Risk Sharing and Strategic Choice*

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Abstract

We undertake a decision-theoretic analysis of a model of bilateral risk sharing, conceptualized in two stages: in the first stage agents choose risky endowments (Savage acts), and in the second stage they form a risk-sharing arrangement. Only the first-stage choices are observable to the analyst. We formulate axioms that put joint restrictions on best responses in the first stage and a representation result according to which agents behave as if they are risk averse expected-utility maximizers who anticipate the subsequent sharing arrangement. All the parameters of the model, including the sharing arrangement, can be identified from this first-stage choice data.

Keywords: Risk sharing, Informal insurance, Choice under risk, Cooperative bargaining theory

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1 Introduction

This project proposes a novel, decision-theoretic approach to the study of informal risk-sharing arrangements between individuals. We assume that individual choices of risky prospects (i.e., of Savage acts) precede a second, risk-sharing stage. The decision-theoretic nature of the study is twofold. One, the analyst only observes first-stage choices and tries to infer properties of the unobserved second stage. Two, we describe first-stage choices via behavioral axioms.

We aim to capture environments in which the motive for risk sharing is mutual insurance stemming from a diversification benefit to pooling risks (as opposed to risk shifting, which could lead one individual to insure another even without any diversification benefit from pooling risks). We argue that this focus on hedging idiosyncratic risks, and our resultant findings, are in line with much of literature on risk sharing in developing economies.[1]

Our theory consists of *i*) behavioral axioms describing observable choices of first-stage Savage acts; and *ii*) a parameterized model that describes the agents' utility functions and the precise rule that governs how risk is allocated depending on the agents' pre-sharing choices of risky acts. We then establish that the axioms tightly characterize the model: a data set satisfies the axioms if and only if there exist parameters under which the model generates that data set.[2] We do not argue that our axioms are unassailable, but rather view them as plausible and empirically testable properties. Our model can then be tested entirely in terms of individual choice data, as opposed to describing outcomes as in cooperative game theory and relying on data about transfers. Moreover, the parameters of the model, including the sharing arrangement, can be uniquely identified (given sufficient choice data), meaning our model can be estimated.

For concreteness, consider the following instantiations of the elements in this environment, using the example of two neighboring farmers for illustration. The objects of observable choice are acts that lead to a consumption outcome for every payoff relevant state of the world, ω. For instance, one act may be planting a crop that produces high yield only when it rains a lot, and another act may be planting a crop that does better when the weather is dry. Suppose that each agent i evaluates consuming the output of

[1]The parsimony of our axiomatization and representation benefits from the focus on diversification, but our approach could be adapted to environments where risk shifting is relevant as well.

[2]This work follows Daley and Sadowski (2017) in applying axioms to behavioral data directly in the multi-agent context in order to characterize aspects of the multi-agent outcome (e.g., the sharing rule), rather than applying axioms exclusively to the single-agent context and importing the single-agent representation into an assumed model of how multiple agents interact.

any act f via a risk averse expected utility function $u_i(f)$.

The key is that agents are not consigned to consume the output of their chosen act. Instead, the acts chosen by Agents 1 and 2, respectively, serve as risky endowments in the subsequent sharing stage, where risk pooling may reduce exposure to idiosyncratic risk. For example, if Agent 1 plants the crop that is better suited to wet conditions, and Agent 2 plants the crop that does well in dry conditions, they might then agree on a transfer from the first to the second when the weather is wet, and vice versa when it is dry, providing some insurance to both. More generally, given respective endowments f and g, a *sharing rule* specifies a feasible state-contingent allocation $a(\omega)$ to Agent 1 and $b(\omega)$ to Agent 2 such that $a(\omega) + b(\omega) \leq f(\omega) + g(\omega)$ for all ω.

There are many possible sharing rules. For example, given (f, g), the three prominent cooperative bargaining solutions of Nash (1950); Kalai and Smorodinsky (1975); Kalai (1977) each prescribe an allocation of the pooled risk. In the case of the Nash bargaining solution (NBS), that allocation is

$$(a^*, b^*) = \arg \max_{(a,b) \in A(f,g)} \left(u_1\left(a\right) - u_1\left(f\right)\right)\left(u_2\left(b\right) - u_2\left(g\right)\right), \tag{1}$$

where $A(f, g)$ is the set of all feasible allocations given choices f and g. Under the NBS, holding fixed Agent 2's choice of act g, it is clear that Agent 1's choice of act f will affect the allocation each agent receives.

On the one hand, the endowments f and g determine the *size* of the possible surplus from mutual insurance. If $f = g$, there is no idiosyncratic risk to hedge through pooling. In the example this could occur in the hypothetical of two farmers with the same technologies and comparable plots of land choosing to grow the same crop. Of course, choices $f \neq g$ that better hedge one another increase the potential surplus from risk pooling.

On the other hand, f affects Agent 1's disagreement value, $u_1(f)$, and under the NBS the *share* of the surplus Agent 1 receives is increasing in this disagreement value. When choosing a risky act in the first stage, forward looking agents will trade off the size of the potential surplus from risk pooling and their individual share.

For many sharing rules, including those based on other prominent bargaining solutions, shares also depend on disagreement values, but the manner in which they do so varies. As a consequence, the first-stage choice of act will depend on the particular sharing rule that governs the second stage, as well as on risk preferences. Our representation and identification theorems together imply that choice data that satisfies our axioms will reveal, via our model and the uniquely identified bargaining solution and risk preferences, the specific tradeoff the agents face.[3]

Mutual insurance through informal risk sharing has been been a topic of ongoing interest in the study of developing economies for several decades. Quoting the seminal work of Townsend (1994):

"[T]hroughout much of the underdeveloped world, [people] live in poor, high-risk environments. Per capita income and per capita consumption are low, and the risk to agriculture from erratic monsoon rains is high. Crop and human diseases are also prevalent. Various policy issues turn on this level of risk and on the presence or absence of risk reduction mechanisms ... In an optimal arrangement, both [agents] would coinsure the fluctuations of each, ..."

The preeminent question of interest in this field is whether pooling these risks among agents provides diversification that efficiently shields them from idiosyncratic shocks. Notable additional examples that focus on this motive for risk sharing include Ligon et al. (2002); Belhaj and Deroïan (2012); Munshi and Rosenzweig (2016). Since risk-sharing arrangements—especially informal ones that are prevalent in many developing economies—are difficult to observe directly, efficiency is often empirically tested based on consumption and income data obtained primarily from surveys. Pareto efficient sharing is rejected if individual consumption varies not only with common, but also with idiosyncratic, income shocks.[4] If this test rejects Pareto efficiency, then pooling must be incomplete, indicating that diversification benefits are left on the table. The test is insensitive to how pooled risk is distributed across agents, in line with a focus on the diversification benefits of risk sharing.

We too focus our analysis on situations where risk sharing serves to provide mutual insurance via diversification. Our approach has a number of potential benefits. First, it can identify properties of the sharing arrangement (e.g., does the sharing arrangement adhere to a known cooperative bargaining solution such as Nash, 1950?) and specifies the distributional consequences of different sharing arrangements beyond whether they are Pareto

[3] In the motivations of our axioms, agents anticipate how they would agree to divide the surplus from risk sharing given endowments f and g, for instance through subsequent bargaining as in the example above. In an alternative timing, bargaining could take place earlier and include the choices of f and g. The timing in which agents take endowments as given when deciding on how to share risk is well suited to situations where the productive choices of f and g are difficult to enforce, for instance due to noncontractable effort, or because the length of time between the choices of f and g and the realization of output is too large to expect an agreement to last that long.

[4] Papers in the substantial literature that employs versions of this test include Cochrane (1991); Mace (1991); Altonji et al. (1992); Townsend (1994); Hayashi et al. (1996); Ravallion and Chaudhuri (1997); Laczó (2015); Bold and Broer (2021); Meghir et al. (2022).

efficient. Second, understanding how agents' pre-sharing choices of risky prospects (e.g., choices of crops to plant, profession, education level, marriage/reproduction/household formation, etc.) depend on the sharing arrangement is crucial when evaluating efficiency of the arrangement not just in terms of transfers at the sharing stage, but also in terms of prior productive choices.[5] For instance, how much do agents prioritize their disagreement values at the cost of social surplus? Third, our axioms provide a basis for testing Pareto efficiency at the sharing stage jointly with auxiliary assumptions about individual risk preferences. Finally, in contexts where consumption and transfer data are hard to observe directly, the pre-sharing choice data on which our approach is based may provide a useful alternative or supplemental data source.

The paper is structured as follows. Section 2 formally introduces the environment. Section 3 explains our general axioms, provides the representation and identification theorems for our benchmark efficient-sharing model, and discusses features of the model in connection to evidence from, and modeling assumptions in, the applied literature. According to our identification result, each sharing arrangement has unique behavioral implications for first-stage choices. As examples, a supplemental appendix provides additional axioms that characterize sharing according to each of the prominent cooperative bargaining solutions of Nash (1950); Kalai and Smorodinsky (1975); Kalai (1977).[6]

Of course, existing evidence suggests that risk sharing is often less than efficient.[7] In Section 4, we relax our axioms to allow for frictions at the sharing stage, modeled as the probability of a "breakdown" of sharing. We provide comparative statics results that relate the magnitude of frictions to the strength of agents' preferences for sharing, as well as their incentives to prioritize their disagreement values over the generation of social surplus.

We believe our results demonstrate that in the risk-sharing context, multi-agent best-response data strikes an appropriate balance between observability, tractability, and power for testing and estimation. If agents choose risky acts in sequence, or if opportunities for each agent to change acts arise stochastically, and if agents heavily discount the future, then static best responses to the risky acts currently held by others may reasonably model

[5] The most basic implication of this trade-off—that a subsequent opportunity for risk sharing should increase individual risk taking—is in line with empirical and experimental evidence (Angelucci et al., 2018; Attanasio et al., 2012).

[6] Hence, in the context of our model, our approach allows us to provide behavioral foundations for these bargaining solutions, which are usually motivated only via the normative appeal of the outcomes they produce.

[7] The null-hypothesis of efficient sharing is rejected in several studies in the Economic Development literature; see Ligon, Thomas, and Worrall (2002) for references, which begin with Townsend (1994).

their behavior. For simultaneous-move games, Section 5 illustrates how the decision-theoretic study of our model can serve as a foundation for further analysis. We formally model the situation mentioned above, where two farmers have to decide which crops to plant, and investigate the inefficiencies induced by different sharing protocols in equilibrium. In particular, we rank the prominent bargaining solutions in terms of the efficiency loss they induce, perhaps providing a novel criterion to compare those solutions.

Throughout, we refer to neighboring farmers for illustration, but our model could be relevant when the risk from other labor income is pooled across households or, as we mention again in Sections 3 and 6, within the household. Throughout, we focus on the bilateral risk-sharing case for ease of exposition. An extension to larger groups (*aka* "syndicates" in Wilson, 1968) is conceptually straightforward. Section 6 concludes by briefly suggesting a potential alternative extension where multiple agents share risk bilaterally along the ties of a network.

2 The Risk-sharing Environment

Overview. There are two agents, $i \in \{1, 2\}$, and three time stages. In the first stage, each agent strategically chooses a risky endowment. In the second stage, the agents have the opportunity to share risk by agreeing to a feasible reallocation given their first-stage choices. In the third stage, uncertainty realizes, transfers are made in accordance with the second-stage agreement, and consumption occurs.

We are explicit about the first-stage choices of risky endowments, corresponding to an assumption that this is the data available to the analyst and therefore the domain of our axioms. In contrast, the procedure by which the agents bargain over risk sharing in the second stage is unspecified as in the cooperative game theory literature (most prominently Nash, 1950). Notice that if the agents fail to agree on a reallocation, then each agent is left to consume her first-stage choice, meaning it serves as the agent's *disagreement allocation*. In the first stage, agents are aware of the manner by which agreements are reached in the second stage (though it is unobservable to the analyst).

First-stage Acts. Let $(\Omega, \mathcal{A}, \mu)$ be an infinitely divisible probability space where $\mathcal{A}$ is the Borel sigma algebra and $\mu \in \Delta(\Omega)$ is a probability measure on Ω. An *act* $f : \Omega \to \mathbb{R}_{++}$ is a mapping that is measurable in $\mathcal{A}$ and assigns a consumption outcomes to each $\omega \in \Omega$.[8] We use $f = g$ to mean the two acts are equal almost surely. Let $\chi(f)$ be the range of f,

[8] Restricting to strictly positive consumption averts well known issues for common utility functions, such as $u(c) = \log(c)$, in which the zero consumption yields $u(0) = -\infty$.

and say that f is simple if $\chi(f)$ is finite. Denote by $\mathcal{F}$ the set of all simple acts.

Theories of individual choice often follow Savage (1954), and employ acts on a state space to study subjective probabilities. In contrast, the probability measure μ in our model is objective, but states and acts are used in order to track the joint distribution generated by the two acts chosen by the two agents—given μ, every pair of acts endows a lottery over *pairs* of outcomes. Infinite divisibility of the probability space means that any simple lottery over pairs of outcomes can be generated this way.

Second-stage Agreements. Let f and g be the acts held by Agents 1 and 2, respectively, at the outset of the second stage. A *sharing rule* $\Gamma : \mathcal{F}^2 \to \mathcal{F}^2$ is a mapping from (f, g) to a consumption allocation (a, b) in the set of feasible reallocations $A(f, g) := \{(a, b) \in \mathcal{F}^2 \,|a + b \leq f + g\}$. Here, "feasibility" signifies only that there are sufficient resources to construct (a, b) from the pooled (f, g) in the abstract, not that such a reallocation among the agents is necessarily attainable. For example, physical or informational frictions may impede the required transfers, but whether such frictions exist is unobservable to the analyst. Looking ahead, our representation in Section 3 will capture the frictionless environment, and the possibility of unobservable frictions will be incorporated in Section 4.

3 Axioms, Model, and Representation

3.1 The Choice Domain

The observable-choice domain for our axioms is best-response data. The analyst observes each agent i's preferences over $\mathcal{F} \times \{0, 1\}$, holding fixed the act held by $j \neq i$ to be $g \in \mathcal{F}$.[9] We denote these preferences by $\left\{\succsim^i_g\right\}_{g\in\mathcal{F}}$. For alternative $\langle f, o\rangle \in \mathcal{F} \times \{0, 1\}$ the interpretation of the second component $o \in \{0, 1\}$ is that

- $\langle f, 1\rangle$ indicates the agents will have the opportunity to pool and reallocate their respectively chosen risky endowments, f and g (as described in the preceding section).
- $\langle f, 0\rangle$ indicates the agents *cannot* pool their risks, and must consume their own risky endowments. We refer to this situation as *autarky*.

For concreteness, $\langle f, 0\rangle \succsim^i_g \langle f', 0\rangle$ means that i prefers f over f' in autarky when agent j holds act g (below we will assume that autarky preferences are independent of g, ruling

[9] Summarizing best responses with conditional preferences is analogous to the standard approach in the decision theory literature, where choice data is summarized by a preference.

out social comparisons such as envy). Similarly, $\langle f, 1\rangle \succsim_g^i \langle f', 1\rangle$ means that i prefers f over f' when agent j holds act g, but now with the understanding that the two agents will have the opportunity to reallocate risk in the second stage. In this case, we should expect i's ranking over acts to depend on the act j holds. Note that the primitive encodes whether agents have the *opportunity* to pool risk; for instance, farmers may be able to pool risk only if they live in the same village, or a pair of marriageable individuals only if they have the potential to form a household with each other. To what extend they actually do so is an empirical question which our axioms can help address. Finally, our primitive also captures i's freedom to opt out of the sharing arrangement entirely: $\langle f, 1\rangle \succsim_g^i \langle f', 0\rangle$ means that i prefers to hold f when j holds g and they have the opportunity to reallocate in the second stage, rather than opting out of the sharing arrangement (e.g., a farmer may choose to relocate to another village, a potential partner may choose to decline a marriage proposal, etc.) and consuming f' in autarky. Note that while i likely also has preferences over the act held by j, we do not observe those preferences because i cannot choose for j. Our *model* (Definition 3.1), however, does induce a complete preference over $\mathcal{F}^2 \times \{0, 1\}$ for each agent.

