Supplement to
"Risk Sharing and Strategic Choice"

(For Online Publication)

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This supplement contains extended formal results for Daley and Sadowski (2024) (henceforth DS24). Specifically, we consider the respective special cases of the model from DS24 wherein the risk-sharing arrangement adheres to one of the three prominent cooperative bargaining solutions of Nash (1950); Kalai and Smorodinsky (1975); Kalai (1977). For each solution, we provide additional behavioral axioms that, together with the general axioms of DS24, characterize the specific model.

The axioms we provide differ from the well known normative desiderata on which these bargaining solutions are usually founded (see Table 1 in DS24) for three related reasons. First, our theory is positive, and hence our axioms aim to capture testable restrictions on behavior, rather than properties of the distribution of surplus. In this view, the normative appeal of each bargaining solution matters only insofar as it leads agents to adopt it in practice. Second, properties that are intuitive on the domain of utility surpluses may not have an appealing counterpart on our domain of first-stage preferences. Third, we look only at bargaining problems that correspond to a risk-sharing situation in the context of our model, precluding the vast majority of possible bargaining problems as considered in the cooperative bargaining literature. In particular, the feasible set of utility surpluses in our model is pinned down by η, $u(f+g)$, $u(f)$ and $u(g)$. The desirable properties discussed above therefore have less bite on our restricted domain. For instance, it can be shown that all three bargaining solutions satisfy Monotonicity on our domain. Because of these differences, novel axioms are needed.

The following representation theorem serves to organize the next three sections, which provide the axioms for the different solutions in turn.

Theorem S.1 (Prominent Solutions) *If preferences $\left\{\succsim_g^i\right\}_{g\in\mathcal{F}}$ for $i\in\{1,2\}$ can be explained by a two-stage model of proportional risk sharing (Axioms 1-6), then*

1. *they satisfy Axioms 7 and 8 if and only if α corresponds to the NBS.*
2. *they satisfy Axiom 9 if and only if α corresponds to the KSS.*
3. *they satisfy Axiom 10 if and only if α corresponds to the EBS.*

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Remark 1 *In the context of the two-stage model with frictions (DS24, Section 4), the KSS and EBS can again be tightly characterized by requiring Axiom 9 or 10, respectively. Characterizing the NBS in the presence of frictions involves appropriate modifications to Axioms 7 and 8.*

We will make use of the following notation. Since $\alpha^1_{f,g} + \alpha^2_{f,g} = 1$, we drop the superscript and let $\alpha^1_{f,g} = \alpha_{f,g}$ and $\alpha^2_{f,g} = 1 - \alpha_{f,g}$. Given (f, g), recall from Section 3.5 of DS24 that $\bar{a}_{f,g}$ and $\bar{b}_{f,g}$ are the best allocations for Agents 1 and 2, respectively, consistent with Voluntary Participation (Axiom 3). Because Pareto efficient sharing arrangements are proportional in our representation, there exists unique $\underline{\alpha}_{f,g}$, $\overline{\alpha}_{f,g}$ such that $\bar{a}_{f,g} = \overline{\alpha}_{f,g}(f + g)$ and $\bar{b}_{f,g} = (1 - \underline{\alpha}_{f,g})(f + g)$. Stated in preference terms,

$$\underline{\alpha}_{f,g} := \sup\{\alpha | \langle f, 0\rangle \succ^1 \langle \alpha(f + g), 0\rangle\} \quad \text{and} \quad \overline{\alpha}_{f,g} := \inf\{\alpha | \langle g, 0\rangle \succ^2 \langle (1 - \alpha)(f + g), 0\rangle\}.$$

It will also be useful to recall from Section 4 of DS24 that $\mathcal{B} \subset \mathcal{F}^2$ is the set of sharing situations in which at least one agent strictly values the ability to share and that $(f\mathbf{p}f', g\mathbf{p}g')$ is the act-pair that generates the same distribution over consumption-pairs as would a randomization that yielded (f, g) with probability $\mathbf{p}$ and (f', g') with probability $1 - \mathbf{p}$.

S.1 Sharing according to the Nash Bargaining Solution

Given our general model, the use of the NBS is tightly characterized by two additional axioms. First, while a sharing arrangement may divide the gains unevenly in general, it seems plausible that when the stakes are very small, gains would be split about evenly. To see why this feature arises when sharing is based on the NBS, recall a critical step in its familiar construction based on the properties in Table 1. That construction begins by showing that for linear Pareto surplus frontiers, Symmetry and Scale Invariance imply that agents' surpluses must be proportional to the frontier's slope. In our context, surplus frontiers are never linear. However, the possible utility frontiers in our model are all smooth, and hence become approximately linear as the surplus from sharing shrinks sufficiently. In terms of first-stage preferences, our first additional axiom therefore requires that for sufficiently small achievable gains from sharing (f, g), each agent likes sharing as much as a final allocation that is sufficiently close to the reallocation that lies "halfway" between their best/worst proportional shares. To formalize this notion, let

$$\widetilde{\alpha}_{f,g} = \frac{\underline{\alpha}_{f,g} + \overline{\alpha}_{f,g}}{2}.$$

The axiom states that if the gains at stake shrink sufficiently, then for any $\varepsilon > 0$ Agent 1 prefers to get share $\widetilde{\alpha}_{f,g}$ of $f + g$ over (getting the autarky equivalent of) sharing f with

g with high probability but having to consume f in autarky with low probability $\varepsilon > 0$, and analogously for Agent 2.

