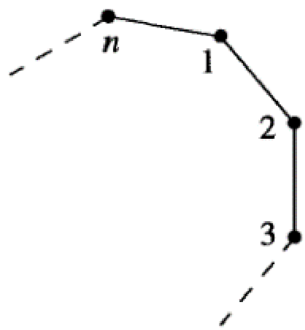
# Reading Assignment 4
## Lecture 11

1. Read the content below about a rigorous description of the Dihedral Groups of Degree 2n for an arbitrary n.

### DIHEDRAL GROUPS

An important family of examples of groups is the class of groups whose elements are symmetries of geometric objects. The simplest subclass is when the geometric objects are regular planar figures.

For each $n \in \mathbb{Z}^+$, $n \geq 3$ let $D_{2n}$ be the set of symmetries of a regular $n$-gon, where a symmetry is any rigid motion of the $n$-gon which can be effected by taking a copy of the $n$-gon, moving this copy in any fashion in 3-space and then placing the copy back on the original $n$-gon so it exactly covers it. More precisely, we can describe the symmetries by first choosing a labelling of the $n$ vertices, for example as shown in the following figure.

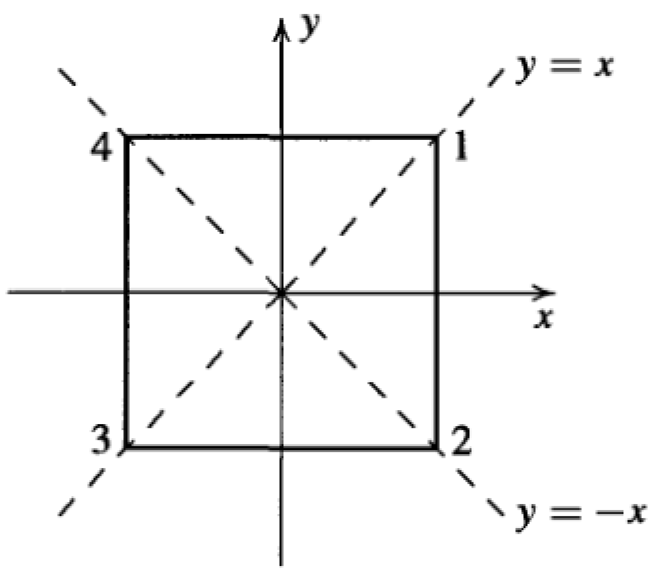



Then each symmetry $s$ can be described uniquely by the corresponding permutation $\sigma$ of $\{1, 2, 3, \ldots, n\}$ where if the symmetry $s$ puts vertex $i$ in the place where vertex $j$ was originally, then $\sigma$ is the permutation sending $i$ to $j$. For instance, if $s$ is a rotation of $2\pi/n$ radians clockwise about the center of the $n$-gon, then $\sigma$ is the permutation sending $i$ to $i+1$, $1 \leq i \leq n-1$, and $\sigma(n) = 1$. Now make $D_{2n}$ into a group by defining $st$ for $s, t \in D_{2n}$ to be the symmetry obtained by first applying $t$ then $s$ to the $n$-gon (note that we are viewing symmetries as functions on the $n$-gon, so $st$ is just function composition — read as usual from right to left). If $s, t$ effect the permutations $\sigma$, $\tau$, respectively on the vertices, then $st$ effects $\sigma \circ \tau$. The binary operation on $D_{2n}$ is associative since composition of functions is associative. The identity of $D_{2n}$ is the identity symmetry (which leaves all vertices fixed), denoted by 1, and the inverse of $s \in D_{2n}$ is the symmetry which reverses all rigid motions of $s$ (so if $s$ effects permutation $\sigma$ on the vertices, $s^{-1}$ effects $\sigma^{-1}$). In the next paragraph we show

$$|D_{2n}| = 2n$$

and so $D_{2n}$ is called the *dihedral group of order* $2n$. In some texts this group is written $D_n$; however, $D_{2n}$ (where the subscript gives the order of the group rather than the number of vertices) is more common in the group theory literature.

