Elections in Double-Member Districts with Nonseparable Voter Preferences

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Abstract

This paper derives electoral equilibria when voters have nonseparable preferences for candidates in double-member districts. When candidates are elected simultaneously, nonseparable voter preferences create multiple equilibria, including some in which candidates adopt extreme positions. These results are robust to limited voter uncertainty about candidate interaction in the legislature. Nonseparable voter preferences create incentives for the formation of political parties and disincentives for candidates to moderate their positions.
Introduction

This paper derives equilibrium conditions for strategic position-taking by candidates when voters have nonseparable preferences. Nonseparable preferences change profoundly the conditions of electoral equilibrium. To illustrate the impact of nonseparable preferences in elections, we examine the simplest interesting case where voters have nonseparable preferences: double-member districts. Cox (1984) derives the conditions of electoral equilibrium for double-member districts when voter preferences are separable. We reevaluate the equilibrium conditions when voter preferences are nonseparable and elections are simultaneous. We compare our results to Cox’s as an example of how nonseparable preferences change electoral equilibrium results. We also consider the case where candidates are elected in staggered elections, such as the United States Senate. We derive the equilibrium conditions for staggered elections, producing results similar to those of Fiorina (1988), Krassa (1989), Alesina and Rosenthal (1989), and Alesina, Fiorina, and Rosenthal (1991). Our results generalize and clarify the conditions under which their results hold. We then evaluate our results when voters are uncertain about how candidates will interact in the legislature. We end by considering the implications of nonseparable voter preferences for the formation of political parties and for the survival of incumbents.

What are nonseparable preferences?

Imagine a voter in a double-member district trying to decide for whom to cast her two votes. Suppose three candidates compete for two seats in the legislature and adopt positions $x_1$, $x_2$, and $x_3$ in $\mathbb{R}^1$ where the voter has single-peaked and symmetric preferences around her ideal point, $v_i$, as in Figure 1.

If the voter evaluates the candidates independently of each other, then she should vote for the two closest candidates. Her first vote will go to candidate 2, her second to candidate 1. The policy advocated by the elected candidates, if the candidates are true to their policy pronouncements, will be somewhere in the space $[x_1, x_2]$. Thus the problem: the voter’s ideal point is not in the segment. By voting for candidates 1 and 2, the voter knows that she has no chance of seeing her preferred policies realized.
Suppose the voter votes for candidates 1 and 3 or for candidates 2 and 3. The policy advocated by her representatives will be somewhere in \([x_1, x_3]\) or \([x_2, x_3]\), and the voter’s ideal policies may be realized. The problem for the voter is to anticipate how her two votes for candidates will be translated into policy outputs. If the two elected candidates adopt the same position, then the voter should expect that point to be the advocated policy. Certainly she should expect that the policy advocated by a pair of representatives should be a convex combination of their announced positions, or that the policy will be somewhere in the space bounded by the candidates’ positions.

In multi-member districts, voter preferences may not be defined over one candidate at a time but over sets of candidates. In Figure 1, we cannot say that the voter’s preference ordering over candidate positions is \(x_1 \succ_i x_2 \succ_i x_3\), where \(\succ_i\) indicates voter \(i\)’s strict preference. To do so is to oversimplify the voter’s preference relation and to ignore the fact that her votes may be interdependent. Instead, a voter’s preference ordering should be defined over \(k\)-tuples of candidate positions (Austen-Smith and Banks 1991). In Figure 1, the voter’s preference ordering would be something like \(\{x_1, x_3\} \succ_i \{x_2, x_3\} \succ_i \{x_1, x_2\}\).

In technical terms, we are suggesting that voters may have nonseparable preferences for candidates. To define separable preferences, we suppose voter \(i\) holds a strict preference, \(\succ_i\), over two different sets of legislator positions \(L\), where

\[
L_1 = (x_1, ..., x_{j-1}, x_j, x_{j+1}, ..., x_m) \succ_i (x_1, ..., x_{j-1}, x'_j, x_{j+1}, ..., x_m) = L_2
\]

If two sets of legislator positions, \(L_1\) and \(L_2\), differ only in \(x'_j\), and if \(L_1 \succ_i L_2\), then the assumption of separable preferences requires that

\[
L_3 = (x'_1, ..., x'_{j-1}, x_j, x''_{j+1}, ..., x''_m) \succ_i (x''_1, ..., x''_{j-1}, x'_j, x''_{j+1}, ..., x''_m) = L_4.
\]

To interpret, if a person has separable preferences and if she prefers \(x_j\) to \(x'_j\), then she does so regardless of the positions of other candidates. If this restriction on preferences is violated and a voter’s preference for candidate \(j\)’s position depends on the positions of other candidates, then the voter’s preferences are nonseparable.

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1Voter preferences are defined over the policy space. A voter then has induced preferences over candidates who adopt positions on the policy space. The induced preferences for candidates that are nonseparable.
Separability is a restriction imposed on preferences in nearly all of the formal models of elections and legislatures. Many researchers recognize the pivotal role of separable preferences in formal models (Kadane 1972; Kramer 1972; Schwartz 1977; Austen-Smith and Banks 1988, 1991; Benoit and Kornhauser 1994), but very little work derives general results describing the impact of nonseparable preferences in those models. In particular, no work has yet to derive the conditions of electoral equilibrium when voters have nonseparable preferences for candidates.

Electoral equilibrium with separable voter preferences

The simplest interesting case where voters may have nonseparable preferences is the double-member district, which provides the institutional context for our analysis. Double-member districts constitute an important pivotal case for the derivation of equilibrium results. We know that a single-member district with two candidates will tend to compel candidates to converge at the position of the median voter (Black 1958; Downs 1957). Single-member districts moderate electoral competition by drawing candidates toward the middle of the distribution of voters. We also know that multi-member districts tend to produce multiple electoral equilibria, including some in which candidates adopt positions in the extremes of the distribution of voters (Downs 1957; Cox 1990a). Double-member districts sit between these two cases. Cox (1984) shows that candidates in a double-member district will converge somewhere in the middle third of the distribution of voters, suggesting that double-member districts are more like single-member districts than like multi-member districts. As our results will show, nonseparable voter preferences make double-member districts look more like multi-member districts.²

Double-member districts are an important class of electoral institutions. They were once the norm in English elections (Cox 1987), and they are currently used to select members of the upper house of the Russian legislature.