Axiom 7 (Small Stakes Symmetry)
Fix any $(f,g) \in \mathcal{B}^C$, and sequences of acts $\{f^n\}_{n=1}^{\infty} \to f$ uniformly and $\{g^n\}_{n=1}^{\infty} \to g$ uniformly. If, for $(a^n, b^n) \in A(f^n, g^n)$, $\langle f^n, 1\rangle \sim_{g^n}^1 \langle a^n, 0\rangle$ and $\langle g^n, 1\rangle \sim_{f^n}^2 \langle b^n, 0\rangle$, then for all $\varepsilon > 0$, there is $N > 0$ such that for all $n > N$

$$\begin{gathered}\langle \widetilde{\alpha}_{f^n,g^n}(f^n+g^n), 0\rangle \succsim_{g^n}^1 \langle f^n \varepsilon a^n, 0\rangle \\ \langle (1-\widetilde{\alpha}_{f^n,g^n})(f^n+g^n), 0\rangle \succsim_{f^n}^2 \langle g^n \varepsilon b^n, 0\rangle.\end{gathered}$$

The second new axiom is the appropriate translation to our environment of a convexity assumption in Peters and Van Damme's (1991) alternative characterization of the NBS via its dependence on disagreement values. They consider bargaining agreements made when the parameters of the bargaining problem are uncertain; for example the agents may be sharing (f,g) or they may be sharing (f',g'), but the resolution of this uncertainty occurs only after the bargaining agreement has been completed. In our context, this randomness is captured by the endowments being $(f\mathbf{p}f', g\mathbf{p}g')$ for some $\mathbf{p} \in [0,1]$. Their assumption is that bargaining from the "endowment" consisting of an after-bargaining randomization of (f,g) and its corresponding bargaining solution $(a,b) = \Gamma(f,g)$ must result in the same solution: (a,b).

Axiom 8 (Disagreement Convexity)
If $f,g,a,b \in \mathcal{F}$, $f+g = a+b$, and $\mathbf{p} \in [0,1]$ then,

$$\left.\begin{array}{l}\langle f,1\rangle \sim_g^1 \langle a,0\rangle \\ \langle g,1\rangle \sim_f^2 \langle b,0\rangle\end{array}\right\} \implies \left\{\begin{array}{l}\langle a\mathbf{p}f,1\rangle \sim_{b\mathbf{p}g}^1 \langle a,0\rangle \\ \langle b\mathbf{p}g,1\rangle \sim_{a\mathbf{p}f}^2 \langle b,0\rangle\end{array}\right.$$

S.2 Sharing according to the Kalai-Smorodinsky Solution

Kalai and Smorodinsky (1975) dropped IIA in favor of Resource Monotonicity. The resulting solution (KSS) relies on each agent's "aspiration payoff": the maximum payoff an agent can get in an agreement that respects disagreement values. In our risk-sharing model with first-stage acts (f,g), aspiration payoffs are $u(\overline{\alpha}_{f,g}(f+g))$ and $u((1-\underline{\alpha}_{f,g})(f+g))$, and the KSS of the second stage is the reallocation $(a,b) \in PS(f,g)$ such that

$$\frac{u(a)-u(f)}{u(\overline{\alpha}_{f,g}(f+g))-u(f)} = \frac{u(b)-u(g)}{u((1-\underline{\alpha}_{f,g})(f+g))-u(g)}. \tag{S.1}$$

The KSS is tightly characterized by one additional axiom. Not surprisingly, the additional axiom captures that the gains from sharing are proportional to the respective best outcomes each agent can get without violating the participation constraint of the other.

Axiom 9 (Proportional Gains)
If $\mathbf{p} \in [0,1]$ *is such that* $\langle \overline{\alpha}_{f,g}(f+g)\mathbf{p}f, 0\rangle \sim_g^1 \langle f, 1\rangle$, *then* $\langle (1-\underline{\alpha}_{f,g})(f+g)\mathbf{p}g, 0\rangle \sim_f^2 \langle g, 1\rangle$.

S.3 Sharing according to the Egalitarian Bargaining Solution

Recall that the EBS is egalitarian with respect to the utility surplus each agent achieves in the bargaining outcome. In our model with first-stage acts (f, g), the EBS of the second stage is the reallocation $(a, b) \in PS(f, g)$ such that $u(a) - u(f) = u(b) - u(g)$.

The EBS relies on a cardinal interpretation of utilities. In the context of our axioms and general representation, η is common, but setting $u_1 = u_2$ is a normalization. Attaching cardinal meaning to this normalization may be appropriate if, for instance, the scaling of the utilities are part of a social norm that dictates how individual outcomes are evaluated for the purpose of sharing. The use of the EBS is tightly characterized by one additional axiom.[1]

Axiom 10 (Symmetric Division) $\langle f\mathbf{\tfrac{1}{2}}g, 1\rangle \sim_g^1 \langle h, 0\rangle$ *if and only if* $\langle f\mathbf{\tfrac{1}{2}}g, 1\rangle \sim_f^2 \langle h, 0\rangle$.

