Def: Let \( F: K_0 \subseteq K_1 \subseteq \cdots \subseteq K_r = K \) be a filtration for \( K \). Let \( \alpha \in H_p(K^a) \) be a non-trivial \( p \)-th homology class. We say

\( \alpha \) is born at \( K_i \) (for \( i \leq a \)) if \( \alpha \in H_p^{i,a} \) but \( \alpha \notin H_p^{i-1,a} \).

\( \alpha \) dies entering \( K_j \) (for \( a < j \)) if \( f_{a,j-1}(\alpha) \neq 0 \) but \( f_{a,j}(\alpha) = 0 \).

\( \alpha \) lives forever if \( f_{a,i}(\alpha) \neq 0 \) for all \( a < i \leq r \).

(\( \alpha \) persist along all the following filtration steps)

There are two ways of visualize/represent the lifetime of \( \alpha \): persistence barcodes, and persistence diagrams.
Class \( \alpha \) is born at \( K_i \) since it is not in the image of \( f_{i-1,i} \).

Class \( \alpha \) dies entering \( K_j \) since this is the first time its image becomes zero.

Class \( \beta \) is born at \( K_i \) and it lives forever.
Remarks:

(1) Not all classes that are born at $K_i$ necessarily die entering some $K_j$. However, more than one may do so.

(2) Let $\alpha \in H_p(K_{j-1})$ die entering $K_j$. Then,

$\alpha$ is born at $K_i \iff \exists i_1, i_2, \ldots, i_k = i \text{ for some } k \geq 1 \text{ s.t.}$

(i) $0 \neq \alpha_{i_l} \in H_p(K_{j-1})$ is born at $X_{i_l}$ for all $l \in \{1, \ldots, k\}$

(ii) $\alpha = \alpha_{i_1} + \cdots + \alpha_{i_k}$

One may interpret the above fact as follows. When a class dies, it may be thought of as a merger of several classes among which the youngest one, $\alpha_{i_k}$, determines the birth point.
Let \( F: K_0 \leq K_1 \leq \cdots \leq K_r = K \) be a filtration for \( K \). For all \( p \geq 0 \), we will assume \( H_p(K_{r+1}) = 0 \).

**Def:** A persistence pair is a pair of the form \((a_i, a_j)\) with \(0 \leq a_i < a_j \leq r\), or of the form \((a_i, \infty)\) with \(0 \leq a_i \leq r\) such that

1. \((a_i, a_j)\) represents a non-trivial homology class that is born at step \(a_i\) of \( F \) and dies at step \(a_j\) of \( F \).
2. \((a_i, \infty)\) represents a non-trivial homology class that is born at step \(a_i\) of \( F \) and lives forever.
**Def:** Let $p > 0$. For $0 < i < j \leq r + 1$, the persistence pairing function $\mu_{p}^{i,j}$ of a persistence pair $(a_{i}, a_{j})$ is given by

\[
\mu_{p}^{i,j} = (\beta_{p}^{i,j-1} - \beta_{p}^{i,j}) - (\beta_{p}^{i-1,j-1} - \beta_{p}^{i-1,j}),
\]

where $a_{r+1} = \infty$.

($\beta_{p}^{i,j-1} - \beta_{p}^{i,j}$) = Number of independent classes born at or before $K_{i}$, and die entering $K_{j}$.

($\beta_{p}^{i-1,j-1} - \beta_{p}^{i-1,j}$) = Number of independent classes born at or before $K_{i-1}$, and die entering $K_{j}$.

$\mu_{p}^{i,j}$ = Number of independent classes born at $K_{i}$ and die entering $K_{j}$.

$j = r + 1 \Rightarrow \mu_{r+1}^{i} = \text{Number of independent classes born at } K_{i} \text{ and remain alive till the end in the filtration } F$. 
Def: Let $\alpha$ be a class with persistence pair $(a_i, a_j)$. If $\mu_{i,j} \neq 0$, we define the persistence of $\alpha$ as its life span and we denote it by $\text{Per}(\alpha)$.

1. If $j < r+1$, then $\text{Per}(\alpha) = a_j - a_i$.
2. If $j = r+1$, then $\text{Per}(\alpha) = \infty$.

Def: The persistence diagram $\text{Dgm}_{mp}(F)$ of a filtration $F$ is obtained by drawing a point $(a_i, a_j)$ with nonzero multiplicity $\mu_{i,j}$, $i < j$, on the extended plane $\overline{\mathbb{R}}^2 := (\mathbb{R} \cup \{+\infty\})^2$ where the points on the diagonal $\Delta = \{(x, x)\}$ are added with infinite multiplicity.
**Def:** The persistence barcode \( \text{bcd}(F) \) of a filtration \( F \) is obtained by where

1. \((a_i, a_j)\) is represented by a semi-open interval \([a_i, a_j)\) called a bar,
2. \((a_i, \infty)\) is represented by a ray \([a_i, \infty)\) called an infinite bar.