²We suspect that nonseparable preferences in districts that elect at least three representatives will not change electoral competition dramatically since candidates already have incentives to diverge and possibly adopt extreme positions when voter preferences are separable. Therefore, double-member districts are the pivotal case where nonseparable preferences will change equilibrium results.
In the United States, fourteen American states use double-member districts to elect some state legislators, and Arizona and New Jersey rely on them exclusively (Grofman 1985:176; Cox 1984:445).

Cox (1984) derives the electoral equilibria in double-member districts with three candidates when voters have separable preferences and elections are simultaneous. His model assumes (with modified notation) that:

1. Each district elects the two candidates who receive the most votes.
2. Candidates compete along a single dimensional alternative space, $X$, which is a closed and connected subset of $\mathbb{R}^1$.
3. Each voter has single-peaked, symmetric preferences and an ideal point $v_i \in X$.
4. Each of N voters casts two votes (the total number of votes is 2N), one for each of two candidates. No partial abstention, or bullet-voting, is allowed.
5. Each candidate adopts a strategy $x_a \in X$ and maximizes the proportion of the vote he or she receives.
6. Candidates can move costlessly in $X$.
7. Each voter votes for the two candidates whose strategies are closest to $v_i$, resolving any ties equiprobably (Cox 1984:445). The utility of voter $i$ for any pair of candidates $a$ and $b$ is:

$$U_i = -|v_i - x_a| - |v_i - x_b|$$

Since the voter’s utility for any candidate is unrelated to the positions of other candidates, the voter’s preferences are separable. The voter treats her first vote as independent of her second.

Cox proves that in a three candidate race with candidate positions $(x_1, x_2, x_3)$, the Nash equilibrium is the strategy profile $x_1 = x_2 = x_3 = x^*$, where $x^*$ inhabits the middle third of the distribution of voter ideal points. To illustrate the equilibrium, suppose that three candidates compete in a space $[0, 1]$ occupied by a uniform distribution of voters. If all three candidates are positioned at 0.35, can any candidate move to win? If one candidate moves to 0.45, then that candidate receives one vote from every voter in $[0.4, 1]$ for a

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3A wealth of theoretical and empirical evidence demonstrates that in double-member districts with simultaneous elections, three candidates usually compete for seats. This is a special case of the $M + 1$ rule, which holds that under many electoral arrangements the optimal number of candidates in an election is one greater than the number of seats available in the legislature (Reed 1990; Taagepera and Shugart 1989; Cox 1993).
total of 0.6 votes. The two candidates at 0.35 each receive one vote from the voters in [0,0.4] and one-half of the second votes from every voter in [0.4, 1], for a total of 0.7 votes. The latter two candidates win and the candidate who moves loses.

What if all three candidates are outside of the middle-third of voter ideal points, say at 0.3? A move by one candidate to within the middle-third of voters, say at 0.34, earns that candidate one vote from every voter in [0.32, 1] for a total of 0.68 votes. The remaining two candidates each get the 0.32 votes to their left and split the 0.68 votes to their right, for a total of 0.66 votes. In this case, the candidate who moves wins.

When voters have separable preferences, candidates are compelled to adopt identical positions in the middle of the distribution of voters. Double-member districts are much like single-member districts: candidates converge to moderate positions. As Cox concludes, “the centrist bias of the single-member district is not unique” (Cox 1984:449-50). An assumption of electoral engineering is that single-member and double-member districts share centripetal properties while multi-member (M at least three?) districts will share centrifugal properties (Cox 1990). This assumption may not be true when voters have nonseparable preferences for candidates.

Electoral equilibrium with nonseparable preferences and simultaneous elections

To derive equilibrium results for candidate competition, we must model voters’ expectations about the policies that pairs of candidates will advocate. We begin by assuming that voters know the candidates’ positions, \( x_a \), \( x_b \), and \( x_c \), and anticipate the policies that will be advocated by pairs of candidates, such as \( x_{ab} \), \( x_{ac} \), and \( x_{bc} \). While different voters may expect different things from the same pair of candidates, we assume that there is a common pattern to voter evaluations of candidate pairs: voters should anticipate that pairs of candidates will advocate policies that are somewhere between the candidates’ positions. The precise form of the interaction between two candidates once in office may depend on several things, such as whether one candidate is an incumbent running against political novices or whether one candidate has accumulated enough seniority to occupy a position of power.
in the legislature. Variables such as incumbency and seniority are likely to be recognized by voters and factored into the expectations that voters hold over how pairs of candidates will interact.

For purposes of deriving equilibria and keeping our model parsimonious, we simplify voters’ expectations about the policies advocated by a pair of candidates by assuming that voters believe the policy outputs of their representatives will be the midpoint of the positions of the candidates who are elected. We do not believe that voters always and everywhere look for the midpoint of candidate positions in double-member district elections. Instead, our assumption captures as simply as possible the idea that voters care about packages of candidates rather than individual candidates. More generally, we can think of voters as anticipating that the policy output from a pair of candidates is represented by a uniform probability distribution on the convex hull of the candidates’ announced positions. This probability distribution has an expectation equal to the midpoint of the announced positions. We assume formally that voters expect that candidates with positions $x_a$ and $x_b$ will advocate the policy $\frac{x_a + x_b}{2}$ in the legislature. In Figure 1 the average of candidates 1 and 3 is very near $v_i$:

$$|v_i - \frac{x_1 + x_3}{2}| < |v_i - \frac{x_2 + x_3}{2}| < |v_i - \frac{x_1 + x_2}{2}|$$

and voter $i$’s induced preference ordering for candidates becomes:

$$\{x_1, x_3\} \succ_i \{x_2, x_3\} \succ_i \{x_1, x_2\}.$$  

The utility-maximizing voter in Figure 1 should vote not for the candidates closest to her (1 and 2), but for the two candidates at opposite ends of the alternative space (1 and 3). Interestingly, in this example the voter’s nonseparable preference ordering for sets of candidates is exactly the reverse of what her preferences would be if they were separable.