Within the representation, the first preference statement in Axiom 10 implies that for acts f, g, and h: $u(h) = \frac{1}{2}u(\alpha_{f,g}(f+g)) + \frac{1}{2}u(\alpha_{g,g}2g) = \frac{1}{2}u(\alpha_{f,g}(f+g)) + \frac{1}{2}u(g)$, where the second equality is implied by Axioms 3 and 4. Writing out the analogous expression for the second preference statement yields $u(h) = \frac{1}{2}u(f) + \frac{1}{2}u((1-\alpha_{f,g})(f+g))$. Since an act h that satisfies the first preference statement can always be found, this implies $u(\alpha_{f,g}(f+g)) - u(f) = u((1-\alpha_{f,g})(f+g)) - u(g)$, which is the utility-surplus characterization of the EBS of the second stage of our model given first-stage acts (f, g).

S.4 Proofs

Throughout this section, suppose that primitive $\{\succsim_g^i\}_{i=1,2}$ satisfies Axioms 1-6, so is represented by a model of proportional risk sharing denoted (η^*, α^*). Let u be the utility function from (2) with $\eta = \eta^*$, and, for $f, g \in F$, define

$$\lambda(\alpha) := -\frac{\partial u\left(\hat{\alpha}(f+g)\right)/\partial\hat{\alpha}}{\partial u((1-\hat{\alpha})(f+g))/\partial\hat{\alpha}}\bigg|_{\hat{\alpha}=\alpha} = \left(\frac{1-\alpha}{\alpha}\right)^{\eta^*}, \tag{S.2}$$

which does not depend on (f, g).

[1] If, alternatively, individual utilities were not normalized, then the model representing Axioms 1-6 would entail $u_1 = c_1u_2 + c_2$ for arbitrary constants $c_1 > 0$ and c_2. In this context, the bargaining solution characterized by Axiom 10 has the agents' surpluses being proportional to their utility scaling: $\Gamma(f, g) = (a, b) \in PS(f, g)$ such that $u_1(a) - u_1(f) = c_1(u_2(b) - u_2(g))$.

Lemma S.4 *For $f, g \in \mathcal{F}$, the proportional allocation $(a, b) = (\alpha^*_{f,g}(f+g), (1-\alpha^*_{f,g})(f+g))$ is the NBS if and only if*

$$\lambda(\alpha^*_{f,g}) = \frac{u(\alpha^*_{f,g}(f+g)) - u(f)}{u((1-\alpha^*_{f,g})(f+g)) - u(g)}. \tag{S.3}$$

Proof. Given the representation, Proposition 3.1 implies that the objective for the NBS can be written as

$$\max_{\alpha \in [0,1]} \left[u\left(\alpha(f+g)\right) - u\left(f\right)\right]\left[u\left((1-\alpha)(f+g)\right) - u\left(g\right)\right].$$

Because $\eta^* > 0$, the objective is concave in α, and the solution is always interior. It is therefore characterized by its first-order condition, which rearranges to (S.3) ■

Lemma S.5 *For $f, g \in \mathcal{F}$ and the allocation $(a, b) = (\alpha^*_{f,g}(f+g), (1-\alpha^*_{f,g})(f+g))$,*

$$\lim_{\mathbf{p} \to 1} \overline{\alpha}_{a\mathbf{p}f, b\mathbf{p}g} = \lim_{\mathbf{p} \to 1} \underline{\alpha}_{a\mathbf{p}f, b\mathbf{p}g} = \alpha^*_{f,g}.$$

Proof. See that

$$u\left(\underline{\alpha}_{a\mathbf{p}f, b\mathbf{p}g}(f+g)\right) = u\left(\underline{\alpha}_{a\mathbf{p}f, b\mathbf{p}g}(a\mathbf{p}f + b\mathbf{p}g)\right) = u(a\mathbf{p}f), \tag{S.4}$$

where the first equality is by $(f+g) = (a+b) = (a\mathbf{p}f + b\mathbf{p}g)$, and the second equality is by definition of $\underline{\alpha}$. Next,

$$\lim_{\mathbf{p} \to 1} u\left(\underline{\alpha}_{a\mathbf{p}f, b\mathbf{p}g}(f+g)\right) = \lim_{\mathbf{p} \to 1} u(a\mathbf{p}f) = u(a) = u(\alpha^*_{f,g}(f+g)),$$

where the first equality is from (S.4), the second equality is from the continuity of u, and the third is by the assignment of a in the lemma's hypothesis. It follows from the continuity and strict monotonicity of $u(\alpha(f+g))$ in α that $\underline{\alpha}_{a\mathbf{p}f, b\mathbf{p}g} \xrightarrow[\mathbf{p} \to 1]{} \alpha^*_{f,g}$. A symmetric argument establishes that $\overline{\alpha}_{a\mathbf{p}f, b\mathbf{p}g} \xrightarrow[\mathbf{p} \to 1]{} \alpha^*_{f,g}$, completing the proof. ■

Proof of Theorem S.1(1).

