Lecture 7

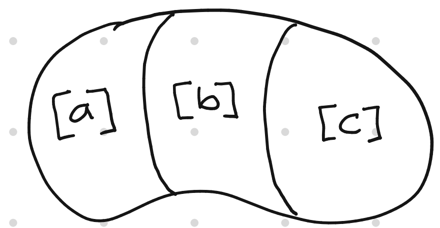
SUMMARY

$A \neq \emptyset \rightsquigarrow$ Relation on A $\rightsquigarrow$ $\sim$ is an e.r $\rightsquigarrow$ $A/\sim \; = \{ [a] \mid a \in A \}$

$a \sim b$

✓ Reflexive
✓ Symmetric
✓ Transitive

quotient set



$A = \{8, 11, 17, 32, 52\} \rightsquigarrow a \sim b \iff$ a, b belong to the same subset in P $\rightsquigarrow$ $\sim$ is an e.r $\rightsquigarrow$ $A/\sim \; = \{[8], [17], [52]\} = P$

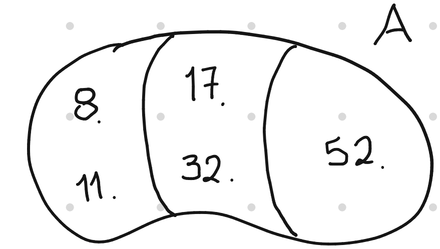
$P = \{\{8, 11\}, \{17, 32\}, \{52\}\}$

$[8] = \{8, 11\}$

$[17] = \{17, 32\}$

$[52] = \{52\}$

A



$\mathbb{Z} \rightsquigarrow a \equiv_n b \iff n \mid (a-b) \rightsquigarrow \equiv_n$ is an e.r $\rightsquigarrow \mathbb{Z}/\equiv_n \; = \{ [a]_n \mid a \in \mathbb{Z} \}$

$[a]_n = \{ a + nk \mid k \in \mathbb{Z} \}$

Ex:

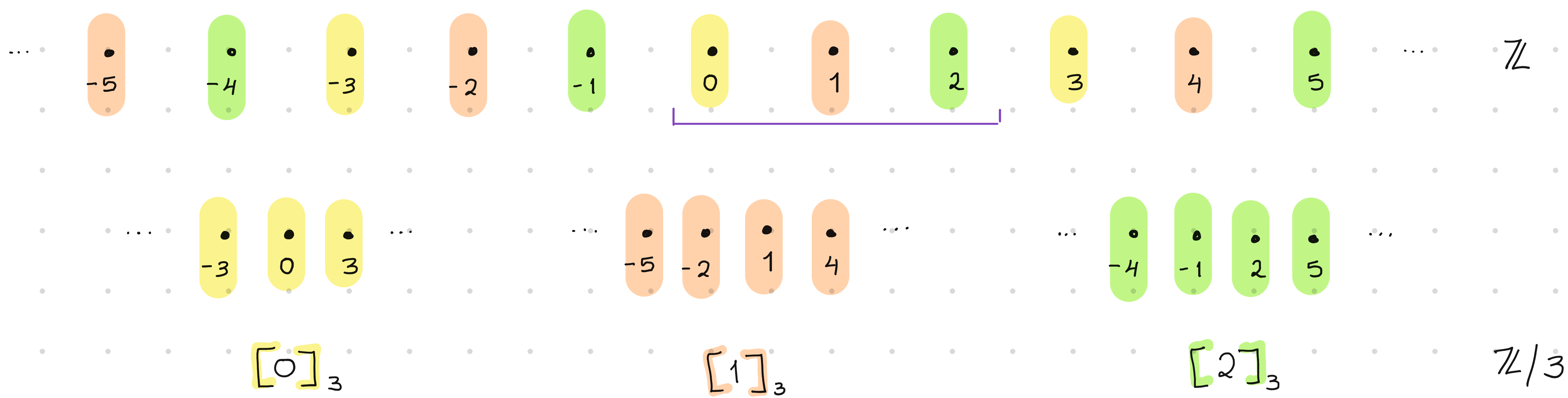
- $[0]_2 = \{ 2k \mid k \in \mathbb{Z} \}$ even $\quad [1]_2 = \{ 1+2k \mid k \in \mathbb{Z} \}$ odd

- $[0]_3 = \{ 3k \mid k \in \mathbb{Z} \}$ multiples of 3 $\quad [0]_3 = \{ \dots, -9, -6, -3, 0, 3, 6, 9, \dots \}$

$[1]_3 = \{ 1+3k \mid k \in \mathbb{Z} \}$

$[2]_3 = \{ 2+3k \mid k \in \mathbb{Z} \} = \{ \dots, -7, -4, -1, 2, 5, 8, 11, \dots \}$

$[3]_3 = \{ 3+3k \mid k \in \mathbb{Z} \} = [0]_3$



Proposition 13: Let $n \in \mathbb{N} \setminus \{0,1\}$. There are exactly n distinct congruence classes, namely $[0], [1], [2], \dots, [n-1]$.

Proof: Let $a \in \mathbb{Z}$. By the Division Algorithm $a = nq + r$ for some $q, r \in \mathbb{Z}$ and $0 \leq r < n$. Therefore, $[a] = [r]$ for some $r = 0, 1, 2, \dots, n-1$.