To see why a voter who thinks this way has nonseparable preferences, consider Figure 2. In this case $\{x_1, x_3\} \succ_i \{x_2, x_3\}$. Now suppose $x_3$ moves to $x_3'$. The preference ordering changes to $\{x_2, x_3'\} \succ_i \{x_1, x_3'\}$. In one case, $\{x_1, \cdot\} \succ_i \{x_2, \cdot\}$; in the other, $\{x_2, \cdot\} \succ_i \{x_1, \cdot\}$. This violates the definition of separable preferences. Since the voter’s induced preference ordering over candidate positions $x_1$ and $x_2$ depends on $x_3$, the voter’s preferences are nonseparable.
To reconsider the equilibrium results under nonseparable voter preferences in a three-candidate field, we adopt the assumptions from Cox’s model with two changes: 4

(5′) Each candidate adopts a strategy \( x_a \in X \) that maximizes the probability of winning. 5

(7′) Each voter votes for the two candidates whose average position is closest to \( v_i \). The utility of voter \( i \) for any pair of candidates \( a \) and \( b \) is:

\[
U_i = -|v_i - \frac{x_a + x_b}{2}|
\]  (2)

In an election with candidates 1, 2, and 3, we represent the outcomes associated with each pair of candidates as \( x_{12}, x_{13}, \) and \( x_{23} \). The midpoint between two candidate combinations, say \( x_{12} \) and \( x_{13} \), is \( \frac{x_{12} + x_{13}}{2} \). We denote the number of voters to the right of a point, say \( x^* \), as \( n(\infty, x^*) \); and the number of voters at a point, say the midpoint of \( x_{12} \) and \( x_{13} \), as \( n(x_{12}\parallel x_{13}) \). The notation \( n(12) \) indicates the number of voters in the open interval from which candidates 1 and 2 receive votes. We designate the voters in \( n(12) + \frac{1}{2}n(x_{12}\parallel x_{13}) \) as I, the voters in \( n(13) + \frac{1}{2}n(x_{13}\parallel x_{23}) \) as II, and the voters in \( n(23) + \frac{1}{2}n(x_{13}\parallel x_{23}) \) as III. Figure 3 illustrates the notation.

\[
V_1(x_1, x_2, x_3) \quad \text{is the number of votes received by candidate 1, given the strategy profile} \quad (x_1, x_2, x_3).
\]

In a three-candidate race, each candidate’s vote is calculated as follows:

\[
V_1(x_1, x_2, x_3) = n(12) + n(x_{12}\parallel x_{13}) + n(13) + \frac{1}{2}n(x_{13}\parallel x_{23}) = I + II
\]

\[
V_2(x_1, x_2, x_3) = n(12) + \frac{1}{2}n(x_{12}\parallel x_{13}) + \frac{1}{2}n(x_{13}\parallel x_{23}) + n(23) = I + III
\]

\[
V_3(x_1, x_2, x_3) = \frac{1}{2}n(x_{12}\parallel x_{13}) + n(13) + n(x_{13}\parallel x_{23}) + n(23) = II + III
\]

4Cox’s assumption (4) holds that voters cannot cast partial or cumulative votes. With nonseparable voter preferences, we could generalize the assumption to allow partial or cumulative voting, but voters with nonseparable preferences will never cast cumulative or partial ballots. We omit the proof but sketch the logic: Voters who have nonseparable preferences in double-member districts evaluate packages of candidates rather than individual candidates, therefore it is always a dominant strategy for a voter to contribute to the vote total of her two most preferred candidates. Empirically, some voters do cast partial ballots in multi-member district elections. Such voters are usually minorities who seek to guarantee the election of minority candidates, and their preferences cannot be captured by the policy-oriented utility function that we adopt.

5We adopt this assumption only to simplify the proofs. Retaining Cox’s Assumption 5 would not alter the results.
Notice that the candidate in the middle, candidate 2, does not get votes from the voters in the middle region, II. Those votes go to candidates 1 and 3. Instead, candidate 2, the moderate, gets the votes from voters at the extremes. Notice also that the candidates at either end of the space cast their votes for the two closest candidates. Their votes will be the same regardless of whether their preferences are separable or nonseparable. But the voters in the middle region who have nonseparable preferences can dramatically alter candidate competition, as we demonstrate with the following result:

**Result 1** When voters have nonseparable preferences for candidates in a double-member district, candidates may adopt any of several positions in equilibrium:

1. all three candidates may adopt the same position in the middle third of the distribution of voters,
2. two candidates may adopt the same position and a third will stand alone, or
3. all three candidates will adopt different positions.

Result 1 describes the Nash equilibria when voters have nonseparable preferences for candidates (see Lemma 1 and Theorem 1 in Appendix for a formal statement of the result). The first set of equilibria holds that all three candidates adopt the same position in the middle third of the distribution of voters, which is the only set of equilibria when voter preferences are separable. To see why this is an equilibrium, assume that voters are distributed uniformly on a space bounded by 0 and 1 and that the total number of votes is 2, the same as in our example with separable preferences. All three candidates are located at 0.5, thus there is a three-way tie with each candidate receiving 0.67 votes. Can any candidate do better by moving? If a candidate, say 3, moves to 0.7, then the midpoint of her position with candidates 1 and 2 is 0.6. Only voters from 0.55 to 1 will vote for candidate 3, so her vote has dropped from 0.67 to 0.45, not a good move. As long as candidates 1 and 2 are located together anywhere between 0.33 and 0.67, candidate 3 can do nothing but move with them. If she breaks from their position, she loses.