Axioms 7-8 $\Rightarrow$ NBS: Fix $f, g \in \mathcal{F}$, and to simplify notation, we drop the subscript on $\alpha^*_{f,g}$. Let $(a, b) = (\alpha^*(f+g), (1-\alpha^*)(f+g)) \in PS(f, g)$. If $(a, b) = (f, g)$, then (a, b) is the unique allocation in $A(f, g)$ that generates non-negative utility surpluses for both agents. Trivially then, (a, b) adheres to the NBS in this case. For the remainder, assume that $(a, b) \neq (f, g)$. The proof proceeds in several steps.

Step 1 *For any* $\mathbf{p} \in [0,1]$, $\alpha^*_{a\mathbf{p}f,b\mathbf{p}g} = \alpha^*$.

$\triangleright$ For any $\mathbf{p} \in [0,1]$, Axiom 8 implies $\langle a\mathbf{p}f, 1\rangle \sim^1_{b\mathbf{p}g} \langle a, 0\rangle$ and $\langle b\mathbf{p}g, 1\rangle \sim^2_{a\mathbf{p}f} \langle b, 0\rangle$, and, in the context of the representation, this is $\alpha^*_{a\mathbf{p}f,b\mathbf{p}g} = \alpha^*$. $\triangleleft$

Step 2 *For any* $\mathbf{p} \in [0,1]$, $u(\alpha^*(a\mathbf{p}f + b\mathbf{p}g)) - u(a\mathbf{p}f) = (1-\mathbf{p})(u(\alpha^*(f+g)) - u(f))$, *and* $u((1-\alpha^*)(a\mathbf{p}f + b\mathbf{p}g)) - u(b\mathbf{p}g) = (1-\mathbf{p})(u((1-\alpha^*)(f+g)) - u(g))$.

$\triangleright$ Because u is linear in probabilities,

$$\begin{aligned} u(\alpha^*(\underbrace{a\mathbf{p}f + b\mathbf{p}g}_{=f+g})) - u(a\mathbf{p}f) &= u(\alpha^*(f+g)) - \mathbf{p}u(\underbrace{a}_{=\alpha^*_{f,g}(f+g)}) - (1-\mathbf{p})u(f) \\ &= (1-\mathbf{p})\left(u(\alpha^*(f+g)) - u(f)\right). \end{aligned}$$

A symmetric argument establishes the claim's second equation. $\triangleleft$

Step 3 $\lim_{\mathbf{p}\to 1} \dfrac{\alpha^* - \underline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g}}{\overline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g} - \alpha^*} = 1.$

$\triangleright$ By hypothesis, $\langle a, 0\rangle \sim_b \langle a, 1\rangle$ and $\langle b, 0\rangle \sim_a \langle b, 1\rangle$, so $(a,b) \in \mathcal{B}^C$. In addition, $a\mathbf{p}f, b\mathbf{p}g$ converge uniformly to a, b, respectively, as $\mathbf{p} \to 1$. By Axiom 7 then, for any $\varepsilon > 0$, and $\mathbf{p} < 1$ large enough,

$$\langle \widetilde{\alpha}_{a\mathbf{p}f,b\mathbf{p}g}(a\mathbf{p}f + b\mathbf{p}g), 0\rangle \succsim^1_{b\mathbf{p}g} \langle (a\mathbf{p}f)\varepsilon a, 0\rangle.$$

In the representation, that is

$$u(\widetilde{\alpha}_{a\mathbf{p}f,b\mathbf{p}g}(\underbrace{a\mathbf{p}f + b\mathbf{p}g}_{=f+g})) \geq \varepsilon u(a\mathbf{p}f) + (1-\varepsilon)u(\underbrace{a}_{=\alpha^*(f+g)})$$

which rearranges to

$$\frac{u(\alpha^*(f+g)) - u(\widetilde{\alpha}_{a\mathbf{p}f,b\mathbf{p}g}(f+g))}{u(\alpha^*(f+g)) - u(a\mathbf{p}f)} \leq \varepsilon,$$

and then, using: $u(a\mathbf{p}f) = \mathbf{p}u(a) + (1-\mathbf{p})u(f) = \mathbf{p}u(\alpha^*(f+g)) + (1-\mathbf{p})u(\underline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g}(f+g))$, to

$$\frac{u(\alpha^*(f+g)) - u(\widetilde{\alpha}_{a\mathbf{p}f,b\mathbf{p}g}(f+g))}{(1-\mathbf{p})(u(\alpha^*(f+g)) - u(\underline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g}(f+g)))} \leq \varepsilon. \tag{S.5}$$

Because $(a,b) \neq (f,g)$ and $\mathbf{p} \in (0,1)$, the denominator of the LHS of (S.5) is positive and hence

$$\frac{u(\alpha^*(f+g)) - u(\widetilde{\alpha}_{a\mathbf{p}f,b\mathbf{p}g}(f+g))}{u(\alpha^*(f+g)) - u(\underline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g}(f+g))} \leq \varepsilon.$$