To find the order $|D_{2n}|$ observe that given any vertex $i$, there is a symmetry which sends vertex 1 into position $i$. Since vertex 2 is adjacent to vertex 1, vertex 2 must end up in position $i+1$ or $i-1$ (where $n+1$ is 1 and $1-1$ is $n$, i.e., the integers labelling the vertices are read mod $n$ ). Moreover, by following the first symmetry by a reflection about the line through vertex $i$ and the center of the $n$-gon one sees that vertex 2 can be sent to either position $i+1$ or $i-1$ by some symmetry. Thus there are $n \cdot 2$ positions the ordered pair of vertices 1, 2 may be sent to upon applying symmetries. Since symmetries are rigid motions one sees that once the position of the ordered pair of vertices 1, 2 has been specified, the action of the symmetry on all remaining vertices is completely determined. Thus there are exactly $2n$ symmetries of a regular $n$-gon. We can, moreover, explicitly exhibit $2n$ symmetries. These symmetries are the $n$ rotations about the center through $2\pi i/n$ radian, $0 \leq i \leq n-1$, and the $n$ reflections through the $n$ lines of symmetry (if $n$ is odd, each symmetry line passes through a vertex and the mid-point of the opposite side; if $n$ is even, there are $n/2$ lines of symmetry which pass through 2 opposite vertices and $n/2$ which perpendicularly bisect two opposite sides). For example, if $n = 4$ and we draw a square at the origin in an $x$, $y$ plane, the lines of symmetry are




the lines $x = 0$ ($y$-axis), $y = 0$ ($x$-axis), $y = x$ and $y = -x$ (note that "reflection" through the origin is not a reflection but a rotation of $\pi$ radians).

Since dihedral groups will be used extensively as an example throughout the text we fix some notation and mention some calculations which will simplify future computations and assist in viewing $D_{2n}$ as an abstract group (rather than having to return to the geometric setting at every instance). Fix a regular $n$-gon centered at the origin in an $x$, $y$ plane and label the vertices consecutively from 1 to $n$ in a clockwise manner. Let $r$ be the rotation clockwise about the origin through $2\pi/n$ radian. Let $s$ be the reflection about the line of symmetry through vertex 1 and the origin (we use the same letters for each $n$, but the context will always make $n$ clear). We leave the details of the following calculations as an exercise (for the most part we shall be working with $D_6$ and $D_8$, so the reader may wish to try these exercises for $n = 3$ and $n = 4$ first):

**(1)** $1, r, r^2, \ldots, r^{n-1}$ are all distinct and $r^n = 1$, so $|r| = n$.

**(2)** $|s| = 2$.

**(3)** $s \neq r^i$ for any $i$.

**(4)** $sr^i \neq sr^j$, for all $0 \leq i, j \leq n-1$ with $i \neq j$, so

$$D_{2n} = \{1, r, r^2, \ldots, r^{n-1}, s, sr, sr^2, \ldots, sr^{n-1}\}$$

i.e., each element can be written *uniquely* in the form $s^k r^i$ for some $k = 0$ or 1 and $0 \leq i \leq n-1$.

**(5)** $rs = sr^{-1}$. [First work out what permutation $s$ effects on $\{1, 2, \ldots, n\}$ and then work out separately what each side in this equation does to vertices 1 and 2.] This shows in particular that $r$ and $s$ do not commute so that $D_{2n}$ is non-abelian.

**(6)** $r^i s = sr^{-i}$, for all $0 \leq i \leq n$. [Proceed by induction on $i$ and use the fact that $r^{i+1}s = r(r^i s)$ together with the preceding calculation.] This indicates how to commute $s$ with powers of $r$.

Having done these calculations, we now observe that the complete multiplication table of $D_{2n}$ can be written in terms $r$ and $s$ alone, that is, all the elements of $D_{2n}$ have a (unique) representation in the form $s^k r^i$, $k = 0$ or 1 and $0 \leq i \leq n-1$, and any product of two elements in this form can be reduced to another in the same form using only "relations" (1), (2) and (6) (reducing all exponents mod $n$). For example, if $n = 12$,

$$(sr^9)(sr^6) = s(r^9 s)r^6 = s(sr^{-9})r^6 = s^2 r^{-9+6} = r^{-3} = r^9.$$