3.2 General Axioms

We now present axioms on the domain just described, which will tightly characterize our behavioral model. Notably, some axioms impose a joint structure on the agents' best responses. We refer to axioms in this section as "general" because they do not prescribe a particular sharing rule (e.g., one that adheres to the NBS), a question we turn to in a supplemental appendix. Our first two axioms specify that all preference relations are well behaved, and that in autarky each agent has standard preferences that are concerned only with her own consumption.

Axiom 1 (Preference)
The binary relations $\left\{\succsim_g^i\right\}_{g\in\mathcal{F}}$ *for* $i \in \{1, 2\}$ *are preference relations on* $\mathcal{F} \times \{0, 1\}$.

Axiom 2 (Expected Utility Preferences in Autarky)
On $\mathcal{F} \times \{0\}$, *the binary relation* $\succsim_g^i$ *is independent of* g *and satisfies the von Neumann Morgenstern (vNM) axioms for a continuous, monotonic expected utility representation.*[10]

[10] Throughout, any reference to expected-utility maximization on the domain of simple acts, $\mathcal{F}$, assumes that only the lotteries induced via the measure μ matter for the ranking of the acts. Formally, for $c \in \mathbb{R}_{++}$ and $f \in \mathcal{F}$, let $p_f(c) = \mu(\{\omega \in \Omega | f(\omega) = c\})$, and assume that the decision maker is indifferent between acts $f, g \in \mathcal{F}$ that induce the same lottery: $p_f = p_g$ implies $f \sim_i g$. The induced ranking of simple lotteries

The next axiom captures voluntary participation: each agent likes being in the sharing arrangement at least as well as her disagreement allocation. Moreover, there is at least one instance in which an agent is made better off by the sharing arrangement compared to autarky.

Axiom 3 (Voluntary Participation) *$\langle f, 1\rangle \succsim_g^i \langle f, 0\rangle$ for all $(f, g) \in \mathcal{F}^2$ and $i \in \{1, 2\}$, with strict preference for some (f, g) and i.*

As discussed at the outset, we consider situations where agents may choose to pool their resources, f and g, if doing so has the potential benefit of mutual insurance via diversification. Note that for scalar $\beta > 0$, the act βf is a "scaled replica" of the act f. Hence the pair $(f, \beta f)$ constitutes a situation without any idiosyncratic shocks to hedge via pooling.[11] The next axiom captures that the agents have no strict preference for sharing over autarky in this case.

Axiom 4 (Hedging) *$\langle f, 0\rangle \succsim_{\beta f}^i \langle f, 1\rangle$ for all $f \in \mathcal{F}$, $\beta > 0$ and $i \in \{1, 2\}$.*

For the next axiom, consider an act-pair (f, g) and remember that Agent 1's preference for f given g is based not on the consumption value of f, but on the second-stage reallocation that will be implemented starting from (f, g). If there is no waste or frictions in the reallocation process, then it cannot be that both agents prefer a feasible final reallocation $(a, b) \in A(f, g)$ to bargaining starting from (f, g). The axiom captures this requirement via its contrapositive.[12]

Axiom 5 (No Waste)
If $\langle a', 0\rangle \succsim_g^1 \langle f, 1\rangle$ and $\langle b', 0\rangle \succsim_f^2 \langle g, 1\rangle$, at least one of them strict, then $(a', b') \notin A(f, g)$.

The final axiom captures that resource constraints are binding, meaning that reallocating resources from the chosen acts is the only "joint production technology" available

is then required to satisfy the von Neumann Morgenstern (1944) axioms of Transitivity, Completeness, Continuity in probabilities and payoffs, and Independence, and to rank larger risk-free amounts over smaller ones.

[11] Consider the following criterion for pooling risks to have even a potential benefit from diversification: *Is the pooled risk less risky than at least one of the original individual risks*? To make this precise, isolate the riskiness of an act apart from its scale via the random variable $R_f := \frac{f}{n(f)}$, where $n : \mathcal{F} \to \mathbb{R}_{++}$ is any normalization factor satisfying $n(f + g) = n(f) + n(g)$ for all f, g. For example, $n(f)$ could be $\mathbb{E}[f]$, or if prices were introduced then R_f could be the gross percentage return of f. Now ask: is $variance(R_{f+g}) < \max\{variance(R_f), variance(R_g)\}$? When f and g are scaled replicas, pooling risk fails this extremely weak criterion for a potential benefit of diversification.

[12] Section 4 relaxes Axiom 5 in order to accommodate possible frictions in the reallocation process.

to the two agents, and that agents do not derive a non-instrumental (for instance, psychological) benefit from participating in risk sharing. Within a utility representation, such an assumption would be clear enough. But we need to capture this idea in terms of our observable domain: $\{\succsim_g^i\}_{g\in\mathcal{F}}$ for $i \in \{1,2\}$. To do so, we define the set of Pareto efficient reallocations of (f,g) as

$$PS\,(f,g) := \{(a,b) \in A(f,g) | \langle a,0\rangle \succsim_g^1 \langle a',0\rangle \ \mathit{or} \ \langle b,0\rangle \succsim_f^2 \langle b',0\rangle \ \forall \ (a',b') \in A\,(f,g)\}.$$

Axiom 6 (Resource Constraint)
If $(a,b) \in PS\,(f,g)$*, then* $\langle a,0\rangle \succsim_g^1 \langle f,1\rangle$ *or* $\langle b,0\rangle \succsim_f^2 \langle g,1\rangle$*.*

3.3 Model Preliminaries

Our behavioral model provides structure on two dimensions: a) the nature of risk-sharing arrangements, and b) how first-stage preferences over acts are determined taking (a) into account. Taking the two components in turn:

a) A particularly parsimonious set of sharing rules are those that deliver each agent a constant share, $\alpha^i \in (0,1)$, of the total output in each state: for first-stage acts (f,g),

$$\Gamma(f,g) = \left(\alpha_{f,g}^1(f+g), \alpha_{f,g}^2(f+g)\right) \in A(f,g), \quad \mathit{where} \ \ \alpha_{f,g}^1 + \alpha_{f,g}^2 = 1.$$

In finance parlance, in such an arrangement the stochastic "cash flows" of f and g are securitized exclusively using equity claims, and each agent holds some fraction of this "market portfolio." We refer to a sharing rule that has this structure (up to changes on μ-measure zero events) as *proportional*. Notice that the shares are constant across states given (f,g), but critically, can vary with the acts (f,g). The choices of first-stage acts can influence the bargaining outcome in the second stage—but the bargaining outcome is some proportional sharing agreement for any chosen pair of acts.

b) As foreshadowed in Section 1 (and by Axiom 2), in our model agents are standard expected utility maximizers that care only about their own consumption. Let u_i be i's utility for consumption c, and slightly abusing notation, denote the linear extension of u_i to $\mathcal{F}$ also by $u_i(f) := \sum_{c\in\chi(f)} u_i(c)\mu(f^{-1}(c))$.

As discussed in the next subsection, the literature on informal risk-sharing often assumes homogeneous risk tolerances, and (among expected utility preferences) constant relative risk aversion (CRRA) has support in empirical evidence. Explicitly, there is $\eta \geq 0$ such that for each agent i, up to a positive affine transformation,

$$u_i(c) = \begin{cases} \frac{c^{1-\eta}-1}{1-\eta} & \eta \neq 1, \eta \geq 0 \\ \ln(c) & \eta = 1. \end{cases} \tag{2}$$

Recall that $\eta = 0$ implies risk neutrality, and greater η implies more aversion to risk. Finally, as is standard, agents are forward looking and have common beliefs about how second-stage bargaining will reallocate from any first-stage acts (f, g).

The following proposition shows that Pareto efficient bargaining generates a tight connection between (a) and (b).[13]

Proposition 3.1 *For two expected-utility maximizers with continuous and monotonic consumption utility functions u_1, u_2, the following are equivalent.*

i) u_1 and u_2 are CRRA utilities with the same η-value.

ii) For all $f, g \in \mathcal{F}$ and $\alpha \in (0,1)$, the proportional reallocation $(\alpha(f+g), (1-\alpha)(f+g))$ is Pareto efficient in $A(f,g)$.

Moreover, if $\eta > 0$, then (i) implies that: iii) for all $f, g \in \mathcal{F}$, if (a,b) is Pareto efficient in $A(f,g)$, then (a,b) is a proportional reallocation of (f,g).

For a specific example, suppose that for both agents, $u_i(c) = \ln(c)$. For any act-pair (f, g), the Egalitarian bargaining solution (EBS) of Kalai (1977) selects the Pareto efficient reallocation (a, b) such that $u_1(a) - u_1(f) = u_2(b) - u_2(g)$ to capture the idea that each agent "gains equally" from their agreement.[14] For this case an explicit solution exists: each agent's share of the total output corresponds to the relative value of consuming her chosen act (i.e., her disagreement allocation). This simple form makes clear each agent's incentive to increase her disagreement value, even at the possible expense of decreasing available surplus.

[13] At least the equivalence between proportional sharing rules and Pareto efficiency when agents have common CRRA utilities and $\eta > 0$ is familiar (Back, 2016, ch. 4).

[14] Unlike other prominent bargaining solutions (Section 3.5), the interpretation of the EBS depends on cardinal utility values.

Lemma 3.1 *If $u_i = \ln$ for both agents and sharing is in accordance with the EBS, then, letting $ce(f)$ denote the certainty equivalent of act f, we have*

$$\alpha^1_{f,g} = \frac{ce(f)}{ce(f) + ce(g)} \quad \text{and} \quad \alpha^2_{f,g} = \frac{ce(g)}{ce(f) + ce(g)}.$$

3.4 Representation Theorem

We now turn to the connection between the model and the axioms. Since η fully describes risk preferences, it is convenient to normalize agent utilities as in (2) and to describe familiar bargaining solutions in terms of those normalized utilities.

Definition 3.1 *A* ***two-stage model of proportional risk sharing*** *is summarized by a pair (η, α) where the parameter $\eta > 0$ specifies a common CRRA utility function u from (2), and α is a proportional sharing rule that respects disagreement values: $u\big(\alpha^1_{f,g}(f+g)\big) \geq u(f)$ and $u\big(\alpha^2_{f,g}(f+g)\big) \geq u(g)$.*

In the model, Agent 1's expected utility of sharing f with g is $u(\alpha^1_{f,g}(f+g))$. Hence, in this situation she prefers f to f' if and only if $u(\alpha^1_{f,g}(f+g)) \geq u(\alpha^1_{f',g}(f'+g))$, and analogously for Agent 2. Notice there are two components that could affect this ranking: the total output $(f+g)$ versus $(f'+g)$, and her individual share $\alpha^1_{f,g}$ versus $\alpha^1_{f',g}$. Recall that our domain includes the possibility of engaging in the sharing arrangement, $o = 1$, or opting out, $o = 0$, so that we have the following.

Definition 3.2 *A model (η, α)* ***explains*** *preferences $\left\{\succsim^i_g\right\}_{g\in\mathcal{F}}$ for $i \in \{1,2\}$, if*

$$\langle f, o\rangle \succsim^i_g \langle f', o'\rangle \iff ou\left(\alpha^i_{f,g}(f+g)\right) + (1-o)u(f) \geq o'u\left(\alpha^i_{f',g}(f'+g)\right) + (1-o')u(f').$$

Theorem 3.1 (Representation) *Preferences $\left\{\succsim^i_g\right\}_{g\in\mathcal{F}}$ for $i \in \{1,2\}$ satisfy Axioms 1-6 if and only if they can be explained by a two-stage model of proportional risk sharing (η, α).*

The theorem establishes our axioms as a complete description of the testable implications of our model. To put that model into an applied context, empirical work on risk sharing in developing economies often follows Townsend (1994) and assumes that agents who share risk have homogeneous risk preferences. It may be the case that individuals in these populations have rather similar risk preferences. In addition, to the extent that there is heterogeneity along this dimension, Attanasio et al. (2012) document that there is assortative matching on risk tolerance in the formation of risk-sharing relationships. In terms of observed risk attitudes, among the prominent classes of expected utility preferences, CRRA appears to be descriptively most accurate (for example, Pope and Just, 1991; Brunnermeier and Nagel, 2008; Chiappori and Paiella, 2011)[15] and most widely assumed (Eeckhoudt et al., 2005).[16]

In alignment with our representation, quite recent and top general-interest publications in the field of Economic Development that assume homogenous CRRA utility functions include Lagakos et al. (2023); Meghir et al. (2022); Brooks and Donovan (2020); Morten (2019); Munshi and Rosenzweig (2016). Indeed, in this literature, the homogenous-CRRA convention appears sufficiently strong that it is often deployed without further justification (similar to footnote 16).

In the Introduction, we discussed that the literature on informal risk sharing in developing countries is primarily concerned with mutual insurance: agents participate in these arrangements in order to hedge risk. Theorem 3.1 tightly connects mutual insurance as a motive for risk sharing to homogeneous CRRA utilities. Our result thereby offers a supporting rationale for the specific combination of sharing motive and other modeling assumptions commonly found in the literature, even if such a rationale is not discussed there.

As also discussed in the Introduction, the literature typically relies on survey data regarding consumption and income to test whether sharing is consistent with Pareto efficiency: Individual consumption should vary with common, but not with idiosyncratic, income shocks.[17] In our two-stage environment, Theorem 3.1 provides an alternative approach to testing efficient risk sharing. It establishes the choice of risky act (e.g., the crops farmers plant) and the choice to participate in informal risk sharing as appropriate behavioral data to test Pareto efficiency at the sharing stage *jointly* with the assumption of homogeneous CRRA preferences. In situations where data on productive choices is available, our results may thus help researchers (i) make progress when there is no reliable consumption or income data, (ii) test the presupposition of homogeneous preferences, and (iii) verify constant relative risk aversion directly in the risk-sharing context. Of course, many applied contexts appear to feature only partial mutual insurance, violating Pareto

[15]Chiappori and Paiella (2011) demonstrated the importance of panel data and found strong support for CRRA in portfolio choice. In the agricultural context Pope and Just (1991) found evidence for CRRA preferences among potato farmers in Idaho.

[16]From their textbook (page 21): "Finally, one set of [expected utility] preferences that has been by far the most used in the literature is the set of power utility functions [CRRA]. Researchers in finance and in macroeconomics are so accustomed to this restriction that many of them do not even mention it anymore when they present their results."

[17]See footnote 4 for prominent works that employed this testing strategy.

efficiency. Theorem 4.1 below axiomatically characterizes a generalization of our model that allows for sharing frictions that result in inefficiency.[18]

Consider one rather obvious example of how a researcher might employ our axioms to test the model: By Voluntary Participation (Axiom 3), for any act-pair, both agents weakly prefer participation in the sharing arrangement to opting out entirely. Hence, any preference for opting out requires that doing so alters the set of available acts. And from No Waste (Axiom 5), if two agents do prefer to opt out of the sharing arrangement, then this must be justified by increased total output at least in some set of states. One way, then, to test our model would be to check whether opting out can be explained by improvements in the set of available actions. For instance, Munshi and Rosenzweig (2016) argued that migration might be a way to improve available actions (e.g., better land or climate, better markets, other job opportunities etc.) but at the cost of losing the local (caste based) risk-sharing arrangement, and that this cost can partly explain low mobility in India.[19]

Discussion of the Proof of the Representation Theorem

As is common in representation results, that the model generates preferences that satisfy the axioms is relatively straightforward. To gain some intuition for how the axioms imply the model, recall that Proposition 3.1 links the Pareto efficiency of proportional sharing rules to homogeneous CRRA utilities. What remains to be shown is that under the axioms, sharing must indeed be both Pareto efficient *and* proportional.

