Introduction to Game Theory
Lecture 7: Bayesian Games

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Games of Incomplete Information: Bayesian Games

- In the games we have studied so far (both simultaneous-move and extensive form games), each player knows the other players’ preferences, or payoff functions. Games of complete information.

- Now we study **games of incomplete information (Bayesian games)**, in which at least some players are not completely informed of some other players’ preferences, or some other characteristics of the other players that are relevant to their decision making.
Example 1: variant of BoS with one-sided incomplete information

- Player 2 knows if she wishes to meet player 1, but player 1 is not sure if player 2 wishes to meet her. Player 1 thinks each case has a 1/2 probability.

- We say player 2 has two **types**, or there are two **states of the world** (in one state player 2 wishes to meet 1, in the other state player 2 does not).
Example 1: solution

This is a Bayesian simultaneous-move game, so we look for the **Bayesian Nash equilibria**. In the Bayesian NE:

- the action of player 1 is optimal, given the actions of the two types of player 2 *and* player 1’s belief about the state of the world;
- the action of each type of player 2 is optimal, given the action of player 1.

The unique pure-strategy equilibrium is \([B, (B, S)]\), in which the first component is player 1’s action, and the second component (in parenthesis) is the pair of actions of the two types of player 2.
Example 2: variant of BoS with two-sided incomplete information

- Now neither player is sure if the other player wishes to meet.

- Player 1’s types: $y_1$ and $n_1$; player 2’s types: $y_2$ and $n_2$. 
Example 2: \([(B,B), (B,S)]\) as NE

- \([(B, B), (B, S)]\) is a pure-strategy NE of the game.

- If player 1 always plays \(B\), certainly player 2 will play \(B\) if her type is \(y_2\) and play \(S\) if \(n_2\). So player 2's \((B, S)\) is indeed best response to player 1's \((B, B)\).
Example 2: [(B,B), (B,S)] as NE (cont.)

- Player 1's (B, B) is also best response to player 2's (B, S).
  - For type $y_1$,
    \[ u_1(B | (B, S)) = \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 0 > u_1(S | (B, S)) = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 1; \]
  - For type $n_1$,
    \[ u_1(B | (B, S)) = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 2 > u_1(S | (B, S)) = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 0. \]
Therefore $[(B, B), (B, S)]$ is a pure-strategy NE.

Is $[(S, B), (S, S)]$ a pure-strategy NE of the game?
Example 2: \([ (S,B), (S,S) ] \) as NE

- If player 2 always plays \( S \), player 1’s best response is indeed \( S \) if her type is \( y_1 \) and \( B \) if \( n_1 \). So player 1’s \((S, B)\) is best response to player 2’s \((S, S)\).
Example 2: NE [(S,B), (S,S)] (cont.)

• Player 2's (S, S) is also best response to player 1's (S, B).
  - For type y2,
    \[ u_2(S|(S, B)) = \frac{2}{3} \cdot 0 + \frac{1}{3} \cdot 2 > u_2(B|(S, B)) = \frac{2}{3} \cdot 0 + \frac{1}{3} \cdot 1 \]
  - For type n2,
    \[ u_2(S|(S, B)) = \frac{2}{3} \cdot 0 + \frac{1}{3} \cdot 2 = u_2(B|(S, B)) = \frac{2}{3} \cdot 1 + \frac{1}{3} \cdot 0 \]
Summary

• A **state** is a complete description of one collection of the players’ relevant characteristics, including their preferences and their information.

• The **type** of a player embodies any private information that is relevant to the player’s decision making, including a player’s payoff function, her beliefs about other player’s pay-off functions, her beliefs about other players’ beliefs about her beliefs, and so on.

• In a Bayesian game, each type of each player chooses an action.

• In NE, the action chosen by each type of each player is optimal, given the actions chosen by every type of every other player.
More information may hurt (1)

- In single-person decision problems, a person cannot be worse off with more information. In strategic interactions, a player may be worse off if she has more information and other players know that she has more information.
- In the following game, each player believes that both states are equally likely. $0 < \epsilon < 1/2$.

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State $\omega_1$

State $\omega_2$

- What is the NE?
• Player 2’s unique best response to each action of player 1 is $L$. Player 1’s unique best response to $L$ is $B$. So the unique NE is $(B, L)$, with each player getting a payoff of 2.