Need to proof that the n classes above are all distinct (this means, we WTS that no two of $0, 1, 2, \dots, n-1$ are congruent module n).

Let $s, t \in \{0, 1, 2, \dots, n-1\}$ so that $s \neq t$. WOLG (without loss of generality) say $0 \leq s < t \leq n-1$. Then $t - s > 0$ and $t - s < n$, i.e. $n \nmid (t-s)$, thus by definition $t \not\equiv_n s$. Then, $[0], [1], [2], \dots, [n-1]$ are all distinct.

Def: The quotient set of $\mathbb{Z}$ by $\equiv_n$ is denoted by $\mathbb{Z}/n$ (or $\mathbb{Z}_n$) (instead of $\mathbb{Z}/\equiv_n$) and it is read $\mathbb{Z}$ mod n.

Remarks:

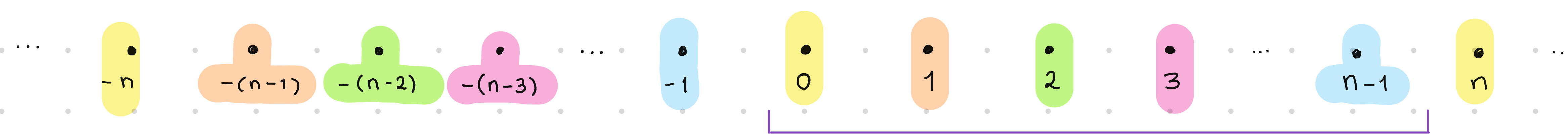
1. $a \equiv_n b$ iff $[a]_n = [b]_n$

$$\{a+nk \mid k \in \mathbb{Z}\} = \{b+nk \mid k \in \mathbb{Z}\}$$

2. Two classes modulo n are either disjoint or identical.

3. $\mathbb{Z}/n = \underbrace{\{[0], [1], [2], \dots, [n-1]\}}_{\text{exactly } n \text{ elements}}$ is a partition of $\mathbb{Z}$.

$$\mathbb{Z} = \bigcup_{a=0}^{n-1} [a] = [0] \cup [1] \cup [2] \cup \cdots \cup [n-1]$$



④ Each congruence class can be written in infinitely many ways.

$$\mathbb{Z}/2 = \{[0], [1]\} \qquad [0] = [2] = [-4] = [2k] \quad \text{for all } k \in \mathbb{Z}$$

$$\mathbb{Z}/7 = \{[0], [1], \dots, [6]\} \qquad [1] = [8] = [-6] = [1+7k] \quad \text{for all } k \in \mathbb{Z}$$

⑤ Be careful! The elements of $\mathbb{Z}/n$ are classes (i.e. subsets of $\mathbb{Z}$), not single integers.

$$5 \notin \mathbb{Z}/6 \qquad [5] \in \mathbb{Z}/6$$

In $\mathbb{Z}$ we have addition and multiplication.

Question: Do we have addition in $\mathbb{Z}/n$?
Do we have multiplication in $\mathbb{Z}/n$?

That'd be weird!
Addition and multiplication between sets.

Answer: Yes, we do!!!

Def: Let $[a], [b] \in \mathbb{Z}/n$. We define

Addition in $\mathbb{Z}/n$: $[a] \oplus [b] := [a+b]$

Multiplication in $\mathbb{Z}/n$: $[a] \odot [b] := [a \cdot b]$

Ex: Consider $\mathbb{Z}/5 = \{[0], [1], [2], [3], [4]\}$

$[0] \oplus [1] = [0+1] = [1]$

$[4] \odot [4] = [16] = [1]$

$16 \equiv_5 1$

Possible problem: Do we get the same answer if we use different representatives? For instance,

$[5] \oplus [6] = [11] = [1]$

$11 \equiv_5 1$

$[9] \odot [-1] = [-9] = [1]$

$-9 \equiv_5 1$

Apparently, we do. But we must prove it!

Theorem 14: The operations of addition and multiplication in $\mathbb{Z}/n$ are well-defined.

Explictly ⇝ They do not depend on the choices of representatives for the classes involved:

If $a, b, c, d \in \mathbb{Z}$ with $[a] = [b]$ and $[c] = [d]$,

then $[a] \oplus [c] = [b] \oplus [d]$ and $[a] \odot [c] = [b] \odot [d]$.

Proof: Read Thm 2.1 and Thm 2.6 in Hungerford's book.

Def: Let $k \in \mathbb{N}$ and $[a] \in \mathbb{Z}/n$.

$$k[a] := \underbrace{[a] \oplus \cdots \oplus [a]}_{k\text{-times}} \qquad [a]^k := \underbrace{[a] \odot \cdots \odot [a]}_{k\text{-times}}.$$

$$0[a] := [0] \qquad [a]^0 := [1]$$

Ex: In $\mathbb{Z}/7$.

$$3[1] = [1] \oplus [1] \oplus [1] = [1+1+1] = [3]$$

$$[2]^4 = [2] \odot [2] \odot [2] \odot [2] = [2^4] = [16] = [2]$$

$$0[6] = [0]$$

$$[3]^0 = [1]$$

Remark: $k[a] = [ka]$ $\quad$ $0[a] = [0 \cdot a]$

$[a]^k = [a^k]$ $\quad$ $[a]^0 = [a^0]$

As you would expect!