Because $\alpha^*, \widetilde{\alpha}_{a\mathbf{p}f,b\mathbf{p}g}$ are both in $(\underline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g}, \overline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g})$, Lemma S.5 implies all four α-terms are converging to one another as $\mathbf{p} \to 1$. Because u is continuous, for any $\gamma > 0$ and $\mathbf{p} < 1$ large enough, we therefore have that

$$\frac{\alpha^* - \frac{1}{2}(\overline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g} + \underline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g})}{\alpha^* - \underline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g}} \leq \frac{\gamma}{2},$$

and hence,

$$\frac{\overline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g} - \alpha^*}{\alpha^* - \underline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g}} \geq 1 - \gamma. \tag{S.6}$$

A symmetric argument using the preferences for Agent 2 implied by Axiom 7 shows that for any $\gamma' > 0$ and $\mathbf{p} < 1$ large enough we have

$$\frac{\overline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g} - \alpha^*}{\alpha^* - \underline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g}} \leq 1 + \gamma'. \tag{S.7}$$

Together, (S.6) and (S.7) imply the claim in Step 3. $\lhd$

Step 4 *The proportional allocation* $(\alpha^*(f+g), (1-\alpha^*)(f+g))$ *satisfies* (S.3).

$\rhd$ For any acts $f', g' \in \mathcal{F}$ and triple $\overline{\alpha} > \alpha' > \underline{\alpha}$, the concavity of u implies

$$\left.\frac{\partial u\left(\alpha(f'+g')\right)}{\partial \alpha}\right|_{\alpha=\underline{\alpha}} > \frac{u\left(\alpha'(f'+g')\right) - u(f')}{\alpha' - \underline{\alpha}} > \left.\frac{\partial u\left(\alpha(f'+g')\right)}{\partial \alpha}\right|_{\alpha=\overline{\alpha}}$$

and

$$-\left.\frac{\partial u\left((1-\alpha)(f'+g')\right)}{\partial \alpha}\right|_{\alpha=\overline{\alpha}} > \frac{u\left((1-\alpha')(f'+g')\right) - u(g')}{(1-\alpha') - (1-\overline{\alpha})} > -\left.\frac{\partial u\left((1-\alpha)(f'+g')\right)}{\partial \alpha}\right|_{\alpha=\underline{\alpha}}.$$

Hence,

$$\lambda(\overline{\alpha}) < \theta(\alpha', \underline{\alpha}, \overline{\alpha}|f', g') := \frac{\frac{u(\alpha'(f'+g'))-u(f')}{\alpha'-\underline{\alpha}}}{\frac{u((1-\alpha')(f'+g'))-u(g')}{(1-\alpha')-(1-\overline{\alpha})}} < \lambda(\underline{\alpha}). \tag{S.8}$$

For any $\mathbf{p} < 1$, we have $\alpha^*_{a\mathbf{p}f,b\mathbf{p}g} \in (\underline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g}, \overline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g})$ and

$$\begin{aligned}\theta(\alpha^*_{a\mathbf{p}f,b\mathbf{p}g}, \underline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g}, \overline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g}|a\mathbf{p}f, b\mathbf{p}g) &= \left(\frac{u\left(\alpha^*(a\mathbf{p}f + b\mathbf{p}g)\right) - u(a\mathbf{p}f)}{u\left((1-\alpha^*)(a\mathbf{p}f + b\mathbf{p}g)\right) - u(b\mathbf{p}g)}\right)\left(\frac{\overline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g} - \alpha^*}{\alpha^* - \underline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g}}\right) \\ &= \left(\frac{u\left(\alpha^*(f+g)\right) - u(f)}{u\left((1-\alpha^*)(f+g)\right) - u(g)}\right)\left(\frac{\overline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g} - \alpha^*}{\alpha^* - \underline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g}}\right),\end{aligned} \tag{S.9}$$

where the first equality is from Step 1 and the second equality is from Step 2.

By Lemma S.5, $\underline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g}$ and $\overline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g}$ both limit to α^* as $\mathbf{p} \to 1$. Using the continuity of λ, (S.8) then implies

$$\begin{aligned}
\lambda(\alpha^*) = \lim_{\mathbf{p}\to 1} \lambda(\overline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g}) = \lim_{\mathbf{p}\to 1} \lambda(\underline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g}) &= \lim_{\mathbf{p}\to 1} \theta(\alpha^*_{a\mathbf{p}f,b\mathbf{p}g}, \underline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g}, \overline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g} | a\mathbf{p}f, b\mathbf{p}g) \\
\text{(using (S.9))} \quad &= \lim_{\mathbf{p}\to 1} \left(\frac{u\left(\alpha^*(f+g)\right) - u(f)}{u\left((1-\alpha^*)(f+g)\right) - u(g)} \right) \left(\frac{\overline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g} - \alpha^*}{\alpha^* - \underline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g}} \right) \\
&= \frac{u\left(\alpha^*(f+g)\right) - u(f)}{u\left((1-\alpha^*)(f+g)\right) - u(g)} \left(\lim_{\mathbf{p}\to 1} \frac{\overline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g} - \alpha^*}{\alpha^* - \underline{\alpha}_{a\mathbf{p}f,b\mathbf{p}g}} \right) \\
\text{(using Step 3)} \quad &= \frac{u\left(\alpha^*(f+g)\right) - u(f)}{u\left((1-\alpha^*)(f+g)\right) - u(g)},
\end{aligned}$$

meaning α^* satisfies (S.3). $\triangleleft$

It follow from Step 4 and Lemma S.4 that $(a, b) = (\alpha^*_{f,g}(f+g), (1-\alpha^*_{f,g})(f+g))$ adheres to the NBS.