• What if player 2 knows the state for sure?
Player 2 has a dominant strategy of $R$ in state $\omega_1$, and a dominant strategy of $M$ in state $\omega_2$. When player 2 is only going to play $R$ or $M$, player 1 has a dominant strategy of $T$. So the unique NE is now $[T, (R, M)]$.

Regardless of the state, player 2’s payoff is now $3\epsilon < 2$, since $\epsilon < 1/2$. 

\[ \begin{array}{c|cc|c|c|c|c} & & & & & & \\
 & L & M & R & & & \\
 T & 1, 2\epsilon & 1, 0 & 1, 3\epsilon & & & \\
 B & 2, 2 & 0, 0 & 0, 3 & & & \\
2 & & & & 1/2 & 1 & 1/2 \\
\end{array} \]
Bayesian games can not only model uncertainty about players’ preferences, but also uncertainty about each other’s knowledge.

In the following, player 1 can distinguish state $\alpha$ from other states, but cannot distinguish state $\beta$ from state $\gamma$; player 2 can distinguish state $\gamma$ from other state, but cannot distinguish state $\alpha$ from state $\beta$.
Information contagion (2)

• Note that player 2’s preferences are the same in all three states, and player 1’s preferences are the same in states $\beta$ and $\gamma$.

• Therefore, in state $\gamma$, each player knows the other player’s preferences, and player 2 knows that player 1 knows her preferences. But player 1 does not know that player 2 knows her preferences (player 1 thinks it might be state $\beta$, in which case player 2 does not know if it is state $\beta$ or $\alpha$).
Information contagion (3)

- If both players are completely informed in state $\gamma$, both $(L, L)$ and $(R, R)$ are NE.

- But this whole Bayesian game has a unique NE. What is it?
  - First consider player 1’s choice in state $\alpha$. (R is dominant.)
  - Next consider player 2’s choice when she knows the state is either $\alpha$ or $\beta$. (R is better than L given 1’s choice in $\alpha$.)
  - Then consider player 1’s choice when she knows the state is either $\beta$ or $\gamma$. (R is better, given 1 and 2’s actions in $\alpha$ and $\beta$.)
  - Finally consider player 2’s choice in state $\gamma$. (R is better.)
Information contagion (4)

- Information contagion leads to the unique NE: \((R, R)\).
- Consider the following extension:

- In state \(\delta\), player 2 knows player 1’s preferences, but player 2 does not know if player 1 knows that player 2 knows player 1’s preferences (player 2 does not know if the state is \(\gamma\) or \(\delta\); if \(\gamma\), player 1 knows it can be \(\beta\); if \(\beta\), player 2 would think it might be \(\alpha\), in which case player 1’s preferences are different.)
- \((R, R)\), however, is still the unique NE.
“They don’t know that we know they know we know...”

- The Rubinstein email game.
- The eye colors puzzle.
- Friends: http://www.youtube.com/watch?v=Fpl4D3_b6DU
Some basic facts about probability

- Let $E$ and $F$ be two events, each occurring respectively with probability $Pr(E)$ and $Pr(F)$. We have the following facts.
- The probability that event $E$ occurs, given that $F$ has occurred, is $Pr(E|F) = \frac{Pr(E,F)}{Pr(F)}$, where $Pr(\cdot|\cdot)$ indicates the **conditional probability**, and $Pr(E,F)$ is the probability that both events occur.
- In other words,
  
  $Pr(E,F) = Pr(F)Pr(E|F) = Pr(F,E) = Pr(E)Pr(F|E)$.

- Further, denote the event that $E$ does not occur as $E^c$ ($c$ means complement), then
  
  $Pr(F) = Pr(E,F) + Pr(E^c,F) = Pr(E)Pr(F|E) + Pr(E^c)Pr(F|E^c)$.