Conversely, three candidates located together outside of the middle third of the distribution of voters cannot be an equilibrium. Suppose $x_1 = x_2 = x_3 = 0.3$. If candidate 3 moves to 0.34, then the midpoint of $x_{12}$ and $x_{13}$ or $x_{23}$ is 0.32. All voters in $[0.31, 1]$ will vote for candidate 3 and either
candidate 1 or 2. All voters in [0, 0.31] will vote for candidates 1 and 2. In this case candidate 3 clearly wins by earning 0.69, which is greater than $\frac{2N}{3}$. The equilibrium where all three candidates adopt the same position in the middle-third of voter ideal points is not confined to a uniform distribution of voters. It can exist in any distribution of voters, as is the case with separable voter preferences.

What makes nonseparable voter preferences important are the second and third sets of equilibria, neither of which can exist when voter preferences are separable. In the second set of equilibria, two candidates adopt the same position, a third stands alone, and none of the candidates occupy the middle third of the distribution of voters. There are two variations within this set of equilibria. In the first variation, the stand-alone candidate wins while the other two tie, and voters must be distributed such that more voters are in II than in either I or III. To see why this is an equilibrium, consider the following example: ten voters are at $x_{13}$ and five voters are at each of $x_1$ and $x_3$. Candidate 3 wins five votes at $x_3$ and ten votes at $x_{13}$, for a total of fifteen, one more than the 14 needed to guarantee victory, so she has no desire to move. Can either candidate 1 or 2 move? If 2 moves while 1 does not, then the ten votes at $x_{13}$ stay with candidates 1 and 3. Candidate 2 can get at most the other ten votes. The same applies to candidate 1 if she tries to move while candidate 2 stays. In general, the second set of equilibria will hold for distributions of voters that have more voters between the two candidates’ positions than on either side of them. Centrist distributions of voters are quite common in politics, thus we should expect these equilibria in many real-world cases.

The second set of equilibria has a variation where all three candidates tie. In this case, identically sized groups of voters are positioned at $x_1$, $x_{13}$, and $x_3$. Candidate 3 cannot get the voters at $x_1$ unless she moves to $x_1$, in which case she still has a three-way tie. For one of the other candidates, say 2, to have an incentive to move away from $x_1 = x_2$, she must expect to beat candidate 1 or candidate 3. Since any move between $x_1$ and $x_3$ loses all of the $\frac{N}{3}$ voters at $x_{13}$, then 2 will still tie candidates 1 and 3. If 2 moves to $x_3$, she is in a tie for second place with candidate 3 since candidate 1 now wins all of the voters at $x_1$ and $x_{13}$. If 2 moves to the right of $x_3$, then she loses the $\frac{N}{3}$ voters at $x_1$ and remains in a three-way tie. The distribution of voters required for this set of equilibria is a bit extreme politically, and we should not expect to find many real-world cases with three identically-sized groups


of voters.

In the third set of equilibria (see Lemma 1 in Appendix), all three candidates can adopt different positions, which is unexpected given the purported similarities between single-member and double-member districts. As an illustration, suppose voters and candidates are distributed as in Figure 4.

Figure 4 includes 20 voters with a total of 40 votes. Three conditions constrain the distribution of voters: no voters inhabit \([x_{12}, x_{13}]\), less than one third of the voters sit to the left of \(x_{1}\) or to the right of \(x_{2}\), and the number of voters to the left of \(x_{12}\) equals the number to the right of \(x_{13}\). Given this array of candidates and voters, no candidate can move to increase her probability of winning. Candidate 1 is winning since she gets one vote from every voter. Candidates 2 and 3 are tied for second place, so each seeks a move that will increase her share of the vote. But since one of these equally sized groups, or half of the total voters, is stationed close to \(x_{12}\), then regardless of where 3 moves, she cannot get any of these votes. The same holds for 2, who cannot win any of the votes near \(x_{13}\). There is one more possibility: candidate 3 could move to \(x_{2}\). But doing so would mean that candidates 2 and 3 get the votes of those voters to the right of \(x_{12}\), which is less than one-third of the voters. Candidate 1 still receives at least two-thirds of the votes while 2 and 3 are still tied. The move to \(x_{2}\) does nothing for candidate 3.

The third set of equilibria will hold only for very restrictive distributions of voter ideal points. Voters must be broken into two identically-sized groups with no voters between them. The conditions of the third set of equilibria correspond to a polarized electorate, which may be quite common in politics, especially in small groups. The first and second sets of equilibria will coexist in centrist distributions of voters, which are certainly common in politics. With a uniform distribution of voters, only the first set of equilibria is possible.

Assuming nonseparable voter preferences maintains the set of equilibria derived by Cox, while adding two other sets of Nash equilibria. Cox finds that the set of equilibria in the three-candidate case is symmetric about the ideal point of the median voter and limited to the middle third of the distribution of voter ideal points (Cox 1984:447). Without the assumption that voter preferences are separable, this centrist bias is weakened. Extreme candidate positions can be in equilibrium. The result also shows that when voters have nonseparable preferences, candidates are no longer compelled to adopt
identical positions. Our result is simple: when voters base their preferences on combinations of candidates rather than individual candidates, then the centrist bias of double-member districts breaks down.

**Voter uncertainty**

Thus far we have treated as certain the voters’ estimates of the policy outputs of pairs of candidates. Now we add a measure of voter uncertainty by discounting any candidate combination by the variance between candidate positions. The idea behind our formulation is simple: voters are averse to voting for a pair of candidates whose policy positions are far from each other. To capture this notion, we alter our assumption about the voter utility function in simultaneous elections to:

\[(7')\] Each voter votes for the combination of two candidates whose average position is closest to \(v_i\), discounting the variance between the candidates. This yields the following utility function for voter \(i\) over any pair of candidates \(a\) and \(b\):

\[
U_i = -|v_i - \frac{x_a + x_b}{2}| - \delta|x_a - x_b|, \quad \delta \geq 0
\]  

(3)

The voter’s utility is a function of both the distance from her position to the candidates’ midpoint and the variance between the two candidates. The \(\delta\) in the voter utility function captures the idea that as the range spanned by a pair of candidates increases, a voter becomes more uncertain about the policy output of the pair and more averse to voting for that pair. As \(\delta\) increases, a voter is more averse to candidate variance. Voters may have to make trade-offs between distance and variance when evaluating pairs of candidates. We do not include in the theorems the values of \(\delta\) for which the equilibria hold since this varies across configurations of candidates. We can, however, offer the following statement about how uncertainty affects nonseparable preferences:

**Result 2** As voters become more uncertain of how pairs of candidates will interact in the legislature, the effects of nonseparable preferences are attenuated and candidates will behave as though voter preferences are separable.

