Reading Assignment 3
Lecture 9

1. Watch the following videos about Abstract Algebra and the definition of Group.
   Abstract Algebra
   Definition of Group

2. Read the content below about Symmetries of a Square and the Dihedral Groups.

Symmetries of a Square

Suppose we remove a square region from a plane, move it in some way, then put the square back into the space it originally occupied. Our goal in this chapter is to describe all possible ways in which this can be done. More specifically, we want to describe the possible relationships between the starting position of the square and its final position in terms of motions. However, we are interested in the net effect of a motion, rather than in the motion itself. Thus, for example, we consider a $90°$ rotation and a $450°$ rotation as equal, since they have the same net effect on every point. With this simplifying convention, it is an easy matter to achieve our goal.

To begin, we can think of the square region as being transparent (glass, say), with the corners marked on one side with the colors blue, white, pink, and green. This makes it easy to distinguish between motions that have different effects. With this marking scheme, we are now in a position to describe, in simple fashion, all possible ways in which a square object can be repositioned. See Figure 1.1. We now claim that any motion—no matter how complicated—is equivalent to one of these eight. To verify this claim, observe that the final position of the square is completely determined by the location and orientation (i.e., face up or face down) of any particular corner. But, clearly, there are only four locations and two orientations for a given corner, so there are exactly eight distinct final positions for the corner.

R_0 = Rotation of 0° (no change in position)

R_{90} = Rotation of 90° (counterclockwise)

R_{180} = Rotation of 180°

R_{270} = Rotation of 270°

H = Flip about a horizontal axis

V = Flip about a vertical axis

D = Flip about the main diagonal

D' = Flip about the other diagonal

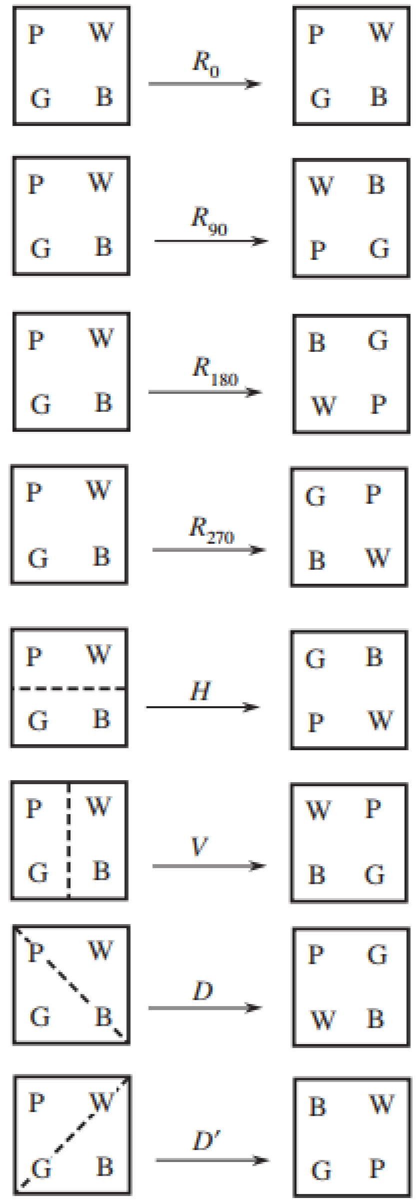
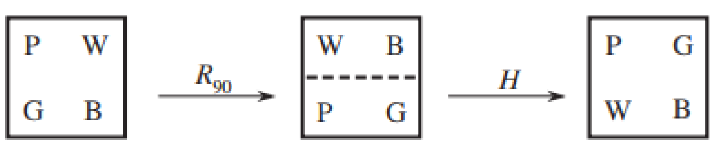


Figure 1.1 Symmetries of a square.

Let's investigate some consequences of the fact that every motion is equal to one of the eight listed in Figure 1.1. Suppose a square is repositioned by a rotation of 90° followed by a flip about the horizontal axis of symmetry.



Thus, we see that this pair of motions—taken together—is equal to the single motion D. This observation suggests that we can compose two motions to obtain a single motion. And indeed we can, since the eight motions may be viewed as functions from the square region to itself, and as such we can combine them using function composition.

With this in mind, we write $H\ R_{90} = D$ because in lower level math courses function composition $f \circ g$ means "g followed by f." The eight motions R_0, R_{90}, R_{180}, R_{270}, H, V, D, and D', together with the operation composition, form a mathematical system called the *dihedral group of order 8* (the order of a group is the number of elements it contains). It is denoted by D_4. Rather than introduce the formal definition of a group here, let's look at some properties of groups by way of the example D_4.