- Then, $Pr(E|F) = \frac{Pr(E)Pr(F|E)}{Pr(E)Pr(F|E) + Pr(E^c)Pr(F|E^c)}$.
Bayes’ rule

• More generally, let $E_1, E_2, \ldots, E_n$ be a collection of exclusive events (meaning exactly one of these events must occur).
• Then the probability of a particular event $E_k$ conditional on event $F$ is

$$Pr(E_k|F) = \frac{Pr(F|E_k)Pr(E_k)}{\sum_{j=1}^{n} Pr(F|E_j)Pr(E_j)}.$$  

• This is an extremely important formula, called Bayes’ rule, which enables us to calculate the posterior probability about an event based on the prior probability and new information/evidence.
• **Prior belief**: a player’s initial belief about the probability of an event (i.e., $Pr(E_k)$).
• **Posterior belief**: a player’s updated belief after receiving new information/evidence (i.e., $Pr(E_k|F)$).
A model of juries: setup (1)

- A number of jurors need to decide to convict or acquit a defendant. A *unanimous verdict is required for conviction*.
- Each juror comes to the trial with a prior belief that the defendant is guilty with probability $\pi$. Then they receive a piece of information. But each may interpret the information differently.
- If a juror interprets the information as evidence of guilt, we say she receives a signal $g$ (or, the juror’s type is $g$); if a juror interprets the information as evidence of innocence, we say she receives a signal $c$ (or, the juror’s type is $c$, $c$ stands for clean).
- Denote the event that the defendant is actually guilty as $G$; denote the event that the defendant is actually innocent (clean) as $C$. 
A model of juries: setup (2)

- When $G$ occurs (the defendant is actually guilty), the probability that a given juror receives the signal $g$ is $p$, $p > 1/2$; in other words, $Pr(g|G) = p > 1/2$.

- When $C$ occurs, the probability that a given juror receives the signal $c$ is $q$, $q > 1/2$; in other words, $Pr(c|C) = q > 1/2$.

- Each juror’s payoffs:

\[
= \begin{cases} 
0, & \text{if guilty defendant convicted or innocent defendant acquitted;} \\
-z, & \text{if innocent defendant convicted;} \\
-(1 - z), & \text{if guilty defendant acquitted.}
\end{cases}
\]  

(1)

- $z$ is cost of convicting an innocent defendant (type I error), and $1 - z$ is the cost of acquitting a guilty defendant (type II error).
First consider the case in which there is only one juror.

Suppose the juror receives the signal $c$ (she interprets the information as evidence of innocence), the probability she thinks the defendant is actually guilty is

$$P(G|c) = \frac{Pr(c|G)Pr(G)}{Pr(c|G)Pr(G) + Pr(c|C)Pr(C)} = \frac{(1-p)\pi}{(1-p)\pi + q(1-\pi)}.$$

By (??), the juror will acquit the defendant if $(1-z)P(G|c) \leq z(1 - P(G|c))$, or

$$z \geq P(G|c) = \frac{(1-p)\pi}{(1-p)\pi + q(1-\pi)}.$$
• Suppose the juror receives the signal $g$, a similar calculation yields that she will convict the defendant if

$$z \leq \frac{p\pi}{p\pi + (1 - q)(1 - \pi)}.$$

• Therefore the juror optimally acts according to her interpretation of the information if

$$\frac{(1 - p)\pi}{(1 - p)\pi + q(1 - \pi)} \leq z \leq \frac{p\pi}{p\pi + (1 - q)(1 - \pi)}.$$
Two jurors

- Now suppose there are two jurors. Is there an equilibrium in which each juror votes according to her signal?
- Suppose juror 2 votes according to her signal: vote to acquit if her signal is $c$ and vote to convict if her signal is $g$.
- If juror 2’s signal is $c$, then juror 1’s vote does not matter for the outcome (unanimity is required for conviction).
- So juror 1 can ignore the possibility that juror 2’s signal may be $c$, and assume it is $g$.
- We want to see when juror 1 will vote to acquit when her signal is $c$. When juror 1’s signal is $c$ and juror 2’s signal is $g$, juror 1 thinks the probability that the defendant is guilty is

$$
P(G|c,g) = \frac{Pr(c,g|G)Pr(G)}{Pr(c,g|G)Pr(G) + Pr(c,g|C)Pr(C)}$$

$$= \frac{(1 - p)p\pi}{(1 - p)p\pi + q(1 - q)(1 - \pi)}.$$
Two jurors (cont.)