For example, in Result 1 the first set of equilibria prescribes that candidates will adopt identical positions in the middle third of distribution of
voters. The equilibria hold for any $\delta \geq 0$. Any candidate who breaks away from the pack gets less than $\frac{2N}{3}$ of the vote when voters dismiss the distance between candidates. Including aversion to candidate variance does not help a break-away candidate get more votes. The two candidates who stay together get all of the votes of at least one-third of the voters (to the side opposite the break-away candidate), and they split the votes of the remaining two-thirds, thereby earning at least two-thirds of the total vote. Values of $\delta$ above zero will only detract from the votes of the break-away candidate. No candidate will have an incentive to move away from the others if all are identically positioned in the middle third of the distribution of voters.

The same is not true for the case where two candidates adopt identical positions and one candidate stands alone. We know that any voter to the left of $x_{12|13}$ will vote for candidate set $\{1, 2\}$. Some voters to the right of that point may vote for $\{1, 2\}$ instead of $\{1, 3\}$ or $\{2, 3\}$ since candidates 1 and 2 minimize voter uncertainty by adopting the same position. We want to know for what values of $\delta$ voters to the right of $x_{12|13}$ will continue to cast a vote for candidate 3. To calculate this, we set $x_{12|13} = 0$, $x_1 = x_2 = -1$, $x_{13} = 1$, and $x_3 = 3$. For a voter $\varepsilon$ to the right of $x_{12|13}$:

\[
U_i(12) = -|\varepsilon + 1| - \delta|0 - 0| = -\varepsilon - 1
\]

\[
U_i(13) = U_i(23) = -|1 - \varepsilon| - \delta|1 - 3| = -1 + \varepsilon - 4\delta
\]

Since the voter’s objective is to maximize the utility of a candidate combination, we look for values of $\delta$ that make $U_i(13) = U_i(23) > U_i(12)$:

\[-1 + \varepsilon - 4\delta > -\varepsilon - 1\]

\[\delta < \frac{\varepsilon}{2}\]

As long as a voter’s discount term over candidate variance, $\delta$, is less than half of the value of the distance from $v_i$ to $x_{12|13}$, the voter will vote for candidate 3. This preserves the set of equilibria.

The third set of equilibria, where all three candidates adopt divergent positions, is maintained when $\delta < \frac{1}{4}$ for all voters at $x_{12|13}$. For the equilibria to hold, two equal-sized groups must be stationed at or close to $x_{12}$ and $x_{13}$. The voters at $x_{12}$ are already voting for the two closest candidates, 1 and 2, so different values of $\delta$ will not alter their votes. Voters at $x_{13}$ may cast votes for $\{1, 2\}$, $\{2, 3\}$, or $\{1, 3\}$ depending on their values of $\delta$. Let us set
\( x_1 = -5, \ x_{12} = -2, \ x_{12\mid13} = -1, \ x_{13} = 0, \ x_2 = 1, \ x_3 = 5 \). For a voter at \( x_{13} \):

\[
\begin{align*}
U_i(13) &= -|0 - 0| - \delta|10| = -10\delta \\
U_i(12) &= -|0 + 2| - \delta|6| = -2 - 6\delta \\
U_i(23) &= -|0 - 3| - \delta|4| = -3 - 4\delta
\end{align*}
\]

When \( \delta < \frac{1}{2} \), voters at \( x_{13} \) will gain more by voting for 1 and 3 than for either 1 and 2 or 2 and 3. This preserves the third set of equilibria.

When \( \delta \) rises above \( \frac{1}{2} \), then fewer voters act as though their preferences are nonseparable. Region II will quickly shrink, and the candidate in the middle will get the votes in the middle. All candidates will want to be in the middle, and the equilibrium will collapse to \((x_1, x_2, x_3)\), where \( x_1 = x_2 = x_3 = x^* \) and \( x^* \) is in the middle third of the distribution of voters. The equilibrium will be as Cox’s result prescribes. Uncertainty over candidate combinations can negate the effects of nonseparable preferences above certain values of \( \delta \).

**Implications of the results**

Political scientists have put much stock in the belief that candidate strategies in elections and, therefore, public policy, can be engineered by manipulating variables such as the number of legislators elected from a district. The conventional wisdom maintains that single-member districts induce candidates to converge to the position of the median voter (Downs 1957; Black 1958). Double-member districts act much like single-member districts when voter preferences are separable: candidates converge somewhere in the middle-third of the distribution of voters (Cox 1984, 1990a). District magnitudes of three or more tend to reduce the incentives for candidates to converge. In these districts, candidates will often adopt extreme positions (Cox 1990a). As a matter of constitutional engineering, single and double-member districts should be adopted by polities that desire centrist candidates and policies, while districts of larger magnitude should be adopted by polities that want to remove barriers to non-centrist candidates and parties.

The results in this paper suggest that the conventional wisdom needs modification when voter preferences are nonseparable. Candidates can diverge and adopt extreme positions in equilibrium in double-member districts when voter preferences are nonseparable. Double-member districts will look
more like multi-member districts than like single-member districts. Polities that want to create incentives for centrist candidates may need to think twice about double-member districts.

Nonseparable voter preferences will likely change other aspects of candidate competition since candidates will have incentives to campaign together or to announce their positions later in the campaign season. When voters have separable preferences, the dominant strategy of every candidate, ignorant of other candidates’ strategies, is to move to the middle of the distribution of voters. But when voters have nonseparable preferences and are distributed with a mass of voters in the middle, then any candidate that moves to the middle risks being flanked and beaten by the other two candidates (see statement 2 in Theorem 1 in Appendix). Candidates cannot move safely to the middle unless they move with a partner. When voter preferences are nonseparable, a pair of candidates positioned in the middle will always have at least a two-thirds probability of winning. Two candidates will have an incentive to agree to the same platform somewhere in the middle third of the distribution of voters. Candidate partnership sounds much like a political party, and it is not something one would expect in a world of separable voter preferences where each candidate wins or loses on her own platform.

