So far what we have considered are pure strategy (纯粹策略) equilibria, in which players choose deterministic actions.
Mixed strategies and von Neumann-Morgenstern preferences

- So far what we have considered are pure strategy (纯粹策略) equilibria, in which players choose deterministic actions.
- Now we consider mixed strategy (混合策略) equilibria, in which players can randomize over their actions.
- Which means we need to deal with preferences regarding lotteries (彩票), i.e., the vNM preferences (冯诺依曼—摩根斯坦偏好)—preferences regarding lotteries over action profiles that may be represented by the expected value of a payoff function over action profiles.
In pure strategy equilibria, we deal with **ordinal preferences** (序数性偏好), which only specify the order of your preferences, not how much you prefer one item over another.

With **vNM preferences**, the payoff numbers in a game state the intensity of your preferences, not just the order, and you can take expectations over the numbers.
von Neumann-Morgenstern Expected Utilities

- The following tables represent the same game with ordinal preferences but different games with vNM preferences.

<table>
<thead>
<tr>
<th></th>
<th>S</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>S</td>
<td>2, 2</td>
<td>0, 3</td>
</tr>
<tr>
<td>B</td>
<td>3, 0</td>
<td>1, 1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>S</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>S</td>
<td>8, 8</td>
<td>0, 11</td>
</tr>
<tr>
<td>B</td>
<td>11, 0</td>
<td>1, 1</td>
</tr>
</tbody>
</table>

- With vNM preferences, we can derive **expected utilities** over lotteries: \( U(p_1, \ldots, p_K) = \sum_{k=1}^{K} p_k u(a_k) \), where \( a_k \) is the \( k \)th outcome of the lottery, and \( p_k \) is the probability that \( a_k \) will happen.

- You prefer the lottery \((p_1, \ldots, p_K)\) to the lottery \((p'_1, \ldots, p'_K)\) only if \( \sum_{k=1}^{K} p_k u(a_k) > \sum_{k=1}^{K} p'_k u(a_k) \).
A mixed strategy of a player in a strategic game is a probability distribution over the player’s actions, denoted by $\alpha_i(a_i)$; e.g., $\alpha_i(\text{left}) = 1/3$, $\alpha_i(\text{right}) = 2/3$.

A pure strategy is a mixed strategy that assigns probability 1 to a particular action.
A **mixed strategy** of a player in a strategic game is a probability distribution over the player’s actions, denoted by $\alpha_i(a_i)$; e.g., $\alpha_i(\text{left}) = 1/3$, $\alpha_i(\text{right}) = 2/3$.

- A pure strategy is a mixed strategy that assigns probability 1 to a particular action.

The mixed strategy profile $\alpha^*$ in a strategic game is a **mixed strategy Nash equilibrium** if

$$U_i(\alpha_i^*, \alpha_{-i}^*) \geq U_i(\alpha_i, \alpha_{-i}^*), \forall \alpha_i \text{ and } i,$$

where $U_i(\alpha)$ is player $i$’s expected payoff with the mixed strategy profile $\alpha$.

- Using best response functions, $\alpha^*$ is a mixed strategy NE iff $\alpha_i^*$ is in $B_i(\alpha_i^*)$ for every player $i$. 
Matching Pennies reconsidered

- There is no pure strategy Nash equilibrium in Matching Pennies

<table>
<thead>
<tr>
<th></th>
<th>head</th>
<th>tail</th>
</tr>
</thead>
<tbody>
<tr>
<td>Player 1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>head</td>
<td>1, −1</td>
<td>−1, 1</td>
</tr>
<tr>
<td>tail</td>
<td>−1, 1</td>
<td>1, −1</td>
</tr>
</tbody>
</table>

- But there is a mixed strategy NE for the game with the above vNM preferences: (\(\text{head}, \frac{1}{2} \); \(\text{tail}, \frac{1}{2} \)), (\(\text{head}, \frac{1}{2} \); \(\text{tail}, \frac{1}{2} \))

- **Theorem** (Nash 1950): Every finite strategic game with vNM preferences has a mixed strategy Nash equilibrium.
Strict domination with mixed strategies

- Player $i$'s mixed strategy $\alpha_i$ **strictly dominates** her action $a'_i$ if $U_i(\alpha_i, a_{-i}) > u_i(a'_i, a_{-i})$ for every list $a_{-i}$ of the other players’ actions. $a'_i$ is **strictly dominated**.

- In the following game, player 1 has no action that is dominated by a pure strategy. But action $T$ is dominated by the mixed strategy $(M, p; B, 1 - p)$, with $\frac{1}{4} < p < \frac{2}{3}$.

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>Player 1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>T</td>
<td>1, $a$</td>
<td>1, $b$</td>
</tr>
<tr>
<td>M</td>
<td>4, $c$</td>
<td>0, $d$</td>
</tr>
<tr>
<td>B</td>
<td>0, $e$</td>
<td>3, $f$</td>
</tr>
</tbody>
</table>

- A strictly dominated action is not used in any mixed strategy Nash equilibrium.
Weak domination with mixed strategies

- Player $i$’s mixed strategy $\alpha_i$ **weakly dominates** action $a'_i$ if $U_i(\alpha_i, a_{-i}) \geq U_i(a'_i, a_{-i})$ for every list $a_{-i}$ of the other players’ actions, and $U_i(\alpha_i, a_{-i}) > U_i(a'_i, a_{-i})$ for some list $a_{-i}$ of the other players’ actions.

- A weakly dominated action, however, may be used in a mixed strategy NE.

