Introduction to Game Theory
Lecture 2: Strategic Game and Nash Equilibrium (cont.)

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Best response functions: example

- In simple games we can examine each action profile in turn to see if it is a NE. In more complicated games it is better to use "best response functions".

- Example:

<table>
<thead>
<tr>
<th></th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>L</td>
</tr>
<tr>
<td>Player 1</td>
<td>T</td>
</tr>
<tr>
<td></td>
<td>B</td>
</tr>
</tbody>
</table>

- What are player 1’s best response(s) when player 2 chooses L, M, or R?
Best response functions: definition

- Notation:

\[ B_i(a_{-i}) = \{ a_i \in A_i : U_i(a_i, a_{-i}) \geq U_i(a'_i, a_{-i}) \text{ for all } a'_i \in A_i \}. \]

- I.e., any action in \( B_i(a_{-i}) \) is **at least as good** for player \( i \) as every other action of player \( i \) when the other players’ actions are given by \( a_{-i} \).

- Example:

<table>
<thead>
<tr>
<th>Player 1</th>
<th>( L )</th>
<th>( M )</th>
<th>( R )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T )</td>
<td>1, 1</td>
<td>1, 0</td>
<td>0, 1</td>
</tr>
<tr>
<td>( B )</td>
<td>1, 0</td>
<td>0, 1</td>
<td>1, 0</td>
</tr>
</tbody>
</table>

\[ B_1(L) = \{ T, B \}, \ B_1(M) = \{ T \}, \ B_1(R) = \{ B \} \]
Using best response functions to define Nash equilibrium

- Definition: the action/strategy profile $a^*$ is a NE of a strategic game iff every player’s action is a best response to the other players’ actions: $a_i^*$ is in $B_i(a^*_{-i})$ for every player $i$.

- If each player has a single best response to each list $a_{-i}$ of the other players’ actions, then $a_i = b_i(a^*_{-i})$ for every $i$. 
Using best response functions to find Nash equilibrium

- **Method:**
  - find the best response function of each player
  - find the action profile in which each player’s action is a best response to the other player’s action

- **Example:**

<table>
<thead>
<tr>
<th></th>
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<th>M</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>T</strong></td>
<td>1,2</td>
<td>2,1</td>
<td>1,0</td>
</tr>
<tr>
<td><strong>M</strong></td>
<td>2,1</td>
<td>0,1</td>
<td>0,0</td>
</tr>
<tr>
<td><strong>B</strong></td>
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<td>0,0</td>
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</table>
Example 39.1 Two people are involved in a synergistic relationship. If both devote more effort to the relationship, they are both better off. For any given effort of individual $j$, the return to individual $i$’s effort first increases, then decreases. Specifically, an effort level is a nonnegative number, and each individual $i$’s preferences are represented by the payoff function $u_i = e_i(c + e_j - e_i)$, where $e_i$ is $i$’s effort level, $e_j$ is the other individual’s effort level, and $c > 0$ is a constant.
Solving the example

- \( u_i = -e_i^2 + (c + e_j)e_i \), a quadratic function (section 17.3); inverted U-shape
- \( u_i = 0 \) if \( e_i = 0 \) or if \( e_i = c + e_j \), so anything in between will give \( i \) a positive payoff
- Symmetry of quadratic functions means that \( b_i(e_j) = \frac{1}{2}(c + e_j) \)
- Similarly, \( b_j(e_i) = \frac{1}{2}(c + e_i) \)
In you know a little calculus

- \( U_i = e_i(c + e_j - e_i) \)

- First order condition: \( \frac{\partial u_i}{\partial e_i} = c + e_j - 2e_i = 0 \Rightarrow \)

  \[ e_i = \frac{c + e_j}{2} \quad (1) \]

- Similarly,

  \[ e_j = \frac{c + e_i}{2} \quad (2) \]

- Plugging (2) into (1), we have \( e_i^* = e_j^* = c \)
Exercise 42.2 (b) Two people are engaged in a joint project. If each person $i$ puts in effort $x_i$, a nonnegative number equal to at most 1, which costs her $x_i$, each person will get a utility $4x_1x_2$. Find the NE of the game. Is there a pair of effort levels that yields higher payoffs for both players than do the NE effort levels?
Best response functions in graph
Strict Domination (强占优)

- Player $i$’s action $a'_i$ strictly dominates action $a''_i$ if

$$u_i(a'_i, a_{-i}) > u_i(a''_i, a_{-i})$$

for every list $a_{-i}$ of the other players’ actions. In this case the action $a''_i$ is strictly dominated.