<u>NBS $\Rightarrow$ Axioms 7-8</u>: Consider the model (η, α^*) where α^* satisfies the NBS, and $f, g, \{(f^n, g^n)\}_{n=1}^{\infty}$ all satisfying the hypotheses of Axiom 7. If $\underline{\alpha}_{f^n,g^n} = \overline{\alpha}_{f^n,g^n}$, then since the NBS respects disagreement values, we have $\alpha^*_{f^n,g^n} = \widetilde{\alpha}_{f^n,g^n} = \underline{\alpha}_{f^n,g^n} = \overline{\alpha}_{f^n,g^n}$, and the implication of Axiom 7 is immediately satisfied. For the remainder, consider any subsequence for which $\underline{\alpha}_{f^n,g^n} < \overline{\alpha}_{f^n,g^n}$ for all n. Using (S.2), define

$$\rho := \lim_{n\to\infty} \lambda(\alpha^*_{f^n,g^n} | f^n, g^n) = \lim_{n\to\infty} \left(\frac{1-\alpha^*_{f^n,g^n}}{\alpha^*_{f^n,g^n}} \right)^{\eta} = \left(\frac{1-\alpha^*_{f,g}}{\alpha^*_{f,g}} \right)^{\eta}.$$

From Lemma S.4,

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \frac{u(\alpha^*_{f^n,g^n}(f^n+g^n)) - u\left(f^n\right)}{u((1-\alpha^*_{f^n,g^n})(f^n+g^n)) - u\left(g^n\right)} \\
&= \lim_{n\to\infty} \frac{u(\alpha^*_{f^n,g^n}(f^n+g^n)) - u\left(\underline{\alpha}_{f^n,g^n}(f^n+g^n)\right)}{u((1-\alpha^*_{f^n,g^n})(f^n+g^n)) - u\left((1-\overline{\alpha}_{f^n,g^n})(f^n+g^n)\right)},
\end{aligned}$$

where the second equality is by definition of $\underline{\alpha}, \overline{\alpha}$. By the Product Rule of Limits, this can be rewritten as

$$\rho = \lim_{n\to\infty} \frac{u(\alpha^*_{f^n,g^n}(f^n+g^n)) - u\left(\underline{\alpha}_{f^n,g^n}(f^n+g^n)\right)}{\alpha^*_{f^n,g^n} - \underline{\alpha}_{f^n,g^n}} \times \lim_{n\to\infty} \frac{(1-\alpha^*_{f^n,g^n}) - (1-\overline{\alpha}_{f^n,g^n})}{u((1-\alpha^*_{f^n,g^n})(f^n+g^n)) - u\left((1-\overline{\alpha}_{f^n,g^n})(f^n+g^n)\right)} \times \lim_{n\to\infty} \frac{\alpha^*_{f^n,g^n} - \underline{\alpha}_{f^n,g^n}}{\overline{\alpha}_{f^n,g^n} - \alpha^*_{f^n,g^n}}. \quad \text{(S.10)}$$

Evaluating the limit of the first factor, note that for any convergent subsequence $\{(f^n, g^n)\}_n$ in a sufficiently small neighborhood of the strictly positive acts (f, g), $\underline{\alpha}_{f^n,g^n}$ is bounded away from 0 because $u(\underline{\alpha}_{f^n,g^n}(f^n + g^n)) = u(f^n)$. Let $\alpha^- := \inf_n(\underline{\alpha}_{f^n,g^n})$. According to the NBS, $u(\alpha^*_{f^n,g^n}(f^n + g^n)) > u(f^n) = u(\underline{\alpha}_{f^n,g^n}(f^n + g^n))$, it follows that $\alpha^- \leq \underline{\alpha}_{f^n,g^n} < \alpha^*_{f^n,g^n} < \overline{\alpha}_{f^n,g^n}$, where the last inequality comes from the parallel oberservation for Agent 2. Hence, for the concave CRRA utility u and any $\alpha' > \alpha'' \geq \alpha^-$, we have that

$$0 \leq \frac{u(\alpha'(f^n+g^n)) - u\left(\alpha''(f^n+g^n)\right)}{\alpha' - \alpha''} \leq \inf_n \left\{ \left.\frac{\partial u(\alpha(f^n+g^n))}{\partial \alpha}\right|_{\alpha=\alpha^-} \right\}.$$

Therefore, $\frac{u(\alpha'(f^n+g^n)) - u(\alpha''(f^n+g^n))}{\alpha'-\alpha''}$ converges to $\frac{u(\alpha'(f+g)) - u(\alpha''(f+g))}{\alpha'-\alpha''}$ uniformly for $\alpha' > \alpha'' \geq \alpha^-$.