Nonseparable voter preferences thus hold important consequences for candidate strategy beyond the simple prediction in Result 1 that candidates may diverge. When voters evaluate candidates as packages, then candidates are likely to run as packages. They are also likely to pay more careful attention to the strategies adopted by other candidates. Double-member districts will not mirror single-member districts, where candidates in a two-candidate race will tend to jump to the median regardless of the strategies adopted by other candidates.

Conclusions

To demonstrate the impact of nonseparable voter preferences on elections, we examine the simple and pivotal case of double-member districts. We revise one of the assumptions behind a model by Cox (1984) to show the impact of nonseparable preferences on electoral equilibrium. We do not argue that Cox’s model is wrong. Rather, Cox’s model is exactly as it should be since each assumption, including separability of preferences, is critical to the re-
result. Separability of preferences is a substantively restrictive assumption, and when that assumption is changed, double-member districts look quite different from the predictions of Cox’s model. But double-member districts are a starting point, not an ending point, for studies of nonseparable preferences in elections. Our results extend to many electoral institutions beyond double-member districts. Voters in single-member districts may have nonseparable preferences if they consider representatives from other districts. Voters in congressional elections may have nonseparable preferences if they consider the position of the president, and vice-versa (Alesina and Rosenthal 1989; Erickson 1988; Fiorina 1988, 1991). Our results will also apply to the selection of members of a committee and to the appointment of judges to a bench.

While the vast majority of research in political science assumes that voter preferences are separable, in many instances this is an unnecessarily restrictive assumption that generates inaccurate predictions about the behavior of voters and candidates. To relax this assumption pushes us toward greater predictive power and greater empirical accuracy in studies of elections. Many of the empirical irregularities in voting behavior and candidate competition that are so often attributed to irrationality, disequilibrium, and instability may actually be perfectly rational, in equilibrium, and stable once we remove assumptions such as separability that restrict voter preferences.
Appendix

We begin with an observation about the region in which voters cast their votes for candidates 1 and 3.

**Remark 1** In a first-two-past-the-post election with three candidates \((x_1, x_2, x_3)\), where \(x_1 < x_2 < x_3\), \(x_{13|23} - x_{12|13} = \frac{x_3 - x_1}{4}\).

Remark 1 shows that the region from which candidates 1 and 3 both receive votes, and from which candidate 2 receives no votes, is equal to one quarter of the distance between \(x_1\) and \(x_3\). This property proves useful in describing the set of Nash equilibria where three candidates adopt different positions.

**Proof of Remark 1:**

\[
\begin{align*}
x_{13} &= \frac{x_1 + x_3}{2} \\
x_{12} &= \frac{x_1 + x_2}{2} \\
x_{23} &= \frac{x_2 + x_3}{2} \\
x_{12|13} &= \frac{2x_1 + x_2 + x_3}{4} \\
x_{13|23} &= \frac{x_1 + x_2 + 2x_3}{4} \\
x_{13|23} - x_{12|13} &= \frac{x_3 - x_1}{4}
\end{align*}
\]

We need some additional notation before moving to the equilibrium conditions: \(\alpha\) indicates the distance that candidate 2’s region of support moves when candidate 2 moves while \(\beta\) is the distance that candidate 3’s region of support moves when candidate 3 moves: \(\alpha = \frac{x_2 - x_2'}{4}\) where \(x_2'\) is any new position for candidate 2, and \(\beta = \frac{x_3 - x_3'}{4}\) where \(x_3'\) is any new position for candidate 3.

**Lemma 1** In a first-two-past-the-post election with three candidates, \((x_1, x_2, x_3)\), where \(x_1 < x_2 < x_3\), is a Nash equilibrium if and only if:
II. Now suppose 3 moves to \( x \), the three candidates are all tied, which implies that I = II = III. Two of the other candidates. Therefore, the situation where two candidates win and one loses cannot be an equilibrium.

First, two of the candidates win and one loses. The losing candidate will always be able to at least secure a tie for second-place by moving to the position of one of the other candidates. Therefore, the situation where two candidates win and one loses cannot be an equilibrium.

Second, the three candidates are all tied, which implies that I = II = III. This cannot be an equilibrium. Suppose 3 moves to \( x + \varepsilon \) and becomes the middle candidate. For 3 to win, either \( I'(x_1, x_2, x_1 + \varepsilon) + \Pi'(x_1, x_2, x_1 + \varepsilon) > I'(x_1, x_2, x_1 + \varepsilon) + \Pi'(x_1, x_2, x_1 + \varepsilon) \) or \( I'(x_1, x_2, x_1 + \varepsilon) + \Pi'(x_1, x_2, x_1 + \varepsilon) > \Pi'(x_1, x_2, x_1 + \varepsilon) + \Pi'(x_1, x_2, x_1 + \varepsilon) \). After 3 moves to \( x_1 + \varepsilon \), the upper bound of \( \Pi' \) becomes \( x_1 + \frac{\varepsilon}{2} \) since the mid-point of \( x_1 \) and \( x_1 + \frac{\varepsilon}{2} \) is \( x_1 + \frac{\varepsilon}{2} \).

Since \( \varepsilon \) can be arbitrarily small, \( x_1 + \frac{\varepsilon}{2} < x_{12|3} \), the lower limit of II. But it is given that II + III > I, so \( II' < III' \) since \( III' \geq II + III \). This proves that 3 will be winning by moving to \( x_1 + \varepsilon \).

