_{f,g})$ and $\Theta_2 = (\underline{\alpha}_{f,g}, 1)$. Hence, for $\Theta_1 \cup \Theta_2 = (0,1)$, it must be that $\underline{\alpha}_{f,g} < \overline{\alpha}_{f,g}$ and there exists $\alpha^* \in \Theta_1 \cap \Theta_2$, meaning

$$\langle a, 0\rangle \succ_b^1 \langle \alpha^*(f+g)\pi f, 0\rangle \text{ and } \langle b, 0\rangle \succ_a^2 \langle (1-\alpha^*)(f+g)\pi g, 0\rangle.$$

By Axiom 2, $\succsim_h^i$ is independent of $h \in \mathcal{F}$ in autarky for $i \in \{1,2\}$. Therefore, for all $h, h' \in \mathcal{F}$,

$$\langle a, 0\rangle \succ_h^1 \langle \alpha^*(f+g)\pi f, 0\rangle \text{ and } \langle b, 0\rangle \succ_{h'}^2 \langle (1-\alpha^*)(f+g)\pi g, 0\rangle. \tag{S.14}$$

Now, let $(a', b') \in A(f,g)$ be such that $a = a'\pi f$ and $b = b'\pi g$, and let

$$\hat{a} := a'\pi\alpha^*(f+g) \ \text{ and } \ \hat{b} := b'\pi(1-\alpha^*)(f+g).$$

Then $\succsim_h^i$ being independent from h in autarky (Axiom 2) implies that (S.14) holds if and only if, for all $h, h' \in \mathcal{F}$,

$$\langle \hat{a}, 0\rangle = \langle a'\pi\alpha^*(f+g), 0\rangle \succ_h^1 \langle \alpha^*(f+g)\pi\alpha^*(f+g), 0\rangle = \langle \alpha^*(f+g), 0\rangle$$

and, analogously, $\langle \hat{b}, 0\rangle \succ_{h'}^2 \langle (1-\alpha^*)(f+g), 0\rangle$.

Separately, by Axiom 4,

$$\begin{aligned}\langle \alpha^*(f+g), 0\rangle &\succsim_{(1-\alpha^*)(f+g)}^1 \langle \alpha^*(f+g), 1\rangle\\ \langle (1-\alpha^*)(f+g), 0\rangle &\succsim_{\alpha^*(f+g)}^2 \langle (1-\alpha^*)(f+g), 1\rangle.\end{aligned}$$

Then, by transitivity of $\succsim_h^i$ for $i \in \{1,2\}$ and any $h \in \mathcal{F}$ (Axiom 1),

$$\begin{aligned}\langle \hat{a}, 0\rangle &\succ^1_{(1-\alpha^*)(f+g)} \langle \alpha^*(f+g), 1\rangle \\ \langle \hat{b}, 0\rangle &\succ^2_{\alpha^*(f+g)} \langle (1-\alpha^*)(f+g), 1\rangle.\end{aligned}$$

By construction $(\hat{a}, \hat{b}) \in A^\pi(\alpha^*(f+g), (1-\alpha^*)(f+g))$. But by Step 4,

$$(\hat{a}, \hat{b}) \notin A^\pi(\alpha^*(f+g), (1-\alpha^*)(f+g)),$$

which is a contradiction. Hence for all $(a,b) \in A^\pi(f,g)$ there is $\alpha \in (0,1)$ such that

$$\langle \alpha(f+g)\pi f, 0\rangle \succsim_b^1 \langle a, 0\rangle \text{ and } \langle (1-\alpha)(f+g)\pi g, 0\rangle \succsim_a^2 \langle b, 0\rangle.$$

Finally, by the definition of $PS^\pi(f,g)$ and because $\succsim_h^i$ is independent of $h \in \mathcal{F}$ in autarky (Axiom 2), we have that for any $(a,b) \in PS^\pi(f,g)$ there is $\alpha \in (0,1)$ such that

$$\langle a, 0\rangle \sim_g^1 \langle \alpha(f+g)\pi f, 0\rangle \text{ and } \langle b, 0\rangle \sim_f^2 \langle (1-\alpha)(f+g)\pi g, 0\rangle. \quad \lhd$$

Step 7 *Autarky preferences have a common CRRA representation.*

$\rhd$ Fix $f \in \mathcal{F}$ and $\beta > 0$. By Axiom 4, $(f, \beta f) \in \mathcal{B}^C$. So, $(f, \beta f) \in PS(f, \beta f)$ by Step 2. Proposition 3.1, then implies u_1, u_2 from the proof of Step 2 must be common CRRA. $\lhd$