To facilitate future computations, we construct an *operation table, the Cayley table* (so named in honor of the prolific English mathematician Arthur Cayley, who first introduced them in 1854) for D_4 below. The circled entry represents the fact that $D = HR_{90}$. (In general, ab denotes the entry at the intersection of the row with a at the left and the column with b at the top.)

	R_0	R_{90}	R_{180}	R_{270}	H	V	D	D'
R_0	R_0	R_{90}	R_{180}	R_{270}	H	V	D	D'
R_{90}	R_{90}	R_{180}	R_{270}	R_0	D'	D	H	V
R_{180}	R_{180}	R_{270}	R_0	R_{90}	V	H	D'	D
R_{270}	R_{270}	R_0	R_{90}	R_{180}	D	D'	V	H
H	H	D	V	D'	R_0	R_{180}	R_{90}	R_{270}
V	V	D'	H	D	R_{180}	R_0	R_{270}	R_{90}
D	D	V	D'	H	R_{270}	R_{90}	R_0	R_{180}
D'	D'	H	D	V	R_{90}	R_{270}	R_{180}	R_0

Notice how orderly this table looks! This is no accident. Perhaps the most important feature of this table is that it has been completely filled in without introducing any new motions. Of course, this is because, as we have already pointed out, any sequence of motions turns out to be the same as one of these eight.

This property is called *closure*, and it is one of the requirements for a mathematical system to be a group. Next, notice that if A is any element of D_4, then $AR_0 = R_0A = A$. Thus, combining any element A on either side with R_0 yields A back again. An element R_0 with this property is called an *identity*, and every group must have one. Moreover, we see that for each element A in D_4, there is exactly one element B in D_4 such that $AB = BA = R_0$. In this case, B is said to be the *inverse* of A and vice versa. For example, R_{90} and R_{270} are inverses of each other, and H is its own inverse. The term *inverse* is a descriptive one, for if A and B are inverses of each other, then B "undoes" whatever A "does," in the sense that A and B taken together in either order produce R_0, representing no change. Another striking feature of the table is that every element of D_4 appears exactly once in each row and column. This feature is something that all groups must have, and, indeed, it is quite useful to keep this fact in mind when constructing the table in the first place.

Another property of D_4 deserves special comment. Observe that $HD \neq DH$ but $R_{90}R_{180} = R_{180}R_{90}$. Thus, in a group, ab may or may not be the same as ba. If it happens that $ab = ba$ for *all* choices of group elements a and b, we say the group is *commutative* or—better yet—*Abelian* (in honor of the great Norwegian mathematician Niels Abel). Otherwise, we say the group is *non-Abelian*.

Thus far, we have illustrated, by way of D_4, three of the four conditions that define a group—namely, closure, existence of an identity, and existence of inverses. The remaining condition required for a group is *associativity*; that is, $(ab)c = a(bc)$ for all a, b, c in the set. To be sure that D_4 is indeed a group, we should check this equation for each of the $8^3 = 512$ possible choices of a, b, and c in D_4. In practice, however, this is rarely done! Here, for example, we simply observe that the eight motions are functions and the operation is function composition. Then, since function composition is associative, we do not have to check the equations.

The Dihedral Groups

The analysis carried out above for a square can similarly be done for an equilateral triangle or regular pentagon or, indeed, any regular n-gon ($n \geq 3$). The corresponding group is denoted by D_n and is called the *dihedral group of order 2n.*

The dihedral groups arise frequently in art and nature. Many of the decorative designs used on floor coverings, pottery, and buildings have one of the dihedral groups as a group of symmetry. Corporation logos are rich sources of dihedral symmetry. Chrysler's logo has D_5 as a symmetry group, and that of Mercedes-Benz has D_3. The ubiquitous five-pointed star has symmetry group D_5.

The phylum Echinodermata contains many sea animals (such as starfish, sea cucumbers, feather stars, and sand dollars) that exhibit patterns with D_5 symmetry. Snowflakes have D_6 symmetry (see Exercise 19).

Chemists classify molecules according to their symmetry. Moreover, symmetry considerations are applied in orbital calculations, in determining energy levels of atoms and molecules, and in the study of molecular vibrations. The symmetry group of a pyramidal molecule such as ammonia (NH_3), depicted in Figure 1.2, is D_3.