By Resource Constraint (Axiom 6), sharing (f, g) does not allow Pareto improvements over what can be achieved via distributions of the total endowment $f + g$ (i.e., final allocations in $A(f, g)$). By No Waste (Axiom 5), no final allocation in $A(f, g)$ Pareto dominates sharing (f, g). Hence, both agents are indifferent between sharing (f, g) and at least some Pareto efficient final reallocation in $A(f, g)$. Next, for arbitrary act h and $\alpha \in (0, 1)$, Hedging (Axiom 4) implies that $\langle \alpha h, 0\rangle \sim^i_{(1-\alpha)h} \langle \alpha h, 1\rangle$. Setting $h = f + g$ then implies that there is no scope for Pareto improvements when first-stage acts (f, g) are proportional. Consequently, final allocations that are shares of the total endowment

[18] Mazzocco and Saini (2012) argued for heterogeneity in risk preferences as a possible explanation for the apparent inefficiency. Recall that our theory does not presuppose homogeneity, but rather suggests how to jointly test homogeneity and efficiency.

[19] See Fafchamps (2011) for a survey of the literature on risk sharing between households, including several papers that document households opting out of sharing arrangements. A potential alternative explanation of opting out would entail some violation of Voluntary Participation (Axiom 3): the risk sharing agreement is individually disadvantageous. For instance, wealthy individuals may opt out if the sharing arrangement is based on social norms that require wealth redistribution.

$f+g$ must be Pareto efficient. Next, notice that the best allocation for agent i would be to give her the entire total output, and the worst would be to give her none of it, which would correspond to $\alpha^i = 1, 0$, respectively. By Continuity and Monotonicity of autarky preferences (Axiom 2), proportional allocations are then order dense in the space of all Pareto efficient allocations in $A(f, g)$. Putting everything together then, we have that sharing any (f, g) corresponds to some proportional (and also Pareto efficient) reallocation of (f, g).

It is worth noting that only expected utility assumptions were imposed on autarky preferences (Axiom 2). Common CRRA preferences in the model are implied by the remaining axioms, which capture plausible qualitative features of behavior in the presence of risk sharing. For instance, the commonality of risk preferences in autarky does not derive from a symmetry assumption, but rather from the Pareto efficiency of risk pooling. In fact, preferences over acts when sharing is possible may not be symmetric across the two agents because the sharing rule may be asymmetric. Similarly, none of our axioms directly impose "differentiability" on preferences, and deriving differentiability of utility functions is a crucial step in establishing that autarky preferences are CRRA.

3.5 Identification and Specific Sharing Arrangements

While the representation result (Theorem 3.1) provides a foundation for testing our model, the following identification result establishes that it is in principle possible to estimate its parameters, and in particular the sharing arrangement, from observable behavior.

Theorem 3.2 (Identification) *If models (η, α) and (η', α') both explain preferences $\{\succsim_g^i\}_{g\in\mathcal{F}}$ for $i \in \{1, 2\}$, then $(\eta, \alpha) = (\eta', \alpha')$.*

Informal sharing arrangements can be difficult to observe directly. In our approach, Theorem 3.2 derives the proportional sharing arrangement as well as the coefficient of risk aversion from the choices of risky actions (e.g., crop choices). By part *(iii)* of Proposition 3.1, the focus on proportional sharing arrangements is without loss, as non-proportional sharing rules cannot be Pareto efficient when individual preferences are homogeneous CRRA.

The class of proportional sharing rules featured in our model is very general: No constraints connect $\alpha_{f,g}$ to any other $\alpha_{f',g'}$, not even symmetry or continuity. Neither of these features would be difficult to capture axiomatically, but since the focus in the empirical literature has been on testing the Pareto efficiency of sharing, and because the set of proportional sharing rules coincides with the set of Pareto efficient sharing rules in the context of common CRRA utilities, the generality is desirable. That said, our identification theorem allows us to probe deeper into the particulars of the sharing arrangement. Next we discuss sharing based on the most prominent bargaining solutions.

3.5.1 Specific Bargaining Solutions

The second stage of our model fits within the class "bargaining problems" studied in cooperative game theory under the header *cooperative bargaining theory* (see Kibris, 2010, for a survey). In this approach, agents' utilities for each feasible bargaining outcome, as well as their disagreement values, are taken as inputs. A *solution* is found by imposing some set of normatively appealing properties regarding the utility surpluses (i.e., utilities in excess of disagreement values) generated for the agents. Chief among these are 1) Pareto Efficiency and 2) Symmetry: in the solution, the surplus to each agent is equal when the Pareto frontier of achievable surpluses is itself symmetric.

Beyond these two properties, three others have been deemed desirable: 3) Scale Invariance says that rescaling one agent's utility should not affect the bargaining solution as interpersonal comparisons of cardinal utilities are meaningless in the context of revealed preferences. As in individual choice, 4) Independence of Irrelevant Alternatives (IIA) requires, roughly, that removing unchosen outcomes from the feasible set should not change the bargaining solution. Finally, 5) Resource Monotonicity requires that if the set of feasible surplus profiles expands (in the sense of set inclusion), every agent should be weakly better off. It is, famously, impossible to satisfy all these requirements simultaneously.

The three most prominent bargaining solutions each satisfy Pareto Efficiency and Symmetry and are tightly characterized by adding two of the three other desiderata, as summarized in the following Table.

	Pareto	Symmetry	Scale inv.	IIA	Monotonicity
Nash, 1950 (NBS)	✓	✓	✓	✓	X
Kalai and Smorodinsky, 1975 (KSS)	✓	✓	✓	X	✓
Kalai, 1977 (EBS)	✓	✓	X	✓	✓

Table 1: Properties of prominent bargaining solutions

The manifestations of both the NBS and EBS in the context of risk sharing have been described above (see (1) and the discussion following Proposition 3.1, respectively). The KSS relies on each agent's "aspiration payoff": the maximum payoff an agent can get in an agreement that respects disagreement values. For first-stage acts (f, g), define

$$\bar{a}_{f,g} = \arg\max_{a \in \mathcal{F}} \ u_1(a), \quad s.t. \ \ u_2(f + g - a) \geq u_2(g),$$

Define $\bar{b}_{f,g}$ analogously for Agent 2. Then the KSS is the allocation $(a, b) \in PS(f, g)$ such that

$$\frac{u_1(a) - u_1(f)}{u_1(\bar{a}_{f,g}) - u_1(f)} = \frac{u_2(b) - u_2(g)}{u_2(\bar{b}_{f,g}) - u_2(g)}. \tag{3}$$

Given our identification result (Theorem 3.2), we can investigate the first-stage behavioral manifestations of each of these bargaining solutions applied in the second stage. In a supplemental appendix, for each bargaining solution, we provide testable behavioral axioms (rather than the normative desiderata described above) that, together with Axioms 1-6, tightly characterize the special case of our general model in which α corresponds to that solution. In Section 5 below, we investigate the equilibrium implications of parametric versions of our model, and how these implications depend on which of the three bargaining solutions governs the sharing arrangement.

4 Inefficient Sharing and Identification of Frictions

As previously mentioned, a large literature has tested the efficiency of informal risk sharing between households in developing and developed countries. Important papers include Altug and Miller (1990); Townsend (1994); Gertler and Gruber (2002); Ligon et al. (2002); Banerjee and Duflo (2007). Most often, full efficiency is rejected. Common explanations of the apparent incompleteness in risk sharing are based on frictions that are unobservable to the analyst, such as private information that constrains the feasibility of transfers (see, for instance, Rogerson, 1985; Ligon, 1998).[20]

This section considers a particularly tractable generalization of our model where frictions can be summarized by a probability of "breakdown" of the sharing arrangement. One interpretation is that this probability captures the strength of the social tie between the agents: the stronger their social tie, the more likely sharing will succeed.

To formalize this idea, we will rely on a notion of randomization over acts. Under the typical assumption that agents are indifferent to the source of randomization, the fact that uncertainty is objective in our model implies this randomization can be captured within our formal domain. To do so, we denote by $f\mathbf{p}f'$ an arbitrary, but fixed, act that generates the same distribution over outcomes as the randomization that yields act f with probability $\mathbf{p}$ and f' otherwise. Furthermore, when $f\mathbf{p}f'$ and $g\mathbf{p}g'$ appear together, they denote two acts such that the distribution over pairs of outcomes is the same as that generated by Agents 1 and 2 simultaneously getting f and g with probability $\mathbf{p}$ and

[20] Mazzocco and Saini (2012) offer an alternative explanation. See footnote 18.

simultaneously getting f' and g' otherwise.[21]

We now introduce π as a pair-specific parameter such that $1-\pi$ measures the probability of a breakdown. Specifically, if $\Gamma(f,g)=(a,b)$, then agents receive the respective allocations a and b with probability π only, and retain their respective acts f and g as the final allocation otherwise. This situation generates the same distribution over outcomes as $(a\pi f, b\pi g)$. Like α and η, the new parameter π is unobservable to the analyst, and its identification is a central question for the analysis.

To distinguish situations where at least one agent values the ability to share from those situations where neither agent does, let $\mathcal{B} := \{(f,g)|\langle f,1\rangle \succ_g^1 \langle f,0\rangle \text{ or } \langle g,1\rangle \succ_f^2 \langle g,0\rangle\}$, and denote by $\mathcal{B}^C$ its complement in $\mathcal{F}^2$. Obviously, if $(f,g)\in\mathcal{B}$ and $\pi<1$ (leaving agents with their endowments with probability $1-\pi$), then No Waste (Axiom 5) will be violated: It would be possible to achieve a Pareto improvement if breakdowns could be avoided. However, if

$$A^{\pi}(f,g) := \left\{(a,b)\in\mathcal{F}^2\,|(a,b)=(a'\pi f, b'\pi g) \text{ for some } (a',b')\in A(f,g)\right\},$$

then No Waste should hold after replacing $A(f,g)$ with $A^{\pi}(f,g)$. Though the analyst does not observe π directly, if No Waste holds for some $A^{\pi}(f,g)$, and if π is independent of the chosen endowments, then it should also hold for $A^{\pi}(\hat{f},\hat{g})$ for any act-pair $(\hat{f},\hat{g})$. The next axiom formalizes this relaxation of No Waste.

Axiom 5′ (Consistent Waste)
If, given π, the following holds for some $(f,g)\in\mathcal{B}$, then it holds for all $(f,g)\in\mathcal{F}^2$:
$\langle a,0\rangle \succsim_g^1 \langle f,1\rangle$ *and* $\langle b,0\rangle \succsim_f^2 \langle g,1\rangle$*, one of them strict, implies* $(a,b)\notin A^{\pi}(f,g)$.

Because Axiom 5′ is weaker than Axiom 5, to guarantee the representation, we need a very minor strengthening of Axiom 3 that rules out that the preference for sharing compared to autarky is always immeasurably slight.

Axiom 3′ (Voluntary Participation 2) $\langle f,1\rangle \succsim_g^i \langle f,0\rangle$ *for all* $(f,g)\in\mathcal{F}^2$ *and* $i\in\{1,2\}$, *and there exist* $f,g,a\in\mathcal{F}$ *and* $i\in\{1,2\}$ *such that* $\langle f,1\rangle \succsim_g^i \langle a,0\rangle \succ_g^i \langle f,0\rangle$.

Turning to the representation, we generalize the definitions of a model and how a model explains data, then state the representation and identification results allowing for frictions. Importantly, π can be identified from observable behavior.

[21] Formally, let $f^{-1}(x)$ denote the event on which $f\in\mathcal{F}$ has outcome x. Then $f\mathbf{p}f'$ and $g\mathbf{p}g'$ denote fixed acts with $\mu\left((f\mathbf{p}f')^{-1}(x)\cap(g\mathbf{p}g')^{-1}(y)\right)=p\mu\left(f^{-1}(x)\cap g^{-1}(y)\right)+(1-p)\mu\left(f'^{-1}(x)\cap g'^{-1}(y)\right)$. The existence of such acts relies on the infinite divisibility of the probability space $(\Omega,\mathcal{A},\mu)$.

Definition 4.1 *A* ***two-stage model of proportional risk sharing with frictions*** *is a triple* (η, α, π) *where* (η, α) *is two-stage model of proportional (frictionless) risk sharing (Definition 3.1) and* $(1-\pi) \in [0,1)$ *is the breakdown probability. A model with frictions* (η, α, π) ***explains*** $\left\{\succsim_g^i\right\}_{g\in\mathcal{F}}$ *for* $i \in \{1,2\}$*, if*

$$\langle f, o\rangle \succsim_g^i \langle f', o'\rangle \iff \pi o u\left(\alpha_{f,g}^i\left(f+g\right)\right)+(1-\pi o)u(f) \geq \pi o' u\left(\alpha_{f',g}^i\left(f'+g\right)\right)+(1-\pi o')u(f').$$

Theorem 4.1 (Representation & Identification with Frictions) *Preferences* $\left\{\succsim_g^i\right\}_{g\in\mathcal{F}}$ *for* $i \in \{1,2\}$ *satisfy Axioms 1, 2, 3′, 4, 5′, and 6 if and only if they can be explained by a two-stage model of proportional risk sharing with frictions* (η, α, π)*. Moreover, if* (η, α, π) *and* (η', α', π') *both explain preferences* $\left\{\succsim_g^i\right\}_{g\in\mathcal{F}}$ *for* $i \in \{1,2\}$*, then* $(\eta, \alpha, \pi) = (\eta', \alpha', \pi')$*.*

The model is silent on whether the second-stage bargaining takes place before or after the agents learn if they will indeed be able to reallocate or if their tie has broken down. In general, the interpretation of the (uniquely identified) sharing arrangement in terms of the model may depend on this timing. However, for many bargaining solutions this distinction is irrelevant, in which case we say the solution is *consequentialist*. For example, the NBS applied to bargaining before learning about breakdown is:

$$\begin{aligned}
(a^*, b^*) &= \arg\max_{(a,b)\in A(f,g)} \left(\pi u_1\left(a\right)+(1-\pi)u_1(f)-u_1\left(f\right)\right)\left(\pi u_2\left(b\right)+(1-\pi)u_2(g)-u_2\left(g\right)\right) \\
&= \arg\max_{(a,b)\in A(f,g)} \left(\pi u_1\left(a\right)-\pi u_1(f)\right)\left(\pi u_2\left(b\right)-\pi u_2\left(g\right)\right) \\
&= \arg\max_{(a,b)\in A(f,g)} \left(u_1\left(a\right)-u_1\left(f\right)\right)\left(u_2\left(b\right)-u_2\left(g\right)\right),
\end{aligned}$$

which coincides with the NBS applied after learning that breakdown has not occurred. Intuitively, since only utility surpluses matter for the NBS, the event where sharing fails is of no consequence, as then the surplus is zero (and the maximizer of a function T also maximizes πT). Similar analyses reveal that the KSS and EBS are also consequentialist.

4.1 Comparative Statics

In this section, we compare pairs of agents in terms of the strength of their ties as measured by π. To that end, consider agents $1A$, $2A$, $1B$, and $2B$. Let $(\eta, \alpha, \pi)^A$ and $(\eta, \alpha, \pi)^B$ denote the models of proportional risk sharing with frictions that describe the behavior of the agents in pairs A and B, respectively.