Step 8 *Let $\mathcal{P} := \{(f, \beta f) | f \in \mathcal{F}, \beta > 0\}$. Then $\mathcal{B}^C = \mathcal{P}$.*

$\rhd$ By Axiom 4, $\mathcal{P} \subseteq \mathcal{B}^C$. Consider now $(f,g) \in \mathcal{B}^C$. By Step 2, $(f,g) \in PS(f,g)$. Then, by Step 7 and Proposition 3.1, (f,g) must be in $\mathcal{P}$. Therefore, $\mathcal{B}^C \subseteq \mathcal{P}$. $\lhd$

Step 9 *To complete the representation result, proceed analogously to Steps 2-6 in the proof of Theorem 3.1.*

$\rhd$ Everything is analogous substituting out Axioms 5 and 6 for their respective analogies from Steps 4 and 5 of the current proof and the observation that the equivalence of (i) and (ii) in Proposition 3.1 remains valid if there is an exogenous probability, $(1-\pi) < 1$, of being unable to risk-share (since expected utilities are linear in probabilities). $\lhd$

Step 10 *The parameters (η, α) are determined analogously to the identification in Theorem 4.1. It remains to establish the uniqueness of π.*

$\rhd$ Consider again $(f,g) \notin PS(f,g)$. Then $u(\alpha_{f,g}(f+g)) > u(f)$ or $u((1-\alpha_{f,g})(f+g)) > u(g)$. Suppose $u(\alpha_{f,g}(f+g)) > u(f)$ and let $h \in \mathcal{F}$ be such that $\langle h, 0\rangle \sim_g^1 \langle f, 1\rangle$. Then $\pi u(\alpha_{f,g}(f+g)) + (1-\pi)u(f)$ is strictly increasing in π and hence the π that

satisfies the requirement $\pi u(\alpha_{f,g}(f+g)) + (1-\pi)u(f) = u(h)$ is unique. The case where $u((1-\alpha_{f,g})(f+g)) > u(g)$ is analogous. $\triangleleft$
This completes the proof of Theorem S.2. $\blacksquare$

Proof of Proposition S.1. First, notice that for any pair of acts (f,g) held by Agents 1 and 2, respectively, all three bargaining solutions depend only on $u(f)$, $u(g)$, and $u(f+g)$. Fix now $\pi > 0$ and act g, with utility $u(g)$, held by Agent 2. Given some set of available acts, $F \subset \mathcal{F}$, Agent 1 seeks to maximize:

$$\pi u(\alpha_{f,g}(f+g)) + (1-\pi)u(f) = \pi \left([(\alpha_{f,g}]^{1-\eta} u(f+g) + u((\alpha_{f,g}) \right) + (1-\pi)u(f),$$

where the equality is from Lemma A.3. With g held fixed, Agent 1 only affects $u(f)$ and $u(f+g)$, so the problem can be rewritten:

$$\max_{w \in W} \ \pi \left(\alpha(w, v(w))^{1-\eta} v(w) + u(\alpha(w, v(w))) \right) + (1-\pi)w \tag{S.15}$$

where $w = u(f)$, $W = \{w : u(f) = w, f \in F\}$, $v(w) = \max\{u(f+g) : u(f) = w, f \in F\}$, and $\alpha(w, v(w)) = \alpha_{f,g}$ given $u(f) = w$, $u(f+g) = v(w)$, and $u(g)$ fixed. Differentiating (S.15) with respect to w yields:

$$\begin{aligned}\pi(1-\eta)v(w)\alpha(w,v(w))^{-\eta}\left(v'(w)\alpha^{(0,1)}(w,v(w)) + \alpha^{(1,0)}(w,v(w))\right) + \pi v'(w)\alpha(w,v(w))^{1-\eta} \\ + \pi u'(\alpha(w,v(w)))\left(v'(w)\alpha^{(0,1)}(w,v(w)) + \alpha^{(1,0)}(w,v(w))\right) + (1-\pi),\end{aligned} \tag{S.16}$$

where the vector in the exponent indicates which argument the partial derivative is being taken with respect to. For all three bargaining solutions under consideration, under the supposition that $u(f) = u(g)$, we have $\alpha = \frac{1}{2}$ and (S.16) becomes

$$[\pi 2^{\eta-1} v'(w) + (1-\pi)] + \pi 2^{\eta}((1-\eta)v(w) + 1)\left(\alpha^{(1,0)}(w,v(w)) + \alpha^{(0,1)}(w,v(w))v'(w)\right). \tag{S.17}$$