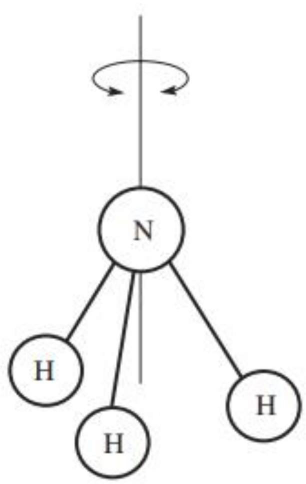


Figure 1.2 A pyramidal molecule with symmetry group D_3.

Mineralogists determine the internal structures of crystals (i.e., rigid bodies in which the particles are arranged in three-dimensional discrete patterns—table salt and table sugar are two examples) by studying two-dimensional x-ray projections of the atomic makeup of the crystals. The symmetry present in the projections reveals the internal symmetry of the crystals themselves. Commonly occurring symmetry patterns are D_4 and D_6 (see Figure 1.3). Interestingly, it is mathematically impossible for a crystal to possess a D_n repeating symmetry pattern with $n = 5$ or $n > 6$.

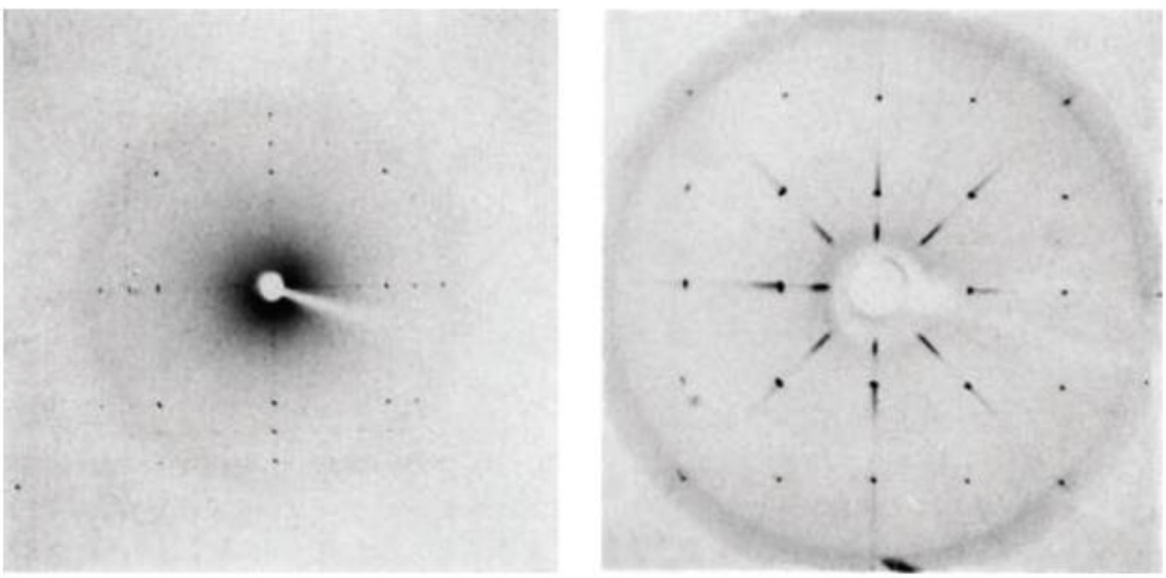


Figure 1.3 X-ray diffraction photos revealing D_4 symmetry patterns in crystals.

The dihedral group of order $2n$ is often called the *group of symmetries of a regular n-gon*. A *plane symmetry* of a figure F in a plane is a function from the plane to itself that carries F onto F and preserves distances; that is, for any points p and q in the plane, the distance from the image of p to the image of q is the same as the distance from p to q. (The term *symmetry* is from the Greek word *symmetros*, meaning "of like measure.") The *symmetry group* of a plane figure is the set of all symmetries of the figure. Symmetries in three dimensions are defined analogously. Obviously, a rotation of a plane about a point in the plane is a symmetry of the plane, and a rotation about a line in three dimensions is a symmetry in three-dimensional space. Similarly, any translation of a plane or of three-dimensional space is a symmetry. A *reflection across a line* L is that function which leaves every point of L fixed and takes any point Q, not on L, to the point Q' so that L is the perpendicular bisector of the line segment joining Q and Q' (see Figure 1.4). Figure 1.4 illustrates the characteristic that distinguishes a reflection from a rotation. The reflected image of a clockwise spiral is a counterclockwise spiral and vice versa. In particular, rotations preserve orientation and reflections reverse orientation. We will use this fact often. Also note that reflections reverse handedness, which is an important fact in chemistry.