Definition 4.2 *Pair A has a* ***stronger preference to share*** *than does pair B if for all* $f, g, f', g' \in \mathcal{F} : \langle f, 1\rangle \succ_g^{1B} \langle f', 0\rangle$ *implies* $\langle f, 1\rangle \succ_g^{1A} \langle f', 0\rangle$*, and* $\langle g, 1\rangle \succ_f^{2B} \langle g', 0\rangle$ *implies* $\langle g, 1\rangle \succ_f^{2A} \langle g', 0\rangle$*, and the converse is not true.*

Theorem 4.2 (Preference for Sharing) *If pair A has a stronger preference to share than does pair B, then* $\pi^A > \pi^B$ *and* $\eta^A = \eta^B$*. Conversely, if* $\alpha^A = \alpha^B$ *and* $\eta^A = \eta^B$*, then* $\pi^A > \pi^B$ *implies that pair A has a stronger preference to share than pair B.*

In words, a stronger preference to share implies a stronger tie and identical risk preferences. The converse is also true if both pairs use the same sharing rule.

Theorem 4.2 observes that stronger ties mean that scope for sharing is more valuable. One might intuit that stronger ties should also induce agents to choose first-stage acts more efficiently: that is, to pay more attention to the value of the total pie, $u(f+g)$, and less to disagreement values, $u(f)$ and $u(g)$. This intuition holds under a mild restriction on the sharing rule, which we formulate in terms of our representation rather than preferences for ease of exposition.

Definition 4.3 *In the context of our representation,*

1. *Proportional sharing rule* α *is* ***sensible*** *if, for any act* g *held by Agent* $j \neq i$*,*

$$u(f) > u(f') \text{ and } u(f+g) > u(f'+g) \implies u(\alpha^i_{f,g}(f+g)) > u(\alpha^i_{f',g}(f'+g)).$$

2. *Given act* g*, act* f *is* ***more selfish*** *than* f' *if* $u(f) > u(f')$ *and* $u(f+g) \leq u(f'+g)$*.*

Under a sensible sharing rule, an agent that simultaneously increases the size of the pie and her own disagreement value must also achieve a better outcome from the sharing arrangement. Notice that the agent's *share* may decrease in this case, but the *utility* of her (possibly smaller) share of the larger pie must not decline under a sensible rule. Note that in our model, the NBS, KSS, and EBS (Section 3.5) all generate sensible rules.

Theorem 4.3 (Efficiency vs. Selfishness) *In the context of our representation, consider pairs A and B such* $\eta^A = \eta^B = \eta$*. A proportional sharing rule is sensible if and only if for any* g *and* $\pi^B < \pi^A$ *:*

$$\langle f', 1\rangle \succsim_g^{1A} \langle f, 1\rangle \text{ and } \langle f, 1\rangle \succ_g^{1B} \langle f', 1\rangle \implies f \text{ is more selfish than } f', \text{ given } g.$$

That is, under a sensible rule, when there is a preference disagreement between Agents $1A$ and $1B$, it is indeed because the agent with the stronger (weaker) tie prefers the act with the larger sharing surplus (disagreement value).

5 Example: Equilibrium Analysis of First-stage

Our representation theorem applies to best-response data. This primitive most naturally aligns with axiomatic decision theories for individual choice, and in many contexts may best correspond to available data. Alternatively, equilibrium analysis requires the analyst to be explicit about both the first-stage game-form (e.g., simultaneous vs. sequential) and also the solution concept (e.g., Nash equilibrium, rationalizability, correlated equilibrium). Moreover, multiple equilibria may satisfy the chosen solution concept—a well-known problem for empirical application. In addition to its own merits, best-response analysis is useful for virtually every solution concept and game form, while avoiding the potential multiplicity problem.

Nevertheless, equilibrium analyses of particular first-stage game-forms should yield additional insights. We now employ our representation to analyze Nash equilibria of a simultaneous-move first stage. The example consists of the model from our representation in the following sharing environment. Two neighboring farmers (agents) each control land of measure 2. There are two equally likely states of the world: R(ainy) or D(ry). There are also two available crops, r and d, each one doing better than the other in its mnemonicly corresponding state of the world. Half of Agent 1's land is already planted with r, half of Agent 2's land is already planted with d.

Both farmers have log utility (i.e., $\eta = 1$) and need to decide how much to specialize versus hedge when deciding how much of their remaining land to dedicate to one crop or the other.[22] The benefit of specialization is due to increasing returns in each crop. Specifically, let r_i (respectively, d_i) be the amount of Agent i's land that is planted with crop r (d). Then, i's production in states R and D are,

$$f(R|r_i, d_i) = r_i^z \quad \text{and} \quad f(D|r_i, d_i) = d_i^z,$$

where $z > 1$ captures the strength of the reward to specialization.[23]

The farmers simultaneously choose their crop allocations and then bargain over risk-sharing. Their tie strength is $\pi > 0$ (probability of breakdown $1 - \pi$). Without loss, pose the game in terms of the degree of specialization each farmer chooses: $s_1 := (r_1 - 1) \in [0, 1]$

[22] The assumption that half their land is already planted allows us to focus on the degree of specialization, and bypass the coordination issue of which farmer should specialize in which crop.

[23] That $f(R|0, 2) = f(D|2, 0) = 0$ violates the specification that $f(\omega) \in \mathbb{R}_{++}$ from Section 2. However, it poses no issues for the analysis that follows and is allowed only for the sake of parsimony. Moreover, all claims regarding efficiency and/or equilibrium are the limits of the corresponding claims for the specification $f_\gamma(R|r_i, d_i) = r_i^z + \gamma d_i^z > 0$ and $f_\gamma(D|r_i, d_i) = \gamma r_i^z + d_i^z > 0$ as $\gamma > 0$ limits to zero.

and $s_2 := (d_2 - 1) \in [0, 1]$.

Fact 5.1 *a) The autarky solution is complete diversification: $s_1 = s_2 = 0$, so each farm is evenly split between the two crops. b) With risk sharing, the efficient individual specialization level is symmetric, denoted s^e, increasing in z and in π, and limits to π as z grows arbitrarily large.*

Suppose now that the sharing rule corresponds to the EBS (Section 3.5). Recalling Lemma 3.1, we have $\alpha^1_{f,g} = \frac{ce(f)}{ce(f)+ce(g)}$, where $ce(f)$ denotes the certainty equivalent of act f. The key tension is clear. Each farmer increases her disagreement value, captured by $ce(f)$, by individually diversifying, which increases the share of the pie she receives conditional on sharing. However, the total pie to share, $u(f + g)$ is largest when the farmers specialize (up to s^e).

The induced first-stage game is symmetric with increasing best response function. Figure 1(a) depicts the farmers' best response function for three different rewards to specialization levels, z.[24] All Nash equilibria are symmetric (lie on the 45°-line), but equilibrium multiplicity is possible. For low z (dashed curve), the reward to specialization is insufficient to overcome the incentive to self-insure, and the unique equilibrium is the autarky solution, $s_1 = s_2 = 0$. An intermediate reward to specialization (solid), gives rise to a coordination issue: the autarky solution is still an equilibrium, but now so too is a profile with a positive degree of specialization $s_1 = s_2 = s^* > 0$. Finally, for high z (dotted), the reward to specialize is large enough that each farmer wants to specialize even if the other is not, and there is a unique equilibrium.[25] Figure 1(b), illustrates the set of equilibria as it varies with the reward to specialization for two different tie strengths.

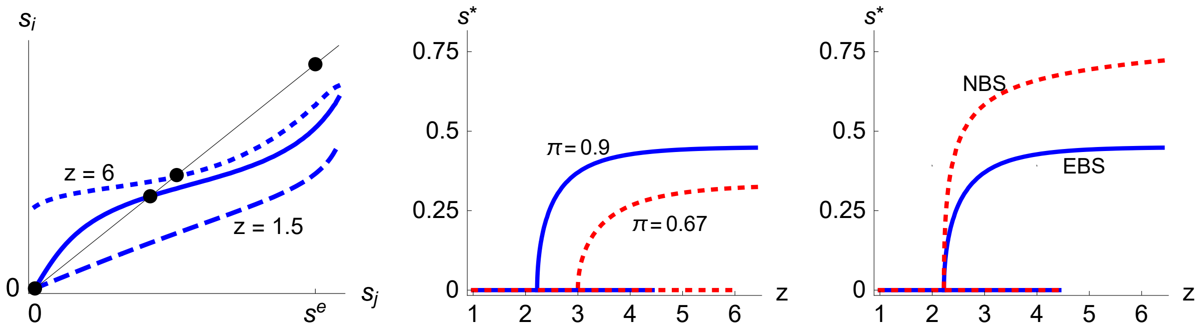
Fact 5.2 *All equilibria are symmetric and at most one involves positive specialization, $s^* > 0$.*

- *The autarky solution, $s_1 = s_2 = 0$, is an equilibrium if and only if $z < \frac{4}{\pi}$.*
- *An equilibrium with positive specialization exists if and only if $z > \frac{2}{\pi}$.*
- *All equilibria involve inefficiently low (high) specialization (self insurance): $s^* < s^e$.*

From Fact 5.2, we can see that increasing the chance of breakdown (decreasing π) increases the parameter space in which the autarky solution is an equilibrium and decreases the parameter space in which an equilibrium with positive specialization exists. Figure

[24]Here the efficient specialization s^e varies so little with z that it undetectable in the figure ($s^e \in (0.87, 0.9)$ for $z \geq 1.5$).

[25]For any z, if the equilibrium is unique, then it is the also the unique rationalizable outcome.



(a) Best Responses ($\pi = 0.9$) (b) Change in tie-strength (c) Change to NBS ($\pi = 0.9$)

Figure 1: Equilibrium behavior behavior in the example.

1(b) illustrates the effect that the tie strength has on the equilibrium actions: both the largest and smallest equilibria increase with π. Underlying this observation is that (a) because the EBS generates a sensible sharing rule, increasing π leads to a more efficient best response to any opponent choice of act (Theorem 4.3), and (b) the best response functions in this game are increasing and symmetric.

It is also worth observing that if instead of simultaneous moves, the first-stage choice of acts were sequential with, say, Agent 2 being able to condition her choice on Agent 1's, then the efficiency of total production would be strictly higher and both agents strictly better off compared to the most efficient equilibrium of the simultaneous-move specification. That is, sequentiality does more than solve the coordination problem in the case of multiplicity, but allows the first-mover to select an even higher level of specialization, confident that the second-mover will follow suit since best-response functions are increasing. In fact, the second-mover earns a higher payoff than the first.

5.1 Varying the Bargaining Solution

Suppose now that instead of the EBS, the farmers shared according to the NBS. In this case, the analytic form of α is intractable. Perhaps surprisingly though, Fact 5.2 remains valid verbatim. However, whenever the positive-specialization equilibrium exists, it involves more efficient specialization under the NBS than under the EBS, as shown in Figure 1(c). That is, the NBS rewards efficient production more than the EBS in this example.

The force underlying this ranking is more general as illustrated in the proposition below. In our context, all three prominent bargaining solutions depend only on $u(f+g)$, $u(f)$, and $u(g)$, which we can think of as efficiency and disagreement values. Because all three solutions generate sensible sharing rules, at any best response the agent has optimally traded off increasing $u(f+g)$ versus increasing $u(f)$. Further, because the space of acts is continuous and both u and α are differentiable (under each of these bargaining solutions), we can consider the local effect from changing f slightly.

Proposition 5.1 *Fix $\pi > 0$ and $f \neq g$ such that $u(f) = u(g)$. Then each agent's local incentive to increase efficiency, $u(f+g)$, at the expense of her disagreement value, $u(f)$ or $u(g)$, is:*

- *strictly greater under both the NBS and the KSS than under the EBS,*
- *strictly greater under the KSS than under the NBS if and only if $\eta < 2$.*[26]

The example and proposition point out an important facet of the two-stage environment. Even though all three of these bargaining solutions result in Pareto efficient reallocation *conditional* on (f, g) (i.e., in the second stage), they generally provide different incentives for, and engender different equilibrium levels of, efficiency in the choice of first-stage acts. One may then ask whether the strength of the incentives provided for this "global" efficiency should be important for the normative evaluation of bargaining solutions, and if any yet unheralded solutions would provide a worthwhile improvement on this dimension.

6 Extensions and Applications

For applied context, this document is framed in terms of informal risk sharing between households, which is particularly important in developing societies. The effects of particular sharing arrangements on the choice of risky actions (e.g., the take-up of novel crops) have received little attention, but may be relevant when evaluating policies. For example, should new crops be made available to all farmers at the same time or only to a subset initially? How would an improvement in the provision of formal insurance affect the agents' production decisions? What are the anticipated effects of increased geographical mobility? Existing tests of efficient risk sharing cannot fully address these questions because they take endowments as exogenous.

As previously mentioned, another instance of informal risk sharing is *within* households, where it is considered one important benefit of marriage, especially in developing societies where individual income is highly volatile (Mazzocco, 2004; Browning, Chiappori, and Weiss, 2014). The "collective model of the household" assumes that decisions

[26] If $f = g$, so there is no scope for risk sharing, then each agent's local incentive to increase efficiency at the expense of her disagreement value is identical under all three bargaining solutions. Notice that $f = g$ is nongeneric among act-pairs with $u(f) = u(g)$.

are made in a Pareto efficient manner, where Pareto weights depend on disagreement values. Indeed it is well documented that, for a given total household income, the relative incomes of spouses affect the pattern of household expenditures and other household decisions (for developing economies see Thomas, 1994; Hoddinott and Haddad, 1995; Duflo, 2003).[27] Baland and Ziparo (2018) pointed out that premarital decisions (e.g, regarding education, occupation, and/or the timing of marriage), affect one's disagreement value and hence the Pareto weights within the collective model. Forward-looking agents should therefore account for these effects. Our model captures this feature, as premarital choices of risky acts determine both the disagreement allocation and individual contributions to household income. Understanding the effect of risk sharing on these choices should be relevant for a number of pressing policy questions, for instance about the effects of policies governing divorce and contraception on female education and participation in the labor force, on reproductive choices, on gender roles, and on the efficiency of marriage as a risk-sharing arrangement.

For clarity of exposition, we focused on bilateral risk sharing. As mentioned at the outset, extending our model and its foundations to risk-sharing groups/syndicates (Wilson, 1968) is conceptually straightforward. In that setting, there are no relationship-specific constraints on how risk can be reallocated. An alternative modeling approach for multi-agent risk-sharing arrangements is to assume bilateral bargaining and transfers along the ties of a network (e.g., Ambrus et al., 2014). In addition to the (unobservable) sharing rule, the analyst must then also determine the underlying network. One approach is the game-theoretical study of network formation (Bramoullé and Kranton, 2007; Bloch, Genicot, and Ray, 2008; Ambrus and Elliott, 2020). In empirical work, often the network is assumed based on non-behavioral data such as family ties, geographic or demographic proximity, or an otherwise documented network of social ties (e.g., Fafchamps and Lund, 2003; De Weerdt and Dercon, 2006). If the (strengths of) network ties that are relevant for risk sharing differ from the assumed social network, this presumption will lead to model misspecification.[28] In contrast, our results allow the identification of sharing frictions, which are the appropriate measure of tie strength in our model, directly from the observable choice of actions. Of course, an extension of our model to sharing on a network would need to resolve how bilateral sharing arrangements between different pairs are sequenced and interact. We leave such an extension as a topic for future research.

[27] Here too, frictions (e.g., due to lack of commitment) appear to be important (Mazzocco, 2007).

[28] For instance, in the context of rural India, Munshi and Rosenzweig (2016) argue that the social network based on caste is more relevant for risk sharing than village membership, which was often used to define social ties in previous studies.

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A Appendix

A.1 Notation and Preliminaries

For a finite collection of acts $\mathcal{F}' \subset \mathcal{F}$, let $\mathcal{P}(\mathcal{F}')$ be the coarsest partition of Ω such that all $f \in \mathcal{F}'$ are measurable: that is, $f(\omega) = f(\omega')$ for all $\omega, \omega' \in S \in \mathcal{P}(\mathcal{F}')$ and $f \in \mathcal{F}'$. Note that $\mathcal{P}(\mathcal{F}')$ is well defined because all acts $f \in \mathcal{F}$ are simple. Further, $\mathcal{P}(\mathcal{F}') \subset \mathcal{A}$ since all $f \in \mathcal{F}$ are measurable in $\mathcal{A}$, which is a σ-algebra. For $f \in \mathcal{F}'$ and $S \in \mathcal{P}(\mathcal{F}')$ we use $f(S)$ to denote the (constant) value of f on S.