- In *Prisoner’s Dilemma*, “confess” strictly dominates “silent”.

<table>
<thead>
<tr>
<th>Suspect 1</th>
<th>silent</th>
<th>confess</th>
</tr>
</thead>
<tbody>
<tr>
<td>silent</td>
<td>0, 0</td>
<td>−2, 1</td>
</tr>
<tr>
<td>confess</td>
<td>1, −2</td>
<td>−1, −1</td>
</tr>
</tbody>
</table>

- If player $i$’s action $a'_i$ strictly dominates every other action of hers, then $a'_i$ is $i$’s strictly dominant action.
Elimination of strictly dominated action

- Not every game has a strictly dominated action. But if there is, it is not used in any Nash equilibrium and so can be eliminated.

- Any strictly dominated action in the following game? Any strictly dominant action?

```
Player 2
   | L  | C  | R  |
---|----|----|----|
Player 1
U  | 7,3| 0,4| 4,4|
M  | 4,6| 1,5| 5,3|
D  | 3,8| 0,2| 3,0|
```
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- Any strictly dominated action in the following game? Any strictly dominant action?

Player 2

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<tr>
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<td>4,6</td>
<td>1,5</td>
<td>5,3</td>
</tr>
<tr>
<td>D</td>
<td>3,8</td>
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⇒ D is strictly dominated by M.
Iterated elimination of strictly dominated action

- Sometimes we can repeat the procedure: eliminate all strictly dominated actions, and then continue to eliminate strategies that are now dominated in the simpler game.
- Are there more than one actions that can be eliminated from the following game?

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<td>1</td>
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Iterated elimination of strictly dominated action

- Sometimes we can repeat the procedure: eliminate all strictly dominated actions, and then continue to eliminate strategies that are now dominated in the simpler game.
- Are there more than one actions that can be eliminated from the following game?

First B and then C can be eliminated.
Weak Domination (弱占优)

- Player $i$'s action $a_i'$ **weakly dominates** action $a_i''$ if

$$u_i(a_i', a_{-i}) \geq u_i(a_i'', a_{-i})$$

for every list $a_{-i}$ of the other players’ actions, and

$$u_i(a_i', a_{-i}) > u_i(a_i'', a_{-i})$$

for some list $a_{-i}$ of the other players’ actions.

- Action $a_i''$ is then **weakly dominated**.
Weak Domination

- Any weakly dominated action in the following game?

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<td>1,2</td>
<td>4,0</td>
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</table>
Any weakly dominated action in the following game?

\[
\begin{array}{c|ccc}
\text{Player 1} & \text{L} & \text{C} & \text{R} \\
\hline
\text{U} & 7, 3 & 0, 4 & 4, 4 \\
\text{M} & 4, 6 & 1, 5 & 5, 3 \\
\text{D} & 3, 8 & 1, 2 & 4, 0 \\
\end{array}
\]

⇒ R weakly dominated by C; D weakly dominated by M.
Weak Domination

- Any weakly dominated action in the following game?

\[
\begin{array}{c|ccc}
& L & C & R \\
\hline
U & 7,3 & 0,4 & 4,4 \\
M & 4,6 & 1,5 & 5,3 \\
D & 3,8 & 1,2 & 4,0 \\
\end{array}
\]

$\Rightarrow$ R weakly dominated by C; D weakly dominated by M.

- If player $i$’s action $a'_i$ weakly dominates every other action of hers, then $a'_i$ is $i$’s **weakly dominant action**.
Example: Voting

There are two candidates A and B for an office, and \(N\) voters, \(N \geq 3\) and odd. A majority of voters prefer A to win.

- Is there a strictly dominated action? A weakly dominated action?
- What are the Nash equilibria of the game? Hint: Let \(N_A\) denote the number of voters that vote for A, and \(N_B\) the number of voters that vote for B, \(N_A + N_B = N\), then
  - What if \(N_A = N_B + 1\) or \(N_B = N_A + 1\), and some citizens who votes for the winner actually prefers the loser?
  - What if \(N_A = N_B + 1\) or \(N_B = N_A + 1\), and nobody who votes for the winner actually prefers the loser?
  - Can it happen that \(N_A = N_B + 2\) or \(N_B = N_A + 2\)?
  - What if \(N_A \geq N_B + 3\) or \(N_B \geq N_A + 3\)?
Solving the voting problem

- What if \( N_A = N_B + 1 \) or \( N_B = N_A + 1 \), and some citizens who votes for the winner actually prefers the loser? \( \Rightarrow \) A citizen like that can unilaterally deviate and make her favorite candidate win. Not a NE.