Next, because the NBS is Pareto efficient, $(f, g) \in \mathcal{B}^C$ implies that $\underline{\alpha}_{f,g} = \overline{\alpha}_{f,g}$. Because u is continuous, we have $\lim_{n\to\infty}[\overline{\alpha}_{f^n,g^n} - \underline{\alpha}_{f^n,g^n}] = 0$. Furthermore, since u is continuously differentiable, and since $\alpha^*_{f^n,g^n} \in (\underline{\alpha}_{f^n,g^n}, \overline{\alpha}_{f^n,g^n})$,

$$\lim_{n\to\infty} \frac{u(\alpha^*_{f^n,g^n}(f+g)) - u\left(\underline{\alpha}_{f^n,g^n}(f+g)\right)}{\alpha^*_{f^n,g^n} - \underline{\alpha}_{f^n,g^n}} = \left.\frac{\partial u\left(\alpha(f+g)\right)}{\partial \alpha}\right|_{\alpha=\alpha^*_{f,g}}. \quad \text{(S.11)}$$

An analogous argument yields that

$$\lim_{n\to\infty} \frac{(1-\alpha^*_{f^n,g^n}) - (1-\overline{\alpha}_{f^n,g^n})}{u((1-\alpha^*_{f^n,g^n})(f+g)) - u\left((1-\overline{\alpha}_{f^n,g^n})(f+g)\right)} = -\left(\left.\frac{\partial u\left((1-\alpha)(f+g)\right)}{\partial \alpha}\right|_{\alpha=\alpha^*_{f,g}} \right)^{-1}. \quad \text{(S.12)}$$

The product of (S.11) and (S.12) is

$$\lambda(\alpha^*_{f,g}) = \left(\frac{1-\alpha^*_{f,g}}{\alpha^*_{f,g}}\right)^{\eta} = \rho.$$

Apply the Order of Limits Theorem (the Moore-Osgood Theorem) to (S.10) and see that

$$\rho = \rho \lim_{n\to\infty} \frac{\alpha^*_{f^n,g^n} - \underline{\alpha}_{f^n,g^n}}{\overline{\alpha}_{f^n,g^n} - \alpha^*_{f^n,g^n}}.$$

Therefore, we have

$$\lim_{n\to\infty} \frac{\alpha^*_{f^n,g^n} - \underline{\alpha}_{f^n,g^n}}{\overline{\alpha}_{f^n,g^n} - \alpha^*_{f^n,g^n}} = 1 = \lim_{n\to\infty} \frac{\overline{\alpha}_{f^n,g^n} - \alpha^*_{f^n,g^n}}{\alpha^*_{f^n,g^n} - \underline{\alpha}_{f^n,g^n}}.$$

Hence,

$$\begin{aligned} 0 = \lim_{n\to\infty} \left(\frac{\overline{\alpha}_{f^n,g^n} - \alpha^*_{f^n,g^n}}{\alpha^*_{f^n,g^n} - \underline{\alpha}_{f^n,g^n}} - \frac{\alpha^*_{f^n,g^n} - \underline{\alpha}_{f^n,g^n}}{\alpha^*_{f^n,g^n} - \underline{\alpha}_{f^n,g^n}} \right) &= \lim_{n\to\infty} \frac{(\overline{\alpha}_{f^n,g^n} + \underline{\alpha}_{f^n,g^n}) - 2\alpha^*_{f^n,g^n}}{\alpha^*_{f^n,g^n} - \underline{\alpha}_{f^n,g^n}} \\ &= \lim_{n\to\infty} \frac{2\widetilde{\alpha}_{f^n,g^n} - 2\alpha^*_{f^n,g^n}}{\alpha^*_{f^n,g^n} - \underline{\alpha}_{f^n,g^n}}. \end{aligned}$$

Dividing both sides by (-2) yields,

$$\lim_{n\to\infty} \frac{\alpha^*_{f^n,g^n} - \widetilde{\alpha}_{f^n,g^n}}{\alpha^*_{f^n,g^n} - \underline{\alpha}_{f^n,g^n}} = 0.$$

Using again the Order of Limits Theorem, as well as Taylor approximation given that each of the α-terms have the same limit, yields

$$0 = \lim_{n\to\infty} \frac{u(\alpha^*_{f^n,g^n}(f^n+g^n)) - u(\widetilde{\alpha}_{f^n,g^n}(f^n+g^n))}{u(\alpha^*_{f^n,g^n}(f^n+g^n)) - u(\underline{\alpha}_{f^n,g^n}(f^n+g^n))} = \lim_{n\to\infty} \frac{u(a^n) - u(\widetilde{\alpha}_{f^n,g^n}(f^n+g^n))}{u(a^n) - u(f^n)}. \tag{S.13}$$

Because $u(a^n) > u(f^n)$, (S.13) implies that for any fixed $\varepsilon > 0$ there exists N such that for all $n > N$,

$$\frac{u(a^n) - u(\widetilde{\alpha}_{f^n,g^n}(f^n+g^n))}{u(a^n) - u(f^n)} \le \varepsilon,$$

which rearranges to $u(\widetilde{\alpha}_{f^n,g^n}(f^n+g^n)) \ge \varepsilon u(f^n) + (1-\varepsilon)u(a^n)$. This is identical to the preference statement, $\langle \widetilde{\alpha}_{f^n,g^n}(f^n+g^n), 0\rangle \succsim^1_{g^n} \langle f^n \varepsilon a^n, 0\rangle$ for all n large enough. An analogous argument establishes the counterpart for Agent 2: $\langle (1-\widetilde{\alpha}_{f^n,g^n})(f^n+g^n), 0\rangle \succsim^2_{f^n} \langle g^n \varepsilon b^n, 0\rangle$ for all n large enough. This is Axiom 7.