For any particular proportional sharing rule, since $\alpha^1_{f,g} + \alpha^2_{f,g} = 1$, we drop the superscript and let $\alpha^1_{f,g} = \alpha_{f,g}$ and $\alpha^2_{f,g} = 1 - \alpha_{f,g}$.

Lemma A.1 *For expected-utility maximizers with differentiable, concave consumption utility functions u_1, u_2, allocation (a, b) is Pareto efficient in $A(f, g)$ according to u_1, u_2 if and only if, $a + b = f + g$ and for all $S, T \in \mathcal{P}(\{f, g, a, b\})$,*

$$\frac{u_1'(a(S))}{u_2'(b(S))} = \frac{u_1'(a(T))}{u_2'(b(T))}. \tag{4}$$

Proof. See Back (2016, ch. 4). ■

Lemma A.2 *Differentiable u is CRRA if and only if for all $c, c' \in \mathbb{R}_{++}$ and $\alpha \in (0, 1)$,*

$$\frac{u'(\alpha c)}{u'(c)} = \frac{u'(\alpha c')}{u'(c')}.$$

Proof. Suppose u is CRRA. Then $u'(c) = k \cdot c^{-\eta}$, for some $k > 0$. Thus, for $\alpha \in (0, 1)$, we have $\frac{u'(\alpha c)}{u'(c)} = \alpha^{-\eta} = \frac{u'(\alpha c')}{u'(c')}$. Now suppose u is not CRRA. Then an expected-utility maximizer with consumption utility function u does not exhibit constant relative risk aversion. So there exist $\hat{c}, l, h, \alpha \in \mathbb{R}_{++}$ with $l < \hat{c}$ and $p \in (0, 1)$ such that $pu(\hat{c} + h) + (1 - p)u(\hat{c} - l) \leq u(\hat{c})$, but $pu(\alpha(\hat{c} + h)) + (1 - p)u(\alpha(\hat{c} - l)) > u(\alpha\hat{c})$ (Nielsen, 2005). This implies

$$\frac{\frac{u(\alpha(\hat{c}))-u(\alpha(\hat{c}-l))}{l}}{\frac{u(\hat{c})-u(\hat{c}-l)}{l}} < \frac{\frac{u(\alpha(\hat{c}+h))-u(\alpha\hat{c})}{h}}{\frac{u(\hat{c}+h)-u(\hat{c})}{h}}.$$

Since u is differentiable, there exist $c \in (\hat{c} - l, \hat{c})$ and $c' \in (\hat{c}, \hat{c} + h)$ for which

$$\frac{u'(\alpha c)}{u'(c)} < \frac{u'(\alpha c')}{u'(c')},$$

establishing the result. ■

Lemma A.3 *If u is CRRA from (2), $f \in \mathcal{F}$, and $\alpha > 0$, then $u(\alpha f) = \alpha^{1-\eta} u(f) + u(\alpha)$.*

Proof. Using (2), we have

$$\begin{aligned}
\alpha^{1-\eta} u(f) + u(\alpha) &= \alpha^{1-\eta} \sum_{c \in \chi(f)} \left(\mu\left(f^{-1}(c)\right) \frac{c^{1-\eta} - 1}{1-\eta} \right) + \frac{\alpha^{1-\eta} - 1}{1-\eta} \\
&= \sum_{c \in \chi(f)} \left(\mu\left(f^{-1}(c)\right) \left(\frac{\alpha^{1-\eta} \cdot c^{1-\eta}}{1-\eta} - \frac{\alpha^{1-\eta}}{1-\eta} \right) \right) + \left(\frac{\alpha^{1-\eta}}{1-\eta} - \frac{1}{1-\eta} \right) \\
&= \sum_{c \in \chi(f)} \mu\left(f^{-1}(c)\right) \frac{(\alpha c)^{1-\eta} - 1}{1-\eta} = u(\alpha f).
\end{aligned}$$

■

A.2 Proofs

Proof Proposition 3.1.

For $(i) \Rightarrow (ii)$: For any $f, g \in \mathcal{F}$ and $\alpha \in (0,1)$, under the proportional reallocation $(a,b) = (\alpha(f+g), (1-\alpha)(f+g))$, we have: $a + b = f + g$ and $a(S)/b(S) = \alpha/(1-\alpha)$ for all $S \in \mathcal{P}(f,g,a,b)$. Hence, for all $S, T \in \mathcal{P}(f,g,a,b)$ and $\eta \geq 0$,

$$\left(\frac{a(S)}{b(S)} \right)^{\eta} = \left(\frac{a(T)}{b(T)} \right)^{\eta}. \tag{5}$$

If u_1, u_2 are CRRA with $\eta_1 = \eta_2 = \eta$, then, using the functional form in (2), we have (5) is equivalent to (4). Thus, Lemma A.1 establishes that (a,b) is Pareto efficient in $A(f,g)$.

For $(i) \Rightarrow (iii)$: Given $f, g, a, b \in \mathcal{F}$ and $S \in \mathcal{P}(f,g,a,b)$, assign $\alpha = \frac{a(S)}{f(S)+g(S)}$ and hence $1 - \alpha = \frac{b(S)}{f(S)+g(S)}$. Suppose (a,b) is Pareto efficient in $A(f,g)$ and, for the purpose of contradiction, that $a \neq \alpha(f+g)$. Then there exists $T \in \mathcal{P}(f,g,a,b)$ such that $\frac{a(T)}{f(T)+g(T)} \neq \alpha$. Then $\frac{a(S)}{b(S)} \neq \frac{a(T)}{b(T)}$, and hence for any $\eta > 0$,

$$\left(\frac{a(S)}{b(S)} \right)^{\eta} \neq \left(\frac{a(T)}{b(T)} \right)^{\eta}. \tag{6}$$

If u_1, u_2 are CRRA with $\eta_1 = \eta_2 = \eta$, then, using the functional form in (2), we have (6) implies a failure of (4). Thus, Lemma A.1 establishes that (a,b) is not Pareto efficient in $A(f,g)$, a contradiction.

For $(ii) \Rightarrow (i)$: Let $\succsim_{u_i}$ be the preference over allocations induced by u_i. We first argue that (ii) implies that $\succsim_{u_1} = \succsim_{u_2}$. To do so, for two events $S, T \in \mathcal{A}$ with $\mu(S) = \mu(T) = \frac{1}{2}$, and $c, c' \in \mathbb{R}_{++}$, let $\langle c; c' \rangle$ denote the act that delivers c on event S and

c' on T. If $\succsim_{u_1} \neq \succsim_{u_2}$, then there exist $c, c', \delta, \delta' \in \mathbb{R}_{++}$ with $\delta < c$ and $\delta' < c'$ such that $u_1(\langle c+\delta; c'-\delta' \rangle) > u_1(\langle c; c' \rangle)$ and $u_2(\langle c-\delta; c'+\delta' \rangle) > u_2(\langle c; c' \rangle)$. Then for $f+g = \langle 2c; 2c' \rangle$, the proportional reallocation $(\frac{1}{2}(f+g), \frac{1}{2}(f+g)) = (\langle c; c' \rangle, \langle c; c' \rangle)$ is Pareto dominated by $(\langle c+\delta; c'-\delta \rangle, \langle c-\delta; c'+\delta' \rangle) \in A(f,g)$. This contradicts all proportional reallocations being Pareto efficient; hence $\succsim_{u_1} = \succsim_{u_2}$, and it is without loss to specify a common $u = u_1 = u_2$.

Next we show that the common u is (weakly) concave. For the purpose of contradiction, suppose not. Then there exists $c, \delta \in \mathbb{R}_{++}$ with $\delta < c$, such that $u(\langle c+\delta; c-\delta \rangle) > u(\langle c; c \rangle)$. Because $\mu(S) = \mu(T) = \frac{1}{2}$, it must also be that $u(\langle c-\delta; c+\delta \rangle) = u(\langle c+\delta; c-\delta \rangle) > u(\langle c; c \rangle)$. Then for $f+g = \langle 2c; 2c \rangle$, the proportional reallocation $(\frac{1}{2}(f+g), \frac{1}{2}(f+g)) = (\langle c; c \rangle, \langle c; c \rangle)$ is Pareto dominated by $(\langle c+\delta; c-\delta \rangle, \langle c-\delta; c+\delta \rangle) \in A(f,g)$. This contradicts all proportional reallocations being Pareto efficient; hence u is concave on $\mathbb{R}_{++}$. As a consequence, left-hand and right-hand derivatives, u'_- and u'_+, exist everywhere on $\mathbb{R}_{++}$ (Rockafellar, 1970, Theorem 23.1). Furthermore, the derivative u'_i exists (u'_- and u'_+ coincide) almost everywhere (Rockafellar, 1970, Theorem 25).

We next show that u is differentiable everywhere. For the purpose of contradiction, suppose that there exists $c_1 \in \mathbb{R}_{++}$ with $u'_-(c_1) \neq u'_+(c_1)$. Now choose $c_2, c_3, c_4 > 0$ such that u is differentiable at each and

$$\frac{c_1}{c_2} = \rho = \frac{c_3}{c_4},$$

which is possible because u is differentiable almost everywhere. Then $\kappa := \frac{u'(c_3)}{u'(c_4)}$ is well defined. Next, for $\xi \in \mathbb{R}_{++}$, let $MRS(\xi) := \frac{u'(\xi c_1)}{u'(\xi c_2)}$ when these derivatives are well defined, which is (at least) almost everywhere.

Now define $\alpha := \frac{\xi c_1}{c_3 + \xi c_1}$, $c := c_3 + \xi c_1$, and $c' := \frac{c_3 + \xi c_1}{\rho}$, and verify that $\alpha c = \xi c_1$, $\alpha c' = \xi c_2$, $(1-\alpha)c = c_3$, and $(1-\alpha)c' = c_4$. It is straightforward to extend Lemma A.1 to show that Pareto efficiency implies (4) whenever the four derivatives therein are well-defined. Therefore, wherever defined, $MRS(\xi) = \kappa$. Let $\xi_n^- \nearrow 1$ and $\xi_n^+ \searrow 1$ be sequences such that $MRS(\xi_n^-)$ and $MRS(\xi_n^+)$ are well defined for all n. Then, $MRS(\xi_n^-) = MRS(\xi_n^+) = \kappa$ for all n. It follows that

$$\lim_{n\to\infty} \frac{u'(\xi_n^- c_1)}{u'(\xi_n^- c_2)} = \lim_{n\to\infty} \frac{u'(\xi_n^+ c_1)}{u'(\xi_n^+ c_2)} = \kappa.$$

At the same time, because u is concave and differentiable at c_2, from Theorem 24.1 in

Rockafellar (1970) we have

$$\lim_{n\to\infty} u'(\xi_n^- c_2) = \lim_{n\to\infty} u'(\xi_n^+ c_2) = u'(c_2).$$

It follows that

$$\lim_{n\to\infty} u'(\xi_n^- c_1) = \kappa u'(c_2) = \lim_{n\to\infty} u'(\xi_n^+ c_1)$$

and again by the same theorem,

$$u'_-(c_1) = \lim_{n\to\infty} u'(\xi_n^- c_1) \quad \text{and} \quad \lim_{n\to\infty} u'(\xi_n^+ c_1) = u'_+(c_1),$$

which contradicts the supposition that $u'_-(c_1) \neq u'_+(c_1)$. Hence u is differentiable everywhere as claimed.

Finally, suppose, contrary to (i), that u_1 is not CRRA. Then by Lemma A.2, there exist $c, c', \alpha, \alpha' \in \mathbb{R}_{++}$ such that

$$\frac{u_1'(\alpha c)}{u_1'(\alpha c')} \neq \frac{u_1'(\alpha' c)}{u_1'(\alpha' c')}.$$

Let τ and $\widetilde{\alpha}$ be such that $(1-\widetilde{\alpha})\tau = 1-\alpha$ and $\widetilde{\alpha}\tau = \alpha'$. Then,

$$\frac{u_1'(\widetilde{\alpha}\tau c)}{u_1'(\widetilde{\alpha}\tau c')} = \frac{u_1'(\alpha' c)}{u_1'(\alpha' c')} \neq \frac{u_1'(\alpha c)}{u_1'(\alpha c')} = \frac{u_2'((1-\alpha)c)}{u_2'((1-\alpha)c')} = \frac{u_2'((1-\widetilde{\alpha})\tau c)}{u_2'((1-\widetilde{\alpha})\tau c')}.$$

Hence, by Lemma A.1, for any $f', g' \in \mathcal{F}$ with $f' + g' = \tau(f+g)$, the reallocation $(\widetilde{\alpha}(f+g), (1-\widetilde{\alpha})(f+g))$ is not Pareto efficient in $A(f', g')$. This contradicts all proportional reallocations being Pareto efficient; hence $u_1 = u_2$ is a CRRA utility (i.e., $\eta_1 = \eta_2$). ■

Proof of Lemma 3.1. If $u_i = u = \ln$ for both agents, then u is CRRA from (2) with $\eta = 1$. Using Lemma A.3, for any $\alpha \in (0,1)$, we have $u(\alpha(f+g)) = u(f+g) + \ln(\alpha)$. Given (f,g) and using Proposition 3.1, the EBS then implies the proportional reallocation satisfying $u(f+g) + \ln(\alpha) - u(f) = u(f+g) + \ln(1-\alpha) - u(g)$. Substituting $\ln(ce(f))$ for $u(f)$ and $\ln(ce(g))$ for $u(g)$, we find the unique solution: $\alpha = ce(f)/(ce(f)+ce(g))$. ■

Proof of Theorem 3.1.

<u>Model $\Rightarrow$ Axioms</u>: Let the model (η, α) explain $\{\succsim_g^i\}_{g\in\mathcal{F}}$. For $o = 0$, the model implies a monotonic expected utility representation that is independent of g for each agent. Axioms 1 and 2 follow immediately. Axiom 3 follows because $u_1\big(\alpha_{f,g}(f+g)\big) \geq u_1(f)$ and $u_2((1-\alpha_{f,g})(f+g)) \geq u_2(g)$ for the proportional sharing rule α (Definition 3.1), with at least one inequality strict whenever $g \neq \beta f$ for any $\beta > 0$ since u_i are CRRA with the

same η (Proposition 3.1). Moreover, Proposition 3.1 implies that proportional sharing of $f+g$ is Pareto efficient, and hence Axioms 5 and 6 follow. It further implies that, when sharing f and βf for $\beta > 0$, the only proportional allocation that respects disagreement values is to maintain the original allocation, which implies Axiom 4.

<u>Axioms $\Rightarrow$ Model</u>: The result is established in several steps. Throughout, assume that the primitive, $\left\{\succsim_g^i\right\}_{g\in\mathcal{F}}$ for $i \in \{1,2\}$, satisfies Axioms 1-6.

Step 1 *For any $f, g \in \mathcal{F}$ and $(a,b) \in PS(f,g)$, there exists $\alpha \in (0,1)$ such that $\langle a, 0\rangle \sim_g^1 \langle \alpha(f+g), 0\rangle$ and $\langle b, 0\rangle \sim_f^2 \langle (1-\alpha)(f+g), 0\rangle$.*

$\triangleright$ Fix $f, g \in \mathcal{F}$ and $(a,b) \in \mathcal{A}(f,g)$. Define

$$\begin{aligned} \Theta_1 &:= \left\{\alpha \in (0,1) \,\middle|\, \langle a, 0\rangle \succ_b^1 \langle \alpha(f+g), 0\rangle \right\} \\ \Theta_2 &:= \left\{\alpha \in (0,1) \,\middle|\, \langle b, 0\rangle \succ_a^2 \langle (1-\alpha)(f+g), 0\rangle \right\}, \end{aligned}$$

and, for the purpose of contradiction, suppose that $\Theta_1 \cup \Theta_2 = (0,1)$. Because $a < f+g$, by monotonicity of autarky preferences (Axiom 2), there exists $\alpha < 1$ large enough such that $\langle \alpha(f+g), 0\rangle \succsim_b^1 \langle a, 0\rangle$ and hence $\Theta_2 \neq \emptyset$. By a symmetric argument, $\Theta_1 \neq \emptyset$.