- But, every finite strategic game has a mixed strategy NE in which no player’s strategy is weakly dominated.
Characterization of mixed strategy NE in finite games

A characterization for finite strategic games: a mixed strategy profile $\alpha^*$ is a mixed strategy NE iff, for each player $i$,

1. the expected payoff, given $\alpha_{-i}^*$, to every action to which $\alpha_i^*$ assigns positive probability is the same
   $\Rightarrow$ otherwise $i$ should just play the more profitable action rather than mixing it with other actions
   - In other words, other players’ equilibrium mixed strategies keep you indifferent between a set of your actions.

2. the expected payoff, given $\alpha_{-i}^*$, to every action to which $\alpha_i^*$ assigns zero probability is lower or at most equal to the expected payoff to any action to which $\alpha^*$ assigns positive probability.
   $\Rightarrow$ otherwise $i$ should play that action
Method for finding all mixed strategy NE

1. Eliminate strictly dominated actions from the game
2. For each player $i$, choose a subset $S_i$ of her set $A_i$ of actions
3. Check if there is a mixed strategy profile $\alpha$ that (1) assigns positive probability only to actions in $S_i$, and (2) satisfies the two conditions in the previous characterization.
4. Repeat the analysis for every other collection of subsets of the players’ sets of actions
Example

- Consider the variant of the Battle of Sexes game below. What are the mixed strategy NE?

<table>
<thead>
<tr>
<th></th>
<th>B</th>
<th>S</th>
<th>X</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>4, 2</td>
<td>0, 0</td>
<td>0, 1</td>
</tr>
<tr>
<td>S</td>
<td>0, 0</td>
<td>2, 4</td>
<td>1, 3</td>
</tr>
</tbody>
</table>

- By inspection, we see there is no dominated strategy to be eliminated. Further, (B, B) and (S, S) are two pure strategy equilibria.
Four possible kinds of mixed strategy equilibrium

- What about an equilibrium in which player 1 plays a pure strategy (B or S), while player 2 plays a strictly mixed strategy? Condition 1 of the characterization impossible to meet.

- Similar reasoning rules out the potential equilibrium in which player 2 plays pure strategy while player 1 randomize over her two actions.

- What about an equilibrium in which player 1 mixes over her two actions, while player 2 mixes over two of her three actions: B & S, B & X, or S & X?

- What about an equilibrium in which player 1 mixes over her two actions, and player 2 mixes over her three actions?
Analyzing the example (1)

- Let player 1’s probability of playing B be $p$ (hence $1 - p$ for $S$)
- Player 2 mixes over B and $S$
  - To satisfy conditions 1 and 2 we need
    \[ 2p = 4(1 - p) \geq p + 3(1 - p). \] Impossible to hold.
- Player 2 mixes over B and $X$
  - To satisfy the two conditions we need
    \[ 2p = p + 3(1 - p) \geq 4(1 - p) \Rightarrow p = \frac{3}{4} \]
  - Next we should examine player 2’s randomization. Let $q$ be her probability of choosing B (hence $1 - q$ for $X$). For player 1 to be indifferent between her two actions (condition 1; condition 2 moot here),
    \[ 4q + 0 = 0 + (1 - q) \Rightarrow q = \frac{1}{5}. \]
  - Thus $((B, \frac{3}{4}; S, \frac{1}{4}), (B, \frac{1}{5}; S, 0; X, \frac{4}{5}))$ is a mixed strategy NE.
Analyzing the example (2)

- Player 2 mixes over S and X
  - In this case player 1 will always choose S. So no NE in which player 1 mixes over B and S.

- Player 2 mixes over B, S, and X
  - For player 2 to be indifferent between her three actions (condition 1; condition 2 moot here), we need
    \[ 2p = 4(1 - p) = p + 3(1 - p) \implies \text{impossible}. \]

- The NE are the two pure strategy equilibria and the strictly mixed strategy NE \((B, \frac{3}{4}; S, \frac{1}{4}), (B, \frac{1}{5}; S, 0; X, \frac{4}{5})\)
A three-player example

- Player 1 chooses between rows, player 2 chooses between columns, and player 3 chooses between tables.
Analyzing the three-player example

- By inspection (A, A, A) and (B, B, B) are two pure-strategy NE.
- If one of the players’ strategy is pure, obviously the other two should choose the first player’s action rather than mix over two or more actions.
- The only remaining case is all three mix over A and B. Let $p$, $q$, and $r$ respectively denote the three players’ probability of choosing A. Then condition 1 of the characterization requires
  \[ qr = 4(1 - q)(1 - r) \]
  \[ pr = 4(1 - p)(1 - r) \]
  \[ pq = 4(1 - p)(1 - q) \]
- Therefore $p = q = r = \frac{2}{3}$ is a mixed strategy NE.
Characterization of mixed strategy NE in infinite games

- Finite games must have a mixed strategy NE. Infinite games may or may not have one.
- Condition 1 of the characterization in finite games does not apply in infinite games because the probabilities are now assigned to sets of actions, not single actions.
- **A characterization for infinite strategic games**: a mixed strategy profile $\alpha^*$ is a mixed strategy NE iff, for each player $i$,
  - for no action $a_i$ does the action profile $(a_i, \alpha^*_{-i})$ yield player $i$ an expected payoff greater than her expected payoff from $\alpha^*$
  - $\alpha^*$ assigns probability zero to the set of actions $a_i$ for which the action profile $(a_i, \alpha^*_{-i})$ yields player $i$ an expected payoff less than her expected payoff from $\alpha^*$. 