Third, one candidate wins and the other two tie for second-place. There are two variations on this outcome: the winning candidate is in the middle (\( V_2 = V_3 \)), or the winning candidate is at one flank (\( V_2 > V_3 \)). If \( V_2 > V_1 = V_3 \), then I + III > I + II = II + III, which implies that I = III > II. Now suppose 3 moves to \( x_2 \). Then 2 and 3 each get one vote from every voter in \( x_{12|2, \infty} \). They also each get one half of the second votes from voters in \(-\infty, x_{12|2}\). But \( n(x_{12|2}, \infty) \geq n(x_{13|23}, \infty) > \frac{N}{3} \). By moving to \( x_2 \), candidate 3 can secure more than \( \frac{2N}{3} \) of the votes since \((\frac{N}{3} + \varepsilon) + \frac{1}{2}(\frac{2N}{3} - \varepsilon) \geq \frac{2N}{3} \).

Thus far we have shown that when condition (1) of Lemma 1 is not satisfied, the candidate profile \((x_1, x_2, x_3)\), where \( x_1 < x_2 < x_3 \), is not an equilibrium. Now we show that when any of the remaining conditions is violated, the candidate profile cannot be an equilibrium.

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When condition (2) is violated, then either
\[ \Pi = n(x_{13} - \alpha, x_{13} - \alpha + \frac{x_1-x_3}{4}) < n[x_{13} - \alpha + \frac{x_1-x_3}{4}, \infty) = III, \]
or
\[ \Pi = n(x_{13} - \alpha, x_{13} + \frac{x_1-x_3}{4} - \alpha) < n(-\infty, x_{13} - \alpha) = I. \] This implies that
either \( V_2 > V_1 \) or \( V_2 > V_3 \). In either case, candidate 2 can find a position \( x'_2 \)
that secures a win.

When condition (3) is violated, by the same reasoning, candidate 3 can secure
a win by finding an \( x'_3 \) such that \( x_1 < x'_3 < x_2 \).

When condition (4) is violated, then 2 can secure a win by moving to \( x_1 \), since
if \( n(-\infty, x_{1|13}) + \frac{1}{2}n(x_{13}) > \frac{N}{3} \), then \( V_1 = V_2 = (\frac{N}{3} + \varepsilon) + \frac{1}{2}(1 - \frac{N}{3} - \varepsilon) > \frac{2}{3}N \).
So this cannot be an equilibrium.

Finally, when condition (5) is violated, by the same reasoning, 3 can win by
moving to \( x_2 \).

Next, we show that when conditions (1) through (5) are satisfied, there
is no unilateral move by any candidate that will improve her chances of
winning. It is straightforward that when conditions (2) and (4) are satisfied,
candidate 2 cannot find an \( x'_2 \in [x_1, x_3] \) to secure a win. Possible strategies
for candidate 2 that are not covered by conditions (2) and (4) are \( x'_2 \geq x_3 \) or
\( x'_2 < x_1 \). If \( x'_2 < x_1 \), then \( x'_{23|13} < x_{13} \). By (2),
\( n(x_{13} - \alpha, x_{13} - \alpha + \frac{x_1-x_3}{4}) \geq n(-\infty, x_{13} - \alpha) \). By (3),
\( n(x_{12} + \beta - \frac{x_1-x_3}{4}, x_{12} + \beta) \geq n(x_{12} + \beta, \infty) \). We know
that \( x_{12} < x_{13} \). This implies that \( n(x_{13}, x_{13} + \frac{x_1-x_3}{4}) = n(x_{12} - \frac{x_1-x_3}{4}, x_{12}) = \frac{N}{3} \).
Therefore, 2 cannot be winning by moving to the left of \( x_1 \). If \( x'_2 = x_3 \),
then \( x'_{13|23} = \frac{3x_{13}+x_1}{4} \) and \( V'_2 = \frac{1}{2}n(-\infty, x_{13|23}) + \frac{1}{2}n(x_{13|23}) + n(x_{13|23}, \infty) \).
But we know that \( n(\frac{3x_{13}+x_1}{4}, \infty) = 0 \), so candidate 2’s expected vote is \( \frac{N}{3} \),
which does not secure a win. If \( x'_2 > x_3 \), then \( V'_2 \leq n(x_{13}, \infty) \leq \frac{N}{3} \). Candidate 2
cannot win. The strategy for candidate 3 not covered by conditions (3) and
(5) is \( x'_3 \leq x_1 \). If \( x'_3 = x_1 \), then, by condition (4),
\( n(-\infty, x_{1|13}) + \frac{1}{2}n(x_{1|13}) \leq \frac{N}{3} \). Since \( x_{1|12} < x_{1|13} \), then \( V'_3 \leq \frac{2N}{3} \). If \( x'_3 < x_1 \), then 3 can secure votes
from the region \( (-\infty, x'_{23|12}) \). But since \( x'_{23|12} < x_{12} \), \( V'_3 \leq \frac{N}{2} \) and 3 loses. \( \square \)

**Theorem 1** In a first-two-past-the-post election with three candidates, \((x_1, x_2, x_3)\)
is a Nash equilibrium if and only if:

1. \( x_1 = x_2 = x_3 = x^* \), \( n(x^*, \infty) \geq \frac{N}{3} \), and \( n(-\infty, x^*) \geq \frac{N}{3} \),
2. \( x_1 = x_2 < x_3 \), \( n(x_{13} - \alpha, x_{13} - \alpha + \frac{x_1-x_3}{4}) \geq n[x_{13} - \alpha + \frac{x_1-x_3}{4}, \infty) \) and
\( n(x_{13} - \alpha, x_{13} - \alpha + \frac{x_1-x_3}{4}) \geq n(-\infty, x_{13} - \alpha) \), where \( 0 < \alpha < \frac{x_1-x_3}{4} \), and
(a) \( V_1 = V_2 < V_3 \), or
(b) \( V_1 = V_2 = V_3 \) and \( n(x_1, \infty) = \frac{2N}{3} \),
or \( x_1 < x_2 < x_3 \) and conditions (1) through (5) in Lemma 1 are satisfied.

Proof of Theorem 1: We first establish sufficiency for a Nash equilibrium.

(1) \( x_1 = x_2 = x_3 = x^* \), \( n(x^*, \infty) \geq \frac{N}{3} \), and \( n(-\infty, x^*) \geq \frac{N}{3} \). Each candidate’s expected vote is \( \frac{2N}{3} \). If 3 moves in any direction, she gets one vote from every voter in the segment bounded by \( (x_{1|3}) \) and the farthest voter in the direction away from the other candidates. This is less than \( \frac{2N}{3} \) of the votes since less than \( \frac{2}{3} \) of the voters are to either side of \( x^* \).