Continuity and monotonicity of autarky preferences in α (which follow immediately from Axiom 2) then imply $\Theta_1 = (0, \overline{\alpha}_{f,g})$ and $\Theta_2 = (\underline{\alpha}_{f,g}, 1)$. Hence, for $\Theta_1 \cup \Theta_2 = (0,1)$, it must be that $\underline{\alpha}_{f,g} < \overline{\alpha}_{f,g}$ and there exists $\alpha^* \in \Theta_1 \cap \Theta_2$, meaning

$$\langle a, 0\rangle \succ_b^1 \langle \alpha^*(f+g), 0\rangle \text{ and } \langle b, 0\rangle \succ_a^2 \langle (1-\alpha^*)(f+g), 0\rangle.$$

By Axiom 2, $\succsim_h^i$ is independent of $h \in \mathcal{F}$ in autarky for $i \in \{1,2\}$. Therefore,

$$\begin{aligned} \langle a, 0\rangle &\succ^1_{(1-\alpha^*)(f+g)} \langle \alpha^*(f+g), 0\rangle \\ \langle b, 0\rangle &\succ^2_{\alpha^*(f+g)} \langle (1-\alpha^*)(f+g), 0\rangle. \end{aligned}$$

Separately, by Axiom 4,

$$\begin{aligned} \langle \alpha^*(f+g), 0\rangle &\succsim^1_{(1-\alpha^*)(f+g)} \langle \alpha^*(f+g), 1\rangle \\ \langle (1-\alpha^*)(f+g), 0\rangle &\succsim^2_{\alpha^*(f+g)} \langle (1-\alpha^*)(f+g), 1\rangle. \end{aligned}$$

Then, by transitivity of $\succsim_h^i$ for $i \in \{1,2\}$ and any $h \in \mathcal{F}$ (Axiom 1),

$$\begin{aligned} \langle a, 0\rangle &\succ^1_{(1-\alpha^*)(f+g)} (\alpha^*(f+g), 1\rangle \\ \langle b, 0\rangle &\succ^2_{\alpha^*(f+g)} \langle (1-\alpha^*)(f+g), 1\rangle. \end{aligned}$$

By Axiom 5, $(a,b) \notin A(\alpha^*(f+g),(1-\alpha^*)(f+g))$. However, $A(\alpha^*(f+g),(1-\alpha^*)(f+g)) = A(f,g)$, producing a contradiction. Hence for all $(a,b) \in A(f,g)$ there is $\alpha \in (0,1)$ such that

$$\langle \alpha (f+g), 0 \rangle \succsim_b^1 \langle a, 0 \rangle \text{ and } \langle (1-\alpha)(f+g), 0 \rangle \succsim_a^2 \langle b, 0 \rangle.$$

Finally, by the definition of $PS(f,g)$ and because $\succsim_h^i$ is independent of $h \in \mathcal{F}$ in autarky (Axiom 2), we then have that for any $(a,b) \in PS(f,g)$ there is $\alpha \in (0,1)$ such that

$$\langle a, 0 \rangle \sim_g^1 \langle \alpha (f+g), 0 \rangle \text{ and } \langle b, 0 \rangle \sim_f^2 \langle (1-\alpha)(f+g), 0 \rangle. \quad \lhd$$

Step 2 *For all $\alpha \in (0,1)$, we have $(\alpha (f+g), (1-\alpha)(f+g)) \in PS(f,g)$.*
$\rhd$ Fix $f, g \in \mathcal{F}$. For the purpose of contradiction, suppose that there exists $\alpha \in (0,1)$ and $(a,b) \in A(f,g)$ that Pareto dominates $(\alpha (f+g), (1-\alpha)(f+g))$ according to autarky preferences. By Step 1, there exists $\hat{\alpha} \neq \alpha$ such that $\langle \hat{\alpha}(f+g), 0 \rangle \sim_g^1 \langle a, 0 \rangle$ and $\langle (1-\hat{\alpha})(f+g), 0 \rangle \sim_f^2 \langle b, 0 \rangle$. However, monotonicity of autarky preferences in α (which follow from Axiom 2) implies that $(\hat{\alpha}(f+g), (1-\hat{\alpha})(f+g))$ does not Pareto dominate $(\alpha (f+g), (1-\alpha)(f+g))$ according to autarky preferences, a contradiction. $\lhd$

Step 3 *For all $f, g \in \mathcal{F}$, there exists $\alpha^* \in (0,1)$ such that:*
$$\langle \alpha^* (f+g), 0 \rangle \sim_g^1 \langle f, 1 \rangle \textit{ and } \langle (1-\alpha^*)(f+g), 0 \rangle \sim_f^2 (g, 1).$$
$\rhd$ Fix $f, g \in \mathcal{F}$, and define

$$\begin{aligned} \Theta_3 &:= \left\{ \alpha \in (0,1) \,\middle|\, \langle \alpha (f+g), 0 \rangle \succsim_g^1 \langle f, 1 \rangle \right\} \\ \Theta_4 &:= \left\{ \alpha \in (0,1) \,\middle|\, \langle (1-\alpha)(f+g), 0 \rangle \succsim_f^2 \langle g, 1 \rangle \right\}. \end{aligned}$$

Step 2 and Axiom 6 together imply that $\Theta_3 \cup \Theta_4 = (0,1)$. For the purpose of contradiction, suppose that $\langle g, 1 \rangle \succ_f^2 \langle (1-\alpha)(f+g), 0 \rangle$ for all $\alpha \in (0,1)$ (i.e., $\Theta_4 = \emptyset$). But then, for α small enough we have the contradiction:

$$\langle \alpha(f+g), 0 \rangle \succsim_g^1 \langle f, 1 \rangle \succsim_g^1 \langle f, 0 \rangle \succ_f^2 \langle \alpha(f+g), 0 \rangle,$$

where the first ranking follows from Step 2 and Axiom 6, the second is by Axiom 3, and the third is from Axiom 2. Hence, $\Theta_4 \neq \emptyset$, and by symmetric argument $\Theta_3 \neq \emptyset$. Continuity and monotonicity of autarky preferences in α (which follow immediately from Axiom 2) then imply $\Theta_3 = [\underline{\alpha}, 1)$ and $\Theta_2 = (0, \overline{\alpha}]$. Hence, for $\Theta_3 \cup \Theta_4 = (0,1)$, it must be that $\underline{\alpha} \leq \overline{\alpha}$ and there exists $\alpha^* \in \Theta_3 \cap \Theta_4$, meaning

$$\langle \alpha^* (f+g), 0 \rangle \succsim_g^1 \langle f, 1 \rangle \text{ and } \langle (1-\alpha^*)(f+g), 0 \rangle \succsim_f^2 \langle g, 1 \rangle.$$

Finally, since $(\alpha^*(f+g),(1-\alpha^*)(f+g)) \in A(f,g)$, by Axiom 5

$$\langle \alpha^*(f+g), 0\rangle \sim_g^1 \langle f, 1\rangle \text{ and } \langle (1-\alpha^*)\,(f+g), 0\rangle \sim_f^2 \langle g, 1\rangle. \quad \triangleleft$$

Step 4 *For $i=1,2$ and any $g \in \mathcal{F}$, there exists an expected-utility representation (with consumption utility function u_i) for $\succsim_g^i$ when $o=0$. Moreover, u_i is independent of g.*

$\triangleright$ Immediate from Axiom 2. $\triangleleft$

Step 5 *There exists a proportional sharing rule, with shares $\alpha_{f,g}^i$, such that for any $f, f', g \in \mathcal{F}$ and $i \in \{1,2\}$ we have:*

$$\langle f, o\rangle \succsim_g^i \langle f', o'\rangle \iff ou\left(\alpha_{f,g}^i\,(f+g)\right) + (1-o)u(f) \geq o'u\left(\alpha_{f',g}^i\,(f'+g)\right) + (1-o')u(f').$$

$\triangleright$ Implied by Steps 3 and 4. $\triangleleft$

Step 6 *There exists a two-stage model of proportional risk sharing (η,α), that explains preferences $\left\{\succsim_g^i\right\}_{g\in\mathcal{F}}$ for $i \in \{1,2\}$.*

$\triangleright$ Given Step 5, all that remains to show is that u_1, u_2 must be CRRA with $\eta_1 = \eta_2$ and that $u_i(\alpha_{f,g}^i(f+g)) \geq u_i(f)$ for all $f, g \in \mathcal{F}$ and $i \in \{1,2\}$. Given Step 5, the latter is implied by Axiom 3. For the former, Steps 2 and 5 imply that any proportional sharing rule is Pareto efficient for expected-utility maximizers with consumption utility functions u_1, u_2 identified in Step 4. Hence, u_1, u_2 must be CRRA with $\eta_1 = \eta_2$ by the equivalence of (i) and (ii) in Proposition 3.1. $\triangleleft$
This completes the proof of Theorem 3.1. ■

Proof of Theorem 3.2. The identification of η from autarky preferences, which satisfy the vNM axioms according to Axiom 2, follows standard arguments. To identify the proportional sharing rule α, let $ce(f,g) \in \mathbb{R}_{++}$ be the autarky certainty equivalent of sharing f with g. That is, if act $h(\omega) = ce(f,g)$ for all $\omega \in \Omega$, then $\langle h, 0\rangle \sim_g^1 \langle f, 1\rangle$. So, $u\left(ce(f,g)\right) = u\left(\alpha_{f,g}(f+g)\right)$. It follows immediately from the representation that $ce(f,g)$ is well defined and, from strict monotonicity of u, that $u\left(\alpha(f+g)\right)$ is strictly monotone in α. Hence, the share $\alpha_{f,g}$ is uniquely identified. ■

Proof of Theorem 4.1.

Definition A.1 *Let $T(f,g,\pi)$ to be the statement: $\langle a, 0\rangle \succsim_g^1 \langle f, 1\rangle$ and $\langle b, 0\rangle \succsim_f^2 \langle g, 1\rangle$, one of them strict, implies $(a,b) \notin A^\pi(f,g)$. So Axiom 5′ is: If $T(f,g,\pi)$ is true for $(f,g) \in \mathcal{B}$, then $T(f',g',\pi)$ is true for all $(f',g') \in \mathcal{F}^2$.*

Model $\Rightarrow$ Axioms: Let $(\eta^*, \alpha^*, \pi^*)$ be a particular model. Immediately, we have $\langle f, 1\rangle \sim_g^1 \langle \alpha^*_{f,g}(f+g)\pi^* f, 0\rangle$ and $\langle g, 1\rangle \sim_f^2 \langle (1-\alpha^*_{f,g})(f+g)\pi^* g, 0\rangle$. Establishing Axioms 1, 2, 3′, 4, and 6 is analogous to the proof of Theorem 3.1. For Axiom 5′, it is straightforward that $T(f,g,\pi)$ is true for all $(f,g) \in \mathcal{F}^2$ when $\pi \leq \pi^*$. Now let $\pi > \pi^*$. For any (f,g),

$$
\begin{aligned}
u(f) &\leq \pi^* u(\alpha^*_{f,g}(f+g)) + (1-\pi^*)u(f) \leq \pi u(\alpha^*_{f,g}(f+g)) + (1-\pi)u(f), \\
u(g) &\leq \pi^* u((1-\alpha^*_{f,g})(f+g)) + (1-\pi^*)u(g) \leq \pi u((1-\alpha^*_{f,g})(f+g)) + (1-\pi)u(g),
\end{aligned}
$$

where the first inequality in each line follows from Axiom 3′, and the second via $\pi > \pi^*$. Moreover, $(f,g) \in \mathcal{B}$ implies the inequalities are strict for at least one agent. Translating back to preference statements, and since $\big(\alpha^*_{f,g}(f+g)\pi f, (1-\alpha^*_{f,g})(f+g))\pi g\big) \in A^\pi(f,g)$, we have that $T(f,g,\pi)$ is false for all $(f,g) \in \mathcal{B}$ when $\pi > \pi^*$, which completes the proof.

Axioms $\Rightarrow$ Model: The result is established in several steps. Throughout, assume that the primitive, $\left\{\succsim_g^i\right\}_{g\in\mathcal{F}}$ for $i \in \{1,2\}$, satisfies Axioms 1, 2, 3′, 4, 5′, and 6.

Step 1 *There exists $(f,g) \in \mathcal{B}$ and $\pi > 0$ such that $T(f,g,\pi)$ is true.*

$\triangleright$ From Axiom 3′ there exist $f, g, a \in \mathcal{F}$ and $i \in \{1,2\}$ such that $\langle f, 1\rangle \succsim_g^i \langle a, 0\rangle \succ_g^i \langle f, 0\rangle$. Because $\langle g, 1\rangle \succsim_f^j \langle g, 0\rangle$ (also by Axiom 3′), we have $(f,g) \in \mathcal{B}$. Without loss, let $i = 1$ and $j = 2$. For the purpose of contradiction, suppose that $T(f,g,\pi)$ is false for all $\pi > 0$. So, for all $\pi > 0$, there exists $(a_\pi, b_\pi) \in A^\pi(f,g)$ such that $\langle a_\pi, 0\rangle \succsim_g^1 \langle f, 1\rangle \succsim_g^1 \langle a, 0\rangle \succ_g^1 \langle f, 0\rangle$. But, as $\pi \to 0$, we have $a_\pi \to f$, and by the continuity of autarky preferences (Axiom 2): $\langle f, 0\rangle \succsim_g^1 \langle a, 0\rangle \succ_g^1 \langle f, 0\rangle$ which is an obvious contradiction. $\triangleleft$

Step 2 *If $(f,g) \in \mathcal{B}^C$, then $(f,g) \in PS(f,g)$.*

$\triangleright$ Let u_1, u_2 be the expected utility functions representing each agent's autarky preferences, which exist by Axiom 2. Fix $(f,g) \in \mathcal{B}^C$. Then Axiom 3′ implies $\langle f, 1\rangle \sim_g^1 \langle f, 0\rangle$ and $\langle g, 1\rangle \sim_f^2 \langle g, 0\rangle$. Therefore, $T(f,g,\pi)$ can be written: $u_1(a) \geq u_1(f)$ and $u_2(b) \geq u_2(g)$, one of them strict, implies $(a,b) \notin A^\pi(f,g)$.