- By (??), juror 1 will vote to acquit the defendant if
  
  \[
  (1 - z)P(G|c, g) \leq z(1 - P(G|c, g)),
  \]

  or

  \[
  z \geq P(G|c) = \frac{(1 - p)p\pi}{(1 - p)p\pi + q(1 - q)(1 - \pi)}.
  \]

- By a similar calculation, if juror 1 receives a signal \( g \), she will vote to convict if

  \[
  z < \frac{p^2\pi}{p^2\pi + (1 - q)^2(1 - \pi)}.
  \]

- Therefore juror 1 optimally votes according to her interpretation of the information

  \[
  \frac{(1 - p)p\pi}{(1 - p)p\pi + q(1 - q)(1 - \pi)} \leq z \leq \frac{p^2\pi}{p^2\pi + (1 - q)^2(1 - \pi)}.
  \]
One juror vs. two jurors

• To recap, when there is only one juror, she acts according to her signal if

\[
\frac{(1 - p)\pi}{(1 - p)\pi + q(1 - \pi)} \leq z \leq \frac{p\pi}{p\pi + (1 - q)(1 - \pi)}.
\]  

(2)

• When there are two jurors, they vote according to their signals if

\[
\frac{(1 - p)p\pi}{(1 - p)p\pi + q(1 - q)(1 - \pi)} \leq z \leq \frac{p^2\pi}{p^2\pi + (1 - q)^2(1 - \pi)}.
\]

(3)

• Compare the left sides of (??) and (??), and recall that \(p > 1/2 > 1 - q\).

• The lowest value of \(z\) with which jurors vote according to their signals is higher in the two-juror case than in the one-juror case. A juror is less worried about convicting an innocent person when there are two jurors.
Many jurors (1)

- Now suppose the number of jurors is $n$. Is there an equilibrium in which each juror votes according to her signal?
- Suppose every juror other than juror 1 votes according to her signal: vote to acquit if her signal is $c$ and vote to convict if her signal is $g$.
- Again juror 1 can ignore the possibility that some other jurors’ signals may be $c$, and assume every other juror’s signal is $g$.
- And again we want to see when juror 1 will vote to acquit when her signal is $c$. When juror 1’s signal $c$ and every other juror’s signal is $g$, juror 1 thinks the probability that the defendant is guilty is

$$P(G|c, g, \ldots, g) = \frac{Pr(c, g, \ldots, g|G)Pr(G)}{Pr(c, g, \ldots, g|G)Pr(G) + Pr(c, g, \ldots, g|C)Pr(C)}$$

$$= \frac{(1 - p)p^{n-1} \pi}{(1 - p)p^{n-1} \pi + q(1 - q)^{n-1}(1 - \pi)}.$$
Many jurors (2)

- By (??), juror 1 will vote for acquittal if 
  $$(1 - z)P(G|c, g, \ldots, g) \leq z(1 - P(G|c, g, \ldots, g))$$, or

  $$z \geq \frac{(1 - p)p^{n-1}\pi}{(1 - p)p^{n-1}\pi + q(1 - q)^{n-1}(1 - \pi)} = \frac{1}{1 + \frac{q}{1-p} \left(\frac{1-q}{p}\right)^{n-1} \left(\frac{1-\pi}{\pi}\right)}.$$

- Given that $p > 1 - q$, the denominator approaches 1 as $n$ increases. So the lower bound on $z$ for which juror 1 votes for acquittal when her signal is $c$ approaches 1 as $n$ increases. In other words, in a large jury, if jurors care even slightly about acquitting a guilty defendant (type II error), then a juror who interprets a piece of information as evidence of innocence will nevertheless vote for conviction.
Many jurors: equilibria

- In other words, in a large jury in which the jurors are concerned about acquitting a guilty defendant, there is no Nash equilibrium in which every juror votes according to her signal.

- Is there a NE in which every juror votes for acquittal regardless of her signal (easy), and is there a NE in which every juror votes for conviction regardless of her signal (slightly harder)?

- There is also a mixed strategy NE for some values of $z$, in which a juror votes for conviction if her signal is $g$, and randomizes between acquittal and conviction if her signal is $c$. Interestingly, *in this NE the probability an innocent defendant is convicted increases as $n$ increases.*