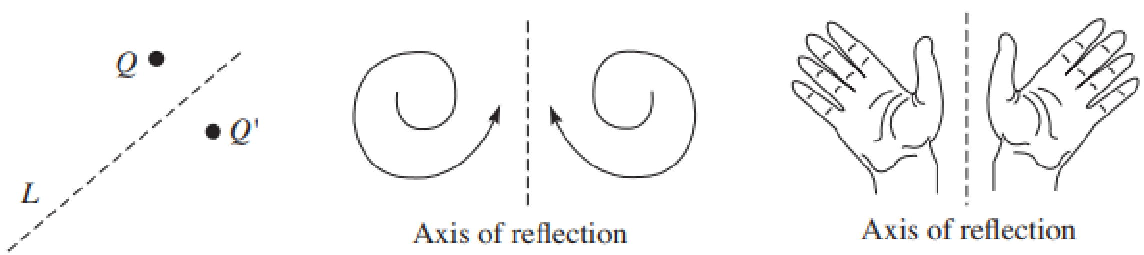


Figure 1.4 Reflected images.

A reflection across a plane in three dimensions is defined analogously. Notice that the restriction of a 180° rotation about a line L in three dimensions to a plane containing L is a reflection across L in the plane. Thus, in the dihedral groups, the motions that we described as flips about axes of symmetry in three dimensions (e.g., H, V, D, D') are reflections across lines in two dimensions. Just as a reflection across a line is a plane symmetry that cannot be achieved by a physical motion of the plane in two dimensions, a reflection across a plane is a three-dimensional symmetry that cannot be achieved by a physical motion of three-dimensional space. A cup, for instance, has reflective symmetry across the plane bisecting the cup, but this symmetry cannot be duplicated with a physical motion in three dimensions.

Many objects and figures have rotational symmetry but not reflective symmetry. A symmetry group consisting of the rotational symmetries of $0°$, $360°/n$, $2(360°)/n, \ldots,$ $(n-1)360°/n$, and no other symmetries, is called a *cyclic rotation group of order n* and is denoted by $\langle R_{360/n} \rangle$. Cyclic rotation groups, along with dihedral groups, are favorites of artists, designers, and nature. Figure 1.5 illustrates with corporate logos the cyclic rotation groups of orders 2, 3, 4, 5, 6, 8, 16, and 20.

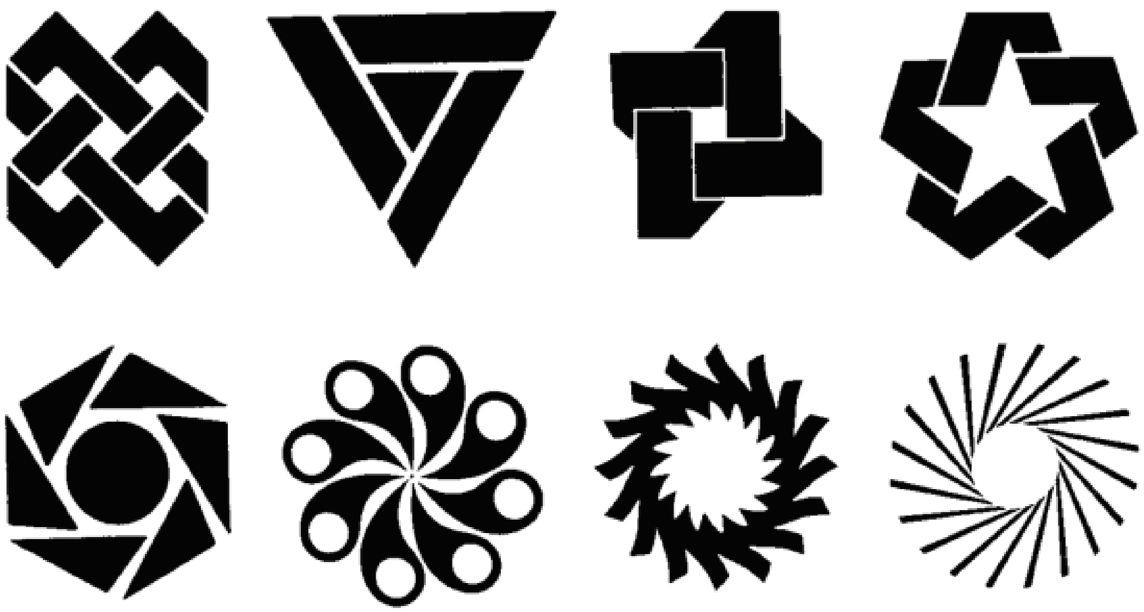


Figure 1.5 Logos with cyclic rotation symmetry groups.