For the purpose of contradiction, suppose $(f,g) \notin PS(f,g)$: there exists $(a^*, b^*) \in A(f,g)$ such that $\langle a^*, 0\rangle \succ^1 \langle f, 0\rangle$ and $\langle b^*, 0\rangle \succsim^2 \langle g, 0\rangle$, or equivalently, $u_1(a^*) > u_1(f)$ and $u_2(b^*) \geq u_2(g)$ (where assigning the strict preference to Agent 1 is without loss). Then, for any $\pi > 0$, we have $(a^*\pi f, b^*\pi g) \in A^\pi(f,g)$ and $u_1(a^*\pi f) = \pi u_1(a^*) + (1-\pi)u_1(f) > u_1(f)$ and analogously for Agent 2 with the inequality weak. Hence, $T(f,g,\pi)$ is false for all $\pi > 0$. However, by Step 1, there exists $(f', g') \in \mathcal{B}$ and $\pi' > 0$ such that $T(f', g', \pi')$ is true. Axiom 5′ then implies that $T(f,g,\pi')$ is true, which is a contradiction. $\triangleleft$

Step 3 *If $(f,g) \in \mathcal{B}^C$, then $T(f,g,\pi)$ is true for all π.*

$\triangleright$ Fix $(f,g) \in \mathcal{B}^C$ and (a,b) such that $\langle a,0\rangle \succ_g^1 \langle f,1\rangle \sim_g^1 \langle f,0\rangle$ and $\langle b,0\rangle \succsim_f^2 \langle g,1\rangle \sim_f^2 \langle g,0\rangle$, where the assignment of the strict preference to Agent 1 is without loss and the indifferences follow from $(f,g) \in \mathcal{B}^C$ and Axiom 3′. Transitivity (Axiom 1) and $(f,g) \in PS(f,g)$ by Step 2 therefore imply $(a,b) \notin A(f,g)$. So, $T(f,g,1)$ is true. Further, $A^\pi(f,g) \subseteq A(f,g)$ for any π, implying that $T(f,g,\pi)$ is also true. $\triangleleft$

Step 4 *There exists $\overline{\pi} \in (0,1]$ such that the following analogy of Axiom 5 holds if and only if $\pi \leq \overline{\pi}$: For all $(f,g) \in \mathcal{F}^2$, the statement $T(f,g,\pi)$ is true.*

$\triangleright$ By Step 1, there exists $(f,g) \in \mathcal{B}$ and $\pi > 0$ such that $T(f,g,\pi)$ is true. By Axiom 5′, then, $T(f',g',\pi)$ is true for all $(f',g') \in \mathcal{F}^2$.

Next, for any $(f,g) \in \mathcal{B}$, the set $\{(a',b')|\langle f,1\rangle \succsim_g^1 \langle a',0\rangle \wedge \langle g,1\rangle \succsim_f^2 \langle b',0\rangle\}$ is closed, because $\{\succsim_g^i\}_{g\in\mathcal{F}}$ are transitive and complete (Axiom 1) and continuous on $\mathcal{F} \times \{0\}$ (Axiom 2). The set $A^\pi(f,g)$ is closed for all $f,g \in \mathcal{F}$ and all $\pi \in [0,1]$. Furthermore, the mapping $H : [0,1] \to 2^{\mathcal{F}^2}$ with $H(\pi) = A^\pi(f,g)$ is continuous and increasing (in the sense that $A^{\pi'}(f,g) \subset A^\pi(f,g)$ for $\pi' < \pi$). The mapping H has a closed graph by the Closed Graph Theorem for Set Valued Functions (Aliprantis and Border, 1999, ch. 7). Hence, the intersection of the graph of H with $[0,1] \times \{(a',b')|\langle f,1\rangle \succsim_g^1 \langle a',0\rangle \wedge \langle g,1\rangle \succsim_f^2 \langle b',0\rangle\}$ is closed. Therefore, there exists a maximal $\overline{\pi}$ such that $A^{\overline{\pi}}(f,g) \subseteq \{(a',b')|\langle f,1\rangle \succsim_g^1 \langle a',0\rangle \wedge \langle g,1\rangle \succsim_f^2 \langle b',0\rangle\}$. Note that the analogy does not hold for π if and only if $A^\pi \not\subset \{(a',b')|\langle f,1\rangle \succsim_g^1 \langle a',0\rangle \wedge \langle g,1\rangle \succsim_f^2 \langle b',0\rangle\}$. Since $A^\pi(f,g) \subseteq A^{\pi'}(f,g)$ if and only if $\pi \leq \pi'$, we have that the analogy holds if and only if $\pi \leq \overline{\pi}$. $\triangleleft$

Step 5 *The $\overline{\pi}$ identified in Step 4 is the unique π at which the analogy of Axiom 5 and the following analogy of Axiom 6 simultaneously hold: For all $f,g \in \mathcal{F}$, $(a,b) \in PS^\pi(f,g)$ implies $\langle a,0\rangle \succsim_g^1 \langle f,1\rangle$ or $\langle b,0\rangle \succsim_f^2 \langle g,1\rangle$, where*

$$PS^\pi(f,g) := \{(a,b) \in \mathcal{F}^2|(a,b) = (a'\pi f, b'\pi g) \text{ for some } (a',b') \in PS(f,g)\}.$$

$\triangleright$ By Step 4, the analogy of Axiom 5 does not hold for any $\pi > \overline{\pi}$. We next argue that the analogy of Axiom 6 holds at $\overline{\pi}$. There are two cases to consider. In the first case, if $\overline{\pi} = 1$, then $PS^{\overline{\pi}}(f,g) = PS(f,g)$, and Axiom 6 and its analogy are equivalent.

In the second case, $\overline{\pi} < 1$. For $(f,g) \in \mathcal{B}^C$, let $(a,b) \in PS^{\overline{\pi}}(f,g)$. Then there exists $(a',b') \in PS(f,g)$ such that $(a,b) = (a'\overline{\pi}f, b'\overline{\pi}g)$. Because $(a',b') \in PS(f,g)$, by autarky preferences being linear in probabilities (Axiom 2), we have $\langle a,0\rangle \succsim_g^1 \langle f,0\rangle$ or $\langle b,0\rangle \succsim_f^2 \langle g,0\rangle$. Moreover, $(f,g) \in \mathcal{B}^C$ implies that $\langle f,0\rangle \succsim_g^1 \langle f,1\rangle$ and $\langle g,0\rangle \succsim_f^2 \langle g,1\rangle$, giving the desired implication by transitivity (Axiom 1).

For act-pairs in $\mathcal{B}$, it is helpful to first consider $\hat{\pi} > \overline{\pi}$. By Step 4, there exists $(f, g) \in \mathcal{B}$ such that $T(f, g, \hat{\pi})$ is false. By the contrapositive of Axiom $5'$, $T(f', g', \hat{\pi})$ must then be false for all $(f', g') \in \mathcal{B}$. For any $(f, g) \in \mathcal{B}$ then, for any $(a, b) \in PS^{\hat{\pi}}(f, g)$, it must be that $\langle a, 0\rangle \succsim^1_g \langle f, 1\rangle$ or $\langle b, 0\rangle \succsim^2_f \langle g, 1\rangle$. Since this holds for all $\hat{\pi} > \overline{\pi}$ and since, by continuity of autarky preferences (Axiom 2), the intersection of weakly better sets with $\mathcal{F} \times 0$ are closed, it follows that $\langle a', 0\rangle \succsim^1_g \langle f, 1\rangle$ or $\langle b', 0\rangle \succsim^2_f \langle g, 1\rangle$ for all $(a', b') \in PS^{\overline{\pi}}(f, g)$, which completes the analogy of Axiom 6 at $\overline{\pi}$.

It remains to show that the analogy of Axiom 6 does not hold for $\pi < \overline{\pi}$. Consider $(f, g) \in \mathcal{B}$, which is nonempty by Axiom $3'$. Then, by Axiom 6, for $(a, b) \in PS(f, g)$, $\langle a, 0\rangle \succsim^1_g \langle f, 0\rangle$ or $\langle b, 0\rangle \succsim^1_f \langle g, 0\rangle$, one of them strict, and hence $(f, g) \notin PS(f, g)$. Let $(a, b) \in PS(f, g)$ be such that $\langle a, 0\rangle \succ^1_g \langle f, 0\rangle$ and $\langle b, 0\rangle \succ^2_f \langle g, 0\rangle$, which is possible because $(f, g) \notin PS(f, g)$ and autarky preferences are monotonic and continuous (Axiom 2). Then for arbitrary π, $a^\pi := a\pi f$, and $b^\pi := b\pi g$, we have $(a^\pi, b^\pi) \in PS^\pi(f, g)$. We make two observations. First, according to the analogy of Axiom 5, $\langle f, 1\rangle \succsim^1_g \langle a^{\overline{\pi}}, 0\rangle$ and $\langle g, 1\rangle \succsim^2_f \langle b^{\overline{\pi}}, 0\rangle$. Second, for $\pi < \overline{\pi}$ the linearity of autarky preferences in probabilities implies that $\langle a^{\overline{\pi}}, 0\rangle \succ^1_g \langle a^\pi, 0\rangle$ and $\langle b^{\overline{\pi}}, 0\rangle \succ^2_f \langle b^\pi, 0\rangle$. Hence, for $\pi < \overline{\pi}$, we established that $(a^\pi, b^\pi) \in PS^\pi(f, g)$ but $\langle f, 1\rangle \succ^1_g \langle a^\pi, 0\rangle$ and $\langle g, 1\rangle \succ^2_f \langle b^\pi, 0\rangle$. That is, the analogy of Axiom 6 does not hold for $\pi < \overline{\pi}$. $\lhd$

Step 6 *For the remainder of the proof, fix $\pi = \overline{\pi}$ as identified in Step 4. For any $f, g \in \mathcal{F}$ and $(a, b) \in PS^\pi(f, g)$ there exists $\alpha \in (0, 1)$ such that $\langle a, 0\rangle \sim^1_g \langle \alpha(f + g)\pi f, 0\rangle$ and $\langle b, 0\rangle \sim^2_f \langle (1 - \alpha)(f + g)\pi g, 0\rangle$.*

$\rhd$ This argument parallels the one for Step 1 in the proof of Theorem 3.1. Fix $f, g \in \mathcal{F}$ and $(a, b) \in \mathcal{A}^\pi(f, g)$. Define

$$\begin{aligned} \Theta_1 &:= \left\{\alpha \in [0,1] \,\middle|\, \langle a, 0\rangle \succ^1_b \langle \alpha(f + g)\pi f, 0\rangle \right\} \\ \Theta_2 &:= \left\{\alpha \in [0,1] \,\middle|\, \langle b, 0\rangle \succ^2_a \langle (1 - \alpha)(f + g)\pi g, 0\rangle \right\}. \end{aligned}$$

and, for the purpose of contradiction, suppose that $\Theta_1 \cup \Theta_2 = (0, 1)$. Because $a < (f + g)\pi f$, by monotonicity of autarky preferences (Axiom 2), there exists $\alpha < 1$ large enough that $\langle \alpha(f+g)\pi f, 0\rangle \succsim^1_b \langle a, 0\rangle$ and hence $\Theta_2 \neq \emptyset$. By symmetric argument, $\Theta_1 \neq \emptyset$.

Continuity and monotonicity of autarky preferences in α (which follow immediately from Axiom 2) imply $\Theta_1 = (0, \overline{\alpha}_{f,g})$ and $\Theta_2 = (\underline{\alpha}_{f,g}, 1)$. Hence, for $\Theta_1 \cup \Theta_2 = (0, 1)$, it must be that $\underline{\alpha}_{f,g} < \overline{\alpha}_{f,g}$ and there exists $\alpha^* \in \Theta_1 \cap \Theta_2$, meaning

$$\langle a, 0\rangle \succ^1_b \langle \alpha^*(f + g)\pi f, 0\rangle \text{ and } \langle b, 0\rangle \succ^2_a \langle (1 - \alpha^*)(f + g)\pi g, 0\rangle.$$

By Axiom 2, $\succsim_h^i$ is independent of $h \in \mathcal{F}$ in autarky for $i \in \{1,2\}$. Therefore, for all $h, h' \in \mathcal{F}$,

$$\langle a, 0\rangle \succ_h^1 \langle \alpha^*(f+g)\pi f, 0\rangle \text{ and } \langle b, 0\rangle \succ_{h'}^2 \langle (1-\alpha^*)(f+g)\pi g, 0\rangle. \tag{7}$$

Now, let $(a', b') \in A(f,g)$ be such that $a = a'\pi f$ and $b = b'\pi g$, and let

$$\hat{a} := a'\pi\alpha^*(f+g) \text{ and } \hat{b} := b'\pi(1-\alpha^*)(f+g).$$

Then $\succsim_h^i$ being independent from h in autarky (Axiom 2) implies that (7) holds if and only if, for all $h, h' \in \mathcal{F}$,

$$\langle \hat{a}, 0\rangle = \langle a'\pi\alpha^*(f+g), 0\rangle \succ_h^1 \langle \alpha^*(f+g)\pi\alpha^*(f+g), 0\rangle = \langle \alpha^*(f+g), 0\rangle$$

and, analogously, $\langle \hat{b}, 0\rangle \succ_{h'}^2 \langle (1-\alpha^*)(f+g), 0\rangle$.

Separately, by Axiom 4,

$$\begin{array}{ccc} \langle \alpha^*(f+g), 0\rangle & \succsim_{(1-\alpha^*)(f+g)}^1 & \langle \alpha^*(f+g), 1\rangle \\ \langle (1-\alpha^*)(f+g), 0\rangle & \succsim_{\alpha^*(f+g)}^2 & \langle (1-\alpha^*)(f+g), 1\rangle. \end{array}$$

Then, by transitivity of $\succsim_h^i$ for $i \in \{1,2\}$ and any $h \in \mathcal{F}$ (Axiom 1),

$$\begin{array}{ccc} \langle \hat{a}, 0\rangle & \succ_{(1-\alpha^*)(f+g)}^1 & \langle \alpha^*(f+g), 1\rangle \\ \langle \hat{b}, 0\rangle & \succ_{\alpha^*(f+g)}^2 & \langle (1-\alpha^*)(f+g), 1\rangle. \end{array}$$

By construction $(\hat{a}, \hat{b}) \in A^\pi(\alpha^*(f+g), (1-\alpha^*)(f+g))$. But by Step 4,

$$(\hat{a}, \hat{b}) \notin A^\pi(\alpha^*(f+g), (1-\alpha^*)(f+g)),$$

which is a contradiction. Hence for all $(a,b) \in A^\pi(f,g)$ there is $\alpha \in (0,1)$ such that

$$\langle \alpha(f+g)\pi f, 0\rangle \succsim_b^1 \langle a, 0\rangle \text{ and } \langle (1-\alpha)(f+g)\pi g, 0\rangle \succsim_a^2 \langle b, 0\rangle.$$

Finally, by the definition of $PS^\pi(f,g)$ and because $\succsim_h^i$ is independent of $h \in \mathcal{F}$ in autarky (Axiom 2), we have that for any $(a,b) \in PS^\pi(f,g)$ there is $\alpha \in (0,1)$ such that

$$\langle a, 0\rangle \sim_g^1 \langle \alpha(f+g)\pi f, 0\rangle \text{ and } \langle b, 0\rangle \sim_f^2 \langle (1-\alpha)(f+g)\pi g, 0\rangle. \quad \triangleleft$$

Step 7 *Autarky preferences have a common CRRA representation.*

$\triangleright$ Fix $f \in \mathcal{F}$ and $\beta > 0$. By Axiom 4, $(f, \beta f) \in \mathcal{B}^C$. So, $(f, \beta f) \in PS(f, \beta f)$ by Step 2. Proposition 3.1, then implies u_1, u_2 from the proof of Step 2 must be common CRRA. $\triangleleft$

Step 8 *Let* $\mathcal{P} := \{(f, \beta f) | f \in \mathcal{F}, \beta > 0\}$. *Then* $\mathcal{B}^C = \mathcal{P}$.

$\triangleright$ By Axiom 4, $\mathcal{P} \subseteq \mathcal{B}^C$. Consider now $(f, g) \in \mathcal{B}^C$. By Step 2, $(f, g) \in PS(f, g)$. Then, by Step 7 and Proposition 3.1, (f, g) must be in $\mathcal{P}$. Therefore, $\mathcal{B}^C \subseteq \mathcal{P}$. $\triangleleft$

Step 9 *To complete the representation result, proceed analogously to Steps 2-6 in the proof of Theorem 3.1.*

$\triangleright$ Everything is analogous substituting out Axioms 5 and 6 for their respective analogies from Steps 4 and 5 of the current proof and the observation that the equivalence of (i) and (ii) in Proposition 3.1 remains valid if there is an exogenous probability, $(1 - \pi) < 1$, of being unable to risk-share (since expected utilities are linear in probabilities). $\triangleleft$

Step 10 *The parameters* (η, α) *are determined analogously to the identification in Theorem 3.2. It remains to establish the uniqueness of* π.