- What if \( N_A = N_B + 1 \) or \( N_B = N_A + 1 \), and nobody who votes for the winner actually prefers the loser? \( \Rightarrow \) The former is a NE, but the latter cannot occur (the supporters of B would be more than half).

- Can it happen that \( N_A = N_B + 2 \) or \( N_B = N_A + 2 \)? \( \Rightarrow \) No, because \( N \) is odd.

- What if \( N_A \geq N_B + 3 \) or \( N_B \geq N_A + 3 \)? \( \Rightarrow \) Yes, NE.
Strategic voting

- There are three candidates, A, B, and C, and no voter is indifferent between any two of them.
- Voting for one’s least preferred candidate is a weakly dominated action. What about voting for one’s second preference? Not dominated.
Strategic voting

- There are three candidates, A, B, and C, and no voter is indifferent between any two of them.
- Voting for one’s least preferred candidate is a weakly dominated action. What about voting for one’s second preference? Not dominated.
- Suppose you prefer A to B to C, and the other citizens’ votes are tied between B and C, with A being a distant third. Then voting for B, your second preference, is your best choice! ⇒ **strategic voting**
- In two-candidate elections you are weakly better off by voting for your favorite candidate, but in three-candidate elections that is not necessarily the case. E.g, Nader supporters in 2000 US election.
A workhorse model of electoral competition. First proposed by Hotelling (1929) and popularized by Downs (1957).

Setup:

- Parties/candidates compete by choosing a policy on the line segment $[0, 1]$. The party with most votes wins; if there is a draw, each party has a 50% chance of winning.
- Parties only care about winning, and will commit to the platforms they have chosen.
- Each voter has a favorite policy on $[0, 1]$; her utility decreases as the winner’s position is further away from her favorite policy. \[ \Rightarrow \textbf{single-peaked} \] preference (单峰偏好)
- Each voter will vote \textbf{sincerely}, choosing the party whose position is closest to her favorite policy.
- There is a median voter position, $m$. 
Two parties

- Suppose there are 2 parties, $L$ and $R$. What is the Nash equilibrium for the parties’ positions?
Two parties

- Suppose there are 2 parties, $L$ and $R$. What is the Nash equilibrium for the parties’ positions?
- The unique equilibrium is both parties choose position $m$.
  - $(m, m)$ is clearly a NE
  - any other action profile is not a NE
- This is the so-called ”median voter theorem” (中位选举人定理).
Three parties

- Suppose there is a continuum of voters, with favorite policies uniformly distributed on $[0, 1]$, and the number of parties is 3 (L, C, R). Do we still have the equilibrium that all parties choose $m$?
Suppose there is a continuum of voters, with favorite policies uniformly distributed on $[0, 1]$, and the number of parties is 3 (L, C, R). Do we still have the equilibrium that all parties choose $m$?

⇒ No. One of the parties can move slightly to the left or the right of the median voter position, and win the election.
Three parties

- Suppose there is a continuum of voters, with favorite policies uniformly distributed on \([0, 1]\), and the number of parties is 3 (L, C, R). Do we still have the equilibrium that all parties choose \(m\)?
  \[\implies\] No. One of the parties can move slightly to the left or the right of the median voter position, and win the election.

- Would the three parties positioning at 0.45, 0.55, 0.6 be a NE?
Three parties

- Suppose there is a continuum of voters, with favorite policies uniformly distributed on $[0, 1]$, and the number of parties is 3 (L, C, R). Do we still have the equilibrium that all parties choose $m$?
  - No. One of the parties can move slightly to the left or the right of the median voter position, and win the election.

- Would the three parties positioning at 0.45, 0.55, 0.6 be a NE?
  - Yes. L wins already; C and R cannot win by moving anywhere.
Condorcet winner

- A **Condorcet winner** in an election is a position, \( x^* \), such that for every other position \( y \) that is different from \( x^* \), a majority of voters prefer \( x^* \) to \( y \).
- The median voter position is a Condorcet winner.
- Not all election games have a Condorcet winner.
  - Condorcet paradox: A prefers X to Y to Z; B prefers Y to Z to X; C prefers Z to X to Y.
- Even if there is a Condorcet winner, it only has guaranteed victory in pairwise comparisons, not necessarily when there are three or more policy alternatives.
The strategic model of the war of attrition

- Examples: animals fighting over prey; interest groups lobbying against each other; countries fighting each other to see who will give up first...