The local incentive to increase w, at the possible expense of efficiency $v(w)$, is therefore given by (S.17). The first term in brackets, $\pi 2^{\eta-1}v'(w) + (1-\pi)$, does not depend on the sharing rule, so does not affect the comparison of the three bargaining solutions. Next, $\pi 2^{\eta}((1-\eta)v(w)+1) > 0$ for all feasible values of $v(w)$ given η, and also does not depend on the sharing rule. Hence, the comparison of the local incentives under the three different bargaining solutions turns on the rankings of $\alpha^{(1,0)}(w,v(w)) + \alpha^{(0,1)}(w,v(w))v'(w)$. A greater value for this term implies a greater local incentive to increase the value of the outside option, $w = u(f)$, at the expense of efficiency, $v(w) = u(f+g)$.

Because all three bargaining solutions are consequentialist (Section 6), they can be characterized independent of π. We first consider the case where $\eta > 1$. For the EBS, α_E

must solve

$$\begin{aligned}\Psi_E(w,v,\alpha) &= \alpha_E(w,v(w))^{1-\eta}v(w) + u(\alpha_E(w,v(w))) - w \\ &\qquad - \big((1-\alpha_E(w,v(w)))^{1-\eta}v(w) + u(1-\alpha_E(w,v(w))) - u(g)\big) = 0.\end{aligned}$$

From the Implicit Function Theorem,

$$\begin{aligned}\alpha_E^{(1,0)} &= \frac{-\Psi_E^{(1,0,0)}}{\Psi_E^{(0,0,1)}} = \frac{-(\alpha_E(1-\alpha_E))^{\eta}}{(\alpha_E^{\eta} + (1-\alpha_E)^{\eta})\left((\eta-1)v(w)-1\right)} \\ \alpha_E^{(0,1)} &= \frac{-\Psi_E^{(0,1,0)}}{\Psi_E^{(0,0,1)}} = \frac{\alpha_E^{\eta+1} - \alpha_E^{\eta} + \alpha_E(1-\alpha_E)^{\eta}}{(\alpha_E^{\eta} + (1-\alpha_E)^{\eta})\left((\eta-1)v(w)-1\right)}.\end{aligned}$$

Next, with $\eta > 1$, we have $u(h) < \frac{1}{\eta-1}$ for any act h. So, $\frac{1}{\eta-1} > u(f+g) \geq u(2f)$, with the second inequality strict if $f \neq g$. Hence, we can parameterize $v(w) = r\left(\frac{1}{\eta-1}\right) + (1-r)u(2f)$, for some $r \in [0,1)$. Recalling that $u(g) = w$ implies $\alpha_E = \frac{1}{2}$, we have

$$\alpha_E^{(1,0)}\Big|_{\alpha_E=\frac{1}{2},u(g)=w} = \frac{1}{4(1-r)(1-(\eta-1)w)}, \quad \text{and} \quad \alpha_E^{(0,1)}\Big|_{\alpha_E=\frac{1}{2},u(g)=w} = 0.$$

Following the same steps for the NBS, α_N must solve

$$\begin{aligned}&\frac{\partial}{\partial \alpha_N}\Big[\big(\alpha_N(w,v(w))^{1-\eta}v(w) + u(\alpha_N(w,v(w))) - w\big) \\ &\qquad \times\big((1-\alpha_N(w,v(w)))^{1-\eta}v(w) + u(1-\alpha_N(w,v(w))) - u(g)\big)\Big] = 0,\end{aligned}$$

and

$$\alpha_N^{(1,0)}\Big|_{\alpha_N=\frac{1}{2},u(g)=w} = \frac{1-\eta}{4(1-\eta-r)(1-(\eta-1)w)}, \quad \text{and} \quad \alpha_N^{(0,1)}\Big|_{\alpha_N=\frac{1}{2},u(g)=w} = 0.$$

Therefore,

$$\left(\alpha_E^{(1,0)} - \alpha_N^{(1,0)}\right)\Big|_{\alpha_E=\alpha_N=\frac{1}{2},u(g)=w} = \frac{\eta r}{4(1-r)(\eta-1+r)(1-(\eta-1)w)} \geq 0$$

for all admissible η, w, r, with the inequality strict for all $f \neq g$ (i.e., $r \neq 0$). This establishes the ranking of the local incentives under the EBS and the NBS.