$\triangleright$ Consider again $(f, g) \notin PS(f, g)$. Then $u(\alpha_{f,g}(f + g)) > u(f)$ or $u((1 - \alpha_{f,g})(f + g)) > u(g)$. Suppose $u(\alpha_{f,g}(f + g)) > u(f)$ and let $h \in \mathcal{F}$ be such that $\langle h, 0\rangle \sim^1_g \langle f, 1\rangle$. Then $\pi u(\alpha_{f,g}(f + g)) + (1 - \pi)u(f)$ is strictly increasing in π and hence the π that satisfies the requirement $\pi u(\alpha_{f,g}(f + g)) + (1 - \pi)u(f) = u(h)$ is unique. The case where $u((1 - \alpha_{f,g})(f + g)) > u(g)$ is analogous. $\triangleleft$

This completes the proof of Theorem 4.1. ■

Proof of Theorem 4.2.

Claim (i): Assume that pair A has a stronger preference to share than does pair B. We first show that $\eta^A = \eta^B$ by establishing that Agents $1A$ and $1B$ have the same preferences in autarky and hence the same consumption utility function. To begin, suppose that $\langle f, 0\rangle \succsim^{1A}_g \langle f', 0\rangle$ for some $f, f', g \in \mathcal{F}$. From Axiom 2, autarky preferences are independent of g, so $\langle f, 0\rangle \succsim^{1A}_{\beta f'} \langle f', 0\rangle \sim^{1A}_{\beta f'} \langle f', 1\rangle$, where the indifference results from Axioms $3'$ and 4 together. We then have that $\langle f, 0\rangle \succsim^{1B}_{\beta f'} \langle f', 1\rangle \sim^{1B}_{\beta f'} \langle f', 0\rangle$, where the strict preference is by the theorem's hypothesis and the indifference again results from Axioms $3'$ and 4 together. It follows for the CRRA utilities u^A and u^B from the respective two-stage models of proportional risk sharing for pairs A and B that $u^A(f) \geq u^A(f')$ implies $u^B(f) \geq u^B(f')$ for all $f, f' \in \mathcal{F}$. Because CRRA utility functions are continuous and monotonic, it must be the case that $u^A = u^B$ or $\eta^A = \eta^B$.

For the purpose of contradiction, suppose now that $\pi^A \leq \pi^B$. The following three cases are exhaustive:

- $\alpha^A_{f,g} < \alpha^B_{f,g}$ for some pair of acts (f, g). Then Agent $1B$ benefits strictly more from sharing (f, g) than Agent $1A$ does. That is, there exist an act f' with $\langle f, 1\rangle \succ^{1B}_g \langle f', 0\rangle$ but $\langle f', 0\rangle \succ^{1A}_g \langle f, 1\rangle$.

- $\alpha^A_{f,g} > \alpha^B_{f,g}$ for some pair of acts (f, g). Then, symmetrically, Agent $2B$ benefits strictly more from sharing (f, g) than Agent $2A$ does. That is, there exist an act f' with $\langle f, 1\rangle \succ^{2B}_g \langle f', 0\rangle$ but $\langle f', 0\rangle \succ^{2A}_g \langle f, 1\rangle$.

- $\alpha^A_{f,g} = \alpha^B_{f,g}$ for all pairs of acts (f, g). Then both agents in pair B benefit weakly more from sharing than those in pair A do. Hence there are no three acts f, g, f' such that either $\langle f, 1\rangle \succ^{1A}_g \langle f', 0\rangle$ and $\langle f', 0\rangle \succsim^{1B}_g \langle f, 1\rangle$, or $\langle f, 1\rangle \succ^{2A}_g \langle f', 0\rangle$ and $\langle f', 0\rangle \succsim^{2B}_g \langle f, 1\rangle$.

In all three cases, pair A does not have a stronger preference to share than pair B, contradicting the hypothesis. Hence, it must be that $\pi^A > \pi^B$.

Claim (ii): Follows immediately from the representation. [Remark: the requirement of $\alpha^A = \alpha^B$ is essential in (ii) even though it was not needed for Claim (i): if $\alpha^A \neq \alpha^B$, the argument in proof of Claim (i) can be applied to show that pair A will not have a stronger preference to share than pair B as long as π^A and π^B are not too different.] ■

Proof of Theorem 4.3. Fix models A and B that differ only in $\pi^A > \pi^B$, with a sensible sharing rule for u. For any $f, g \in \mathcal{F}$, let $\Psi(f, g) := u(\alpha_{f,g}(f + g))$. For the hypothesis, assume that $\langle f', 1\rangle \succsim^{1A}_g \langle f, 1\rangle$ and $\langle f, 1\rangle \succ^{1B}_g \langle f', 1\rangle$. It follows that $\Psi(f', g) \geq \Psi(f, g)$ and $u(f) > u(f')$. For the purpose of contradiction, suppose that $u(f + g) > u(f' + g)$. Together with $u(f) > u(f')$, sensibility then requires that $\Psi(f, g) > \Psi(f', g)$, which is a contradiction. Hence, $u(f + g) \leq u(f' + g)$, and f is more selfish than f'.

For the other direction, fix a model up to π with a sharing rule that is not sensible for u. It is sufficient to find $\pi^A > \pi^B$ and f, f', g such that $\langle f', 1\rangle \succsim^{1A}_g \langle f, 1\rangle$ and $\langle f, 1\rangle \succ^{1B}_g \langle f', 1\rangle$ but f is not more selfish than f', given g. Because the rule is not sensible, there exists f, f', g such that

$$u(f) > u(f') \text{ and } u(f + g) > u(f' + g), \text{ but } \Psi(f, g)) \leq \Psi(f', g)). \tag{8}$$

Using these f, f', g, for π^A large enough and π^B small enough, we have that $\langle f', 1\rangle \succsim^{1A}_g \langle f, 1\rangle$ (due to $\Psi(f', g) \geq \Psi(f, g)$) and $\langle f, 1\rangle \succ^{1B}_g \langle f', 1\rangle$ (due to $u(f) > u(f')$). However, it is clear from (8) that, given g, f is not more selfish than f'. ■

Proof of Proposition 5.1. First, notice that for any pair of acts (f, g) held by Agents 1 and 2, respectively, all three bargaining solutions depend only on $u(f)$, $u(g)$, and $u(f+g)$. Fix now $\pi > 0$ and act g, with utility $u(g)$, held by Agent 2. Given some set of available acts, $F \subset \mathcal{F}$, Agent 1 seeks to maximize:

$$\pi u(\alpha_{f,g}(f+g)) + (1-\pi)u(f) = \pi\left([(\alpha_{f,g}]^{1-\eta}u(f+g) + u((\alpha_{f,g})\right) + (1-\pi)u(f),$$

where the equality is from Lemma A.3. With g held fixed, Agent 1 only affects $u(f)$ and $u(f+g)$, so the problem can be rewritten:

$$\max_{w \in W} \; \pi\left(\alpha(w, v(w))^{1-\eta}v(w) + u(\alpha(w, v(w)))\right) + (1-\pi)w \tag{9}$$

where $w = u(f)$, $W = \{w : u(f) = w, f \in F\}$, $v(w) = \max\{u(f+g) : u(f) = w, f \in F\}$, and $\alpha(w, v(w)) = \alpha_{f,g}$ given $u(f) = w$, $u(f+g) = v(w)$, and $u(g)$ fixed. Differentiating (9) with respect to w yields:

$$\pi(1-\eta)v(w)\alpha(w,v(w))^{-\eta}\left(v'(w)\alpha^{(0,1)}(w,v(w)) + \alpha^{(1,0)}(w,v(w))\right) + \pi v'(w)\alpha(w,v(w))^{1-\eta}$$
$$+\,\pi u'(\alpha(w,v(w)))\left(v'(w)\alpha^{(0,1)}(w,v(w)) + \alpha^{(1,0)}(w,v(w))\right) + (1-\pi), \tag{10}$$

where the vector in the exponent indicates which argument the partial derivative is being taken with respect to. For all three bargaining solutions under consideration, under the supposition that $u(f) = u(g)$, we have $\alpha = \frac{1}{2}$ and (10) becomes

$$[\pi 2^{\eta-1}v'(w) + (1-\pi)] + \pi 2^{\eta}((1-\eta)v(w)+1)\left(\alpha^{(1,0)}(w,v(w)) + \alpha^{(0,1)}(w,v(w))v'(w)\right). \tag{11}$$

The local incentive to increase w, at the possible expense of efficiency $v(w)$, is therefore given by (11). The first term in brackets, $\pi 2^{\eta-1}v'(w) + (1-\pi)$, does not depend on the sharing rule, so does not affect the comparison of the three bargaining solutions. Next, $\pi 2^{\eta}((1-\eta)v(w)+1) > 0$ for all feasible values of $v(w)$ given η, and also does not depend on the sharing rule. Hence, the comparison of the local incentives under the three different bargaining solutions turns on the rankings of $\alpha^{(1,0)}(w,v(w)) + \alpha^{(0,1)}(w,v(w))v'(w)$. A greater value for this term implies a greater local incentive to increase the value of the outside option, $w = u(f)$, at the expense of efficiency, $v(w) = u(f+g)$.

Because all three bargaining solutions are consequentialist (Section 4), they can be characterized independent of π. We first consider the case where $\eta > 1$. For the EBS, α_E

must solve

$$\begin{aligned}\Psi_E(w,v,\alpha) &= \alpha_E(w,v(w))^{1-\eta}v(w)+u(\alpha_E(w,v(w)))-w\\ &\qquad -\big((1-\alpha_E(w,v(w)))^{1-\eta}v(w)+u(1-\alpha_E(w,v(w)))-u(g)\big)=0.\end{aligned}$$

From the Implicit Function Theorem,

$$\begin{aligned}\alpha_E^{(1,0)} &= \frac{-\Psi_E^{(1,0,0)}}{\Psi_E^{(0,0,1)}} = \frac{-(\alpha_E(1-\alpha_E))^{\eta}}{(\alpha_E^{\eta}+(1-\alpha_E)^{\eta})\left((\eta-1)v(w)-1\right)}\\ \alpha_E^{(0,1)} &= \frac{-\Psi_E^{(0,1,0)}}{\Psi_E^{(0,0,1)}} = \frac{\alpha_E^{\eta+1}-\alpha_E^{\eta}+\alpha_E(1-\alpha_E)^{\eta}}{(\alpha_E^{\eta}+(1-\alpha_E)^{\eta})\left((\eta-1)v(w)-1\right)}.\end{aligned}$$

Next, with $\eta>1$, we have $u(h)<\frac{1}{\eta-1}$ for any act h. So, $\frac{1}{\eta-1}>u(f+g)\geq u(2f)$, with the second inequality strict if $f\neq g$. Hence, we can parameterize $v(w)=r\left(\frac{1}{\eta-1}\right)+(1-r)u(2f)$, for some $r\in[0,1)$. Recalling that $u(g)=w$ implies $\alpha_E=\frac{1}{2}$, we have

$$\alpha_E^{(1,0)}\Big|_{\alpha_E=\frac{1}{2},u(g)=w} = \frac{1}{4(1-r)(1-(\eta-1)w)}, \quad \text{and} \quad \alpha_E^{(0,1)}\Big|_{\alpha_E=\frac{1}{2},u(g)=w}=0.$$

Following the same steps for the NBS, α_N must solve

$$\begin{aligned}&\frac{\partial}{\partial\alpha_N}\Big[\big(\alpha_N(w,v(w))^{1-\eta}v(w)+u(\alpha_N(w,v(w)))-w\big)\\ &\qquad\times\big((1-\alpha_N(w,v(w)))^{1-\eta}v(w)+u(1-\alpha_N(w,v(w)))-u(g)\big)\Big]=0,\end{aligned}$$

and

$$\alpha_N^{(1,0)}\Big|_{\alpha_N=\frac{1}{2},u(g)=w} = \frac{1-\eta}{4(1-\eta-r)(1-(\eta-1)w)}, \quad \text{and} \quad \alpha_N^{(0,1)}\Big|_{\alpha_N=\frac{1}{2},u(g)=w}=0.$$

Therefore,

$$\left(\alpha_E^{(1,0)}-\alpha_N^{(1,0)}\right)\Big|_{\alpha_E=\alpha_N=\frac{1}{2},u(g)=w} = \frac{\eta r}{4(1-r)(\eta-1+r)(1-(\eta-1)w)}\geq 0$$

for all admissible η,w,r, with the inequality strict for all $f\neq g$ (i.e., $r\neq 0$). This establishes the ranking of the local incentives under the EBS and the NBS.

The equation that characterizes the KSS, (3), first requires solving for $\bar{a}_{f,g}$ and $\bar{b}_{f,g}$.Because Pareto efficient sharing arrangements are proportional in our representation, there exists unique $\underline{\alpha}_{f,g}$, $\overline{\alpha}_{f,g}$ such that $\bar{a}_{f,g} = \overline{\alpha}_{f,g}(f+g)$ and $\bar{b}_{f,g} = (1-\underline{\alpha}_{f,g})(f+g)$. Specifically,

$$\underline{\alpha}_{f,g} = \left(\frac{1-(\eta-1)w}{1-(\eta-1)v(w)}\right)^{\frac{1}{1-\eta}} \quad \text{and} \quad \overline{\alpha}_{f,g} = 1 - \left(\frac{1-(\eta-1)u(g)}{1-(\eta-1)v(w)}\right)^{\frac{1}{1-\eta}}.$$

Following the same steps, then gives

$$\alpha_K^{(1,0)}\Big|_{\alpha_K=\frac{1}{2},u(g)=w} = \frac{2r\left(\frac{2^{\eta-1}}{1-r}\right)^{\frac{1}{1-\eta}} + 2\left(\frac{2^{\eta-1}}{1-r}\right)^{\frac{1}{1-\eta}} + \left(2-2\left(\frac{2^{\eta-1}}{1-r}\right)^{\frac{1}{1-\eta}}\right)^{\eta} - 2}{4\left(2-2\left(\frac{2^{\eta-1}}{1-r}\right)^{\frac{1}{1-\eta}} - \left(2-2\left(\frac{2^{\eta-1}}{1-r}\right)^{\frac{1}{1-\eta}}\right)^{\eta} + 2r\left(\left(\frac{2^{\eta-1}}{1-r}\right)^{\frac{1}{1-\eta}} - 1\right)\right)((\eta-1)w-1)}$$

$$\alpha_K^{(0,1)}\Big|_{\alpha_K=\frac{1}{2},u(g)=w} = 0.$$

It is then a matter of tedious algebra to establish that $\left(\alpha_E^{(1,0)} - \alpha_K^{(1,0)}\right)\Big|_{\alpha_E=\alpha_K=\frac{1}{2},u(g)=w} \geq 0$ and $\left(\alpha_K^{(1,0)} - \alpha_N^{(1,0)}\right)\Big|_{\alpha_K=\alpha_N=\frac{1}{2},u(g)=w} \geq, =, \leq 0$ for $\eta >, =, < 2$ and, again, with all inequalities strict if $f \neq g$ (i.e, $r \neq 0$). This establishes the ranking of the local incentives under the two comparisons involving the KSS.

The cases of $\eta = 1$ and $\eta < 1$ are handled similarly. The $\eta = 1$ case is straightforward, and for $\eta < 1$ we employ a parameterization similar to that used in the $\eta > 1$ case. In particular, for $\eta < 1$, we have $u(h) \geq \frac{1}{\eta-1}$ for any act h. Since $u(f+g) \geq u(2f)$, it must be that $u(f) = w \leq \overline{w}(v) := \frac{2^{\eta}(1-\eta)v+2^{\eta}-2}{2(1-\eta)}$. So, the appropriate parameterization is to set $w = r\left(\frac{1}{\eta-1}\right) + (1-r)\overline{w}(v)$, for some $r \in [0,1]$. With this, the argument follows the analogous steps. ■