To establish Axiom 8, recall that, given endowment (f, g), under the NBS the objective function for α is the maximization of

$$\left(u(\alpha(f+g)) - u(f)\right)\left(u((1-\alpha)(f+g)) - u(g)\right)$$

Due to the linearity of the CRRA utility u, the objective function for endowment $(a\mathbf{p}f, b\mathbf{p}g)$ with $a\mathbf{p}f + b\mathbf{p}g = f + g$ is

$$(u(\alpha(f+g)) - u(a\mathbf{p}f))\,(u((1-\alpha)(f+g)) - u(b\mathbf{p}g)) = \\ (1-p)^2\,(u(\alpha(f+g)) - u(f))\,(u((1-\alpha)(f+g)) - u(g)).$$

Since the objective functions for the two endowments are identical up to a scalar, the solutions must be the same. This implies Axiom 8. ■

Proof of Theorem S.1(2).

Axiom 9 $\Rightarrow$ KSS: Fix $f, g \in \mathcal{F}$. By definition of $\overline{\alpha}_{f,g}$, we have that $u\left(\alpha_{f,g}(f+g)\right) \in \left[u\left(f\right), u\left(\overline{\alpha}_{f,g}(f+g)\right)\right]$. Hence, there exist $\mathbf{p} \in [0,1]$ such that

$$\mathbf{p}\; u\left(\overline{\alpha}_{f,g}(f+g)\right) + (1-\mathbf{p})u\left(f\right) = u\left(\alpha_{f,g}(f+g)\right).$$

The equivalent preference statement is $\langle \overline{\alpha}_{f,g}(f+g)\mathbf{p}f, 0\rangle \sim_g^1 \langle f, 1\rangle$. It then follows from Axiom 9 that we have $\langle (1-\underline{\alpha}_{f,g})(f+g)\mathbf{p}g, 0\rangle \sim_f^2 \langle g, 1\rangle$. The equivalent utility statement is $\mathbf{p}u\left((1-\underline{\alpha}_{f,g})(f+g)\right) + (1-\mathbf{p})u\left(g\right) = u\left((1-\alpha_{f,g})(f+g)\right)$. Solving each utility statement for $\mathbf{p}$ gives

$$\mathbf{p} = \frac{u\left(\alpha_{f,g}(f+g)\right) - u\left(f\right)}{u\left(\overline{\alpha}_{f,g}(f+g)\right) - u\left(f\right)} = \frac{u\left((1-\alpha_{f,g})(f+g)\right) - u\left(g\right)}{u\left((1-\underline{\alpha}_{f,g})(f+g)\right) - u\left(g\right)}.$$

Since $\alpha_{f,g}$ is also Pareto efficient given u (Proposition 3.1), it corresponds to the KSS.

KSS $\Rightarrow$ Axiom 9: Straightforward by running the argument above in reverse. ■

Proof of Theorem S.1(3).

Axiom 10 $\Rightarrow$ EBS: Fix $f, g \in \mathcal{F}$. By the continuity of u, there exists $h \in \mathcal{F}$ with $u\left(h\right) = \frac{1}{2}u\left(\alpha_{f,g}(f+g)\right) + \frac{1}{2}u\left(g\right) = \frac{1}{2}u(\alpha_{f,g}(f+g)) + \frac{1}{2}u(\alpha_{g,g}2g)$, where the second equality is implied by Axioms 3 and 4. The equivalent preference statement is $\langle f\tfrac{\mathbf{1}}{\mathbf{2}}g, 1\rangle \sim_g^1 \langle h, 0\rangle$. By Axiom 10, we then have that $\langle f\tfrac{\mathbf{1}}{\mathbf{2}}g, 1\rangle \sim_f^2 \langle h, 0\rangle$ and, equivalently,

$$u\left(h\right) = \frac{1}{2}u((1-\alpha_{f,g})(f+g)) + \frac{1}{2}u(\alpha_{f,f}2f) = \frac{1}{2}u\left((1-\alpha_{f,g})(f+g)\right) + \frac{1}{2}u\left(f\right),$$

where again the second equality is implied by Axioms 3 and 4. That $u\left(\alpha_{f,g}(f+g)\right) - u\left(f\right) = u\left((1-\alpha_{f,g})(f+g)\right) - u\left(g\right)$ follows from algebraic rearrangement of the two utility statements above. Since $\alpha_{f,g}$ is also Pareto efficient given u (Proposition 3.1), it corresponds to the EBS.

EBS $\Rightarrow$ Axiom 10: Straightforward by running the argument above in reverse. ■

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