- Model setup
  - Two players, $i$ and $j$, vying for an object, which is respectively worth $v_i$ and $v_j$ to the two players; a 50% chance of obtaining the object is worth $\frac{v_i}{2}$ and $\frac{v_j}{2}$.
  - Time starts at 0 and runs indefinitely; each unit of time that passes before one of the parties concedes costs each player one unit of utility
  - So, a player $i$’s utility is
    
    $$u_i(t_i, t_j) = \begin{cases} 
    -t_i, & \text{if } t_i < t_j; \\
    \frac{1}{2} v_i - t_j, & \text{if } t_i = t_j; \\
    v_i - t_j, & \text{if } t_i > t_j.
    \end{cases}$$
- Player 2’s best response function is (orange)

\[
B_2(t_1) = \begin{cases} 
\{ t_2 : t_2 > t_1 \}, & \text{if } t_1 < v_2; \\
\{ t_2 : t_2 = 0 \text{ or } t_2 > t_1 \}, & \text{if } t_1 = v_2; \\
\{0\}, & \text{if } t_1 > v_2. 
\end{cases}
\]
NE in war of attrition

- \((t_1, t_2)\) is a NE iff \(t_1 = 0\) and \(t_2 \geq v_1\), or \(t_2 = 0\) and \(t_1 \geq v_2\)
- In equilibrium, either player may concede first, including the one who values the object more
- The equilibria is asymmetric, even when \(v_1 = v_2\) (i.e., when the game is symmetric)
  - A game is symmetric if \(u_1(a_1, a_2) = u_2(a_2, a_1)\) for every action pair \((a_1, a_2)\) (if you and your opponent exchange actions, you also exchange your payoffs).
A direct argument

- If \( t_i = t_j \), then either player can increase her payoff by conceding slightly later and obtaining the object for sure; \( v_i - t_i - \epsilon > \frac{1}{2} v_i - t_i \) for a sufficiently small \( \epsilon \).
- If \( 0 < t_i < t_j \), player \( i \) should rather choose \( t_i = 0 \) to reduce the loss.
- If \( 0 = t_i < t_j < v_i \), player \( i \) can increase her payoff by conceding slightly after \( t_j \), but before \( t_i = v_i \).
- The remaining case is \( t_i = 0 \) and \( t_j \geq v_i \), which we can easily verify as a NE.
Oligopolistic competition (寡头竞争): The Cournot model

- Two firms produce the same product. The unit cost of production is $c$. Let $q_i$ be firm $i$’s output, $Q = \sum_{i=1}^{2} q_i$, then the market price $P$ is $P(Q) = \alpha - Q$, where $\alpha$ is a constant.
- Firms choose their output simultaneously. What is the NE?
- Each firm wants to maximize profit. Firm 1’s profit is

$$\pi_1 = P(Q)q_1 - cq_1$$

$$= (\alpha - q_1 - q_2)q_1 - cq_1.$$ 

- Differentiate $\pi_1$ with respect to $q_1$, we know by the first order condition that firm 1’s optimal output (best response) is

$$q_1 = b_1(q_2)\frac{\alpha - q_2 - c}{2} \quad (3)$$
Similarly (since the game is symmetric), firm 2's optimal output is

\[ q_2 = b_2(q_1) = \frac{\alpha - q_1 - c}{2} \]  

Solving equations (3) and (4) together, we have

\[ q_1^* = q_2^* = \frac{1}{3}(\alpha - c). \]
Similarly (since the game is symmetric), firm 2’s optimal output is

\[ q_2 = b_2(q_1) = \frac{\alpha - q_1 - c}{2} \]  

(4)

Solving equations (3) and (4) together, we have

\[ q_1^* = q_2^* = \frac{1}{3}(\alpha - c). \]

If the two firms can collude, they would maximize

\[ PQ - cQ = (\alpha - Q)Q - cQ. \]

And the output

\[ Q = \frac{1}{2}(\alpha - c) < \frac{2}{3}(\alpha - c), \]

the market price would be

\[ \alpha - Q = \alpha - \frac{1}{2}(\alpha - c) > \alpha - \frac{2}{3}(\alpha - c). \]

Competition (instead of collusion) increases total output, and reduces market price.