(2) \( x_1 < x_2 < x_3 \), \( V_3 \geq \frac{2N}{3} \). When \( n(x_{13} - \alpha, x_{13} - \alpha + \frac{x_{13}-x_1}{4}) \geq n(x_{13} - \alpha + \frac{x_{13}-x_1}{4}, \infty) \), and \( n(x_{13} - \alpha, x_{13} - \alpha + \frac{x_{13}-x_1}{4}) \geq n(-\infty, x_{13} - \alpha) \), where \( 0 < \alpha < \frac{x_{13}-x_1}{4} \), then 2 cannot beat 1 or 3. When \( V_3 \geq \frac{2N}{3} \) then 3 has no incentive to move. But when \( V_3 = \frac{2N}{3} \), then \( V_1 = V_2 = V_3 \). Since \( n(x_1, \infty) = \frac{2N}{3} \), then for any move by 3 to \( x_1 + \varepsilon \), 3 can get no more than \( \frac{2N}{3} \). If 3 moves to \( x_1 = x_2 \), she still gets \( \frac{2N}{3} \). Therefore, 3 has no incentive to move.

(3) See Lemma 1.

Next, we establish that no other combination of candidate strategies can produce a Nash equilibrium. There are four remaining cases.

(1) \( x_1 < x_2 < x_3 \) and conditions (1) through (5) of Lemma 1 are not satisfied. See Lemma 1.

(2) \( x_1 = x_2 < x_3 \) and \( V_3 < \frac{2N}{3} \). Candidate 3 can improve her chances of winning by moving to \( x_1 = x_2 \).

(3) \( x_1 = x_2 < x_3 \) and \( V_3 \geq \frac{2N}{3} \). If either \( n(x_{13} - \alpha, x_{13} - \alpha + \frac{x_{13}-x_1}{4}) < n(x_{13} - \alpha + \frac{x_{13}-x_1}{4}, \infty) \), or \( n(x_{13} - \alpha, x_{13} - \alpha + \frac{x_{13}-x_1}{4}) < n(-\infty, x_{13} - \alpha) \), where \( 0 < \alpha < \frac{x_{13}-x_1}{4} \), then 2 can move to beat 1 or 3. The same applies to any move by candidate 1. If \( V_3 = \frac{2N}{3} \) and \( n(x_1, \infty) > \frac{2N}{3} \), then 3 can win by moving to \( x_1 + \varepsilon \).

(4) \( x_1 = x_2 = x_3 = x^* \) and \( n(x^*, \infty) < \frac{N}{3} \), or \( n(-\infty, x^*) < \frac{N}{3} \). Each candidate’s expected vote is \( \frac{2N}{3} \). In the case of \( n(-\infty, x^*) < \frac{N}{3} \), if 3 moves toward the median she gets one vote from all \( x \in [x_{1|3}, \infty) \), which is greater than \( \frac{2N}{3} \).

\[ \square \]

Theorem 2 In a double-member district with at least one voter at the median position \( v^* \), a previously elected legislator at \( x_0 \), and two candidates
competing for one remaining seat, \((x_1, x_2)\) is a Nash equilibrium if and only if \(x_1 = x_2 = 2v^* - x_0\).

We add the following notation to prove Theorem 2: \(x_0\) is the position of the incumbent, \(x_0a\) is the midpoint of \((x_0, x_a)\), and \((x_{0a}, x_{0b})\) is the midpoint of \((x_{0a}, x_{0b})\).

**Proof of Theorem 2:** Consider the possible cases.

(1) \(x_1 \neq x_2\). \((x_{01} \neq x_{02})\)

(a) \(v^* < x_{01} < x_{02}\). Candidate 1 gets votes from all voters in \((-\infty, x_{01}\cup x_{02}]\), which is more than the \(\frac{N}{2} + 1\) votes needed to win. Candidate 2 can win if she moves anywhere in \((-\infty, x_1)\). If \(x_{02} < x_{01} < v^*\), then 2 wins by moving into \((x_1, \infty)\).

(b) \(x_{01} < v^* < x_{02}\). Candidate 1 gets the votes of those voters to her left while 2 gets the votes of the those voters to her right. The two candidates split the votes in the middle. Candidate 1 could increase her expected vote by moving so that \(x_{01}\) moves closer to \(v^*\). Candidate 2 could increase her expected vote by moving so that \(x_{02}\) moves closer to \(x^*\).

(c) \(x_{01} < x_{02} = v^*\). Candidate 2 wins since she gets all of the votes in \((x_{01}\cup x_{02}, \infty)\) plus one-half of the votes at \(x_{01}\cup x_{02}\). Candidate 1 can improve her chances of winning to \(\frac{1}{2}\) by moving to \(v^*\).

(2) \(x_1 = x_2\). \((x_{01} = x_{02})\)

(a) \(x_{01} = x_{02} < v^*\). Each candidate’s expected chance of winning is \(\frac{1}{2}\). If 1 were to move such that \(x_{01}\) moves closer to \(v^*\), then she would gain all of the votes in \([v^*, \infty)\) plus half of the votes in \([x_{02}, x_{01}]\), which is more than the \(\frac{N}{2}\) votes needed to win. The same argument holds for \(v^* < x_{01} = x_{02}\).

(b) \(x_{01} = x_{02} = v^*\). Each candidate has an expected probability of winning of \(\frac{1}{2}\). But if candidate 1 were to move so that \(x_{01} < v^*\), she would have the support of the voters in \((-\infty, x_{01}\cup x_{02}]\), which is less than \(\frac{N}{2}\) needed to win. Similarly, if 2 were to move unilaterally, she would lose. Therefore, neither candidate has an incentive to move from \(\frac{x_{01}+x_{02}}{2} = \frac{x_{0a}+x_{0b}}{2} = v^*\), which implies \(x_1 = x_2 = 2v^* - x_0\). \(\square\)