The equation that characterizes the KSS, (3), first requires solving for $\bar{a}_{f,g}$ and $\bar{b}_{f,g}$.Because Pareto efficient sharing arrangements are proportional in our representation, there exists unique $\underline{\alpha}_{f,g}$, $\overline{\alpha}_{f,g}$ such that $\bar{a}_{f,g} = \overline{\alpha}_{f,g}(f+g)$ and $\bar{b}_{f,g} = (1-\underline{\alpha}_{f,g})(f+g)$. Specifically,

$$\underline{\alpha}_{f,g} = \left(\frac{1-(\eta-1)w}{1-(\eta-1)v(w)}\right)^{\frac{1}{1-\eta}} \quad \text{and} \quad \overline{\alpha}_{f,g} = 1 - \left(\frac{1-(\eta-1)u(g)}{1-(\eta-1)v(w)}\right)^{\frac{1}{1-\eta}}.$$

Following the same steps, then gives

$$\alpha_K^{(1,0)}\Big|_{\alpha_K=\frac{1}{2},u(g)=w} = \frac{2r\left(\frac{2^{\eta-1}}{1-r}\right)^{\frac{1}{1-\eta}} + 2\left(\frac{2^{\eta-1}}{1-r}\right)^{\frac{1}{1-\eta}} + \left(2-2\left(\frac{2^{\eta-1}}{1-r}\right)^{\frac{1}{1-\eta}}\right)^{\eta} - 2}{4\left(2-2\left(\frac{2^{\eta-1}}{1-r}\right)^{\frac{1}{1-\eta}} - \left(2-2\left(\frac{2^{\eta-1}}{1-r}\right)^{\frac{1}{1-\eta}}\right)^{\eta} + 2r\left(\left(\frac{2^{\eta-1}}{1-r}\right)^{\frac{1}{1-\eta}} - 1\right)\right)((\eta-1)w-1)}$$

$$\alpha_K^{(0,1)}\Big|_{\alpha_K=\frac{1}{2},u(g)=w} = 0.$$

It is then a matter of tedious algebra to establish that $\left(\alpha_E^{(1,0)} - \alpha_K^{(1,0)}\right)\Big|_{\alpha_E=\alpha_K=\frac{1}{2},u(g)=w} \geq 0$ and $\left(\alpha_K^{(1,0)} - \alpha_N^{(1,0)}\right)\Big|_{\alpha_K=\alpha_N=\frac{1}{2},u(g)=w} \geq,=,\leq 0$ for $\eta >,=,< 2$ and, again, with all inequalities strict if $f \neq g$ (i.e, $r \neq 0$). This establishes the ranking of the local incentives under the two comparisons involving the KSS.

The cases of $\eta = 1$ and $\eta < 1$ are handled similarly. The $\eta = 1$ case is straightforward, and for $\eta < 1$ we employ a parameterization similar to that used in the $\eta > 1$ case. In particular, for $\eta < 1$, we have $u(h) \geq \frac{1}{\eta-1}$ for any act h. Since $u(f+g) \geq u(2f)$, it must be that $u(f) = w \leq \overline{w}(v) := \frac{2^{\eta}(1-\eta)v+2^{\eta}-2}{2(1-\eta)}$. So, the appropriate parameterization is to set $w = r\left(\frac{1}{\eta-1}\right) + (1-r)\overline{w}(v)$, for some $r \in [0,1]$. With this, the argument follows the analogous steps. ■

References

Daley, B. and P. Sadowski (2025): "Risk Sharing and Strategic Choice," *Working paper.*

Kalai, E. (1977): "Proportional Solutions to Bargaining Situations: Interpersonal Utility Comparisons," *Econometrica*, 45, 1623–1630.

Kalai, E. and M. Smorodinsky (1975): "Other Solutions to Nash's Bargaining Problem," *Econometrica*, 43, 513–518.

Nash, J. F. (1950): "The Bargaining Problem," *Econometrica*, 18, 155–162.

Peters, H. and E. Van Damme (1991): "Characterizing the Nash and Raiffa Bargaining Solutions by Disagreement Point Axioms," *Mathematics of Operations Research*, 16, 447–461.