Lecture 5

❗ A generalization of the concept of equality should behave the same way.

The equality relation has the following properties:

- ✓ $a = a$
- ✓ If $a = b$, then $b = a$.
- ✓ If $a = b$ and $b = c$, then $a = c$.

Def: An equivalence relation on a set A is a relation R that satisfies the following for all $a, b, c \in A$.

(1) Reflexive: aRa $\qquad (a,a) \in R$

(2) Symmetric: If aRb, then bRa. $\qquad (a,b) \in R \Rightarrow (b,a) \in R$

(3) Transitive: If aRb and bRc, then aRc. $\qquad (a,b), (b,c) \in R \Rightarrow (a,c) \in R$

If $a \in A$, the set $[a] := \{ x \in A \mid xRa \} \subseteq A$ is called the equivalence class of a, a is called a representative.

Ex: Let $A = \{1, 2, 3\}$, $R = \{(1,1), (2,2), (3,3), (1,2), (2,1)\}$.

$S = \{(1,1), (1,3), (3,1), (3,3)\}$

Is R an equivalence relation on A? Need to check element by element.

(1) Is R reflexive? $(1,1), (2,2), (3,3) \in R$ ✓

(2) Is R symmetric? $(1,2), (2,1) \in R$ ✓

(3) Is R transitive? $(1,2), (2,1), (1,1) \in R$ ✓

two representatives

Observe: $[1] = [2] = \{1, 2\}$ and $[3] = \{3\}$

$\boxed{\text{Any element of a class is a representative.}}$

Is S an equivalence relation on A?

No, because it is not reflexive, $(2,2) \notin S$.

Ex: Let $A := \{ p(x) \mid p(x) \text{ is a polynomial with real coefficients} \}$. Define the relation $f \sim g$ if $f' = g'$.

Is $\sim$ an equivalence relation on A?

(1) Is $\sim$ reflexive? ✓ $f' = f' \Rightarrow f \sim f$

(2) Is $\sim$ symmetric? ✓ $f \sim g \Rightarrow f' = g' \Rightarrow g' = f' \Rightarrow g \sim f$

(3) Is $\sim$ transitive? ✓ $f \sim g$ and $g \sim h \Rightarrow f' = g'$ and $g' = h' \Rightarrow f' = h'$
$\Rightarrow f \sim h$

$$[f] = \{ g \in A \mid g \sim f \} = \{ g \in A \mid g' = f' \}$$

$$= \{ g \in A \mid g = f + c \text{ with } c \in \mathbb{R} \} = \{ f + c \mid c \in \mathbb{R} \}$$

polynomials with equal derivatives differ by a constant

Ex: Define a relation R on $\mathbb{Z}$ by

$$n R m \iff nm \geq 0$$

Is R an equivalence relation on $\mathbb{Z}$?

(1) Is R reflexive? ✓ $n^2 \geq 0 \Rightarrow n R n$

(2) Is R symmetric? ✓ $n R m \Rightarrow nm \geq 0 \underset{\text{commutativity}}{\Rightarrow} mn \geq 0 \Rightarrow m R n$

(3) Is R transitive? ✗ $nm \geq 0$ and $mp \geq 0 \overset{?}{\underset{\text{NO}}{\Longrightarrow}} np \geq 0$

$-1 R 0$ and $0 R 1$, but $-1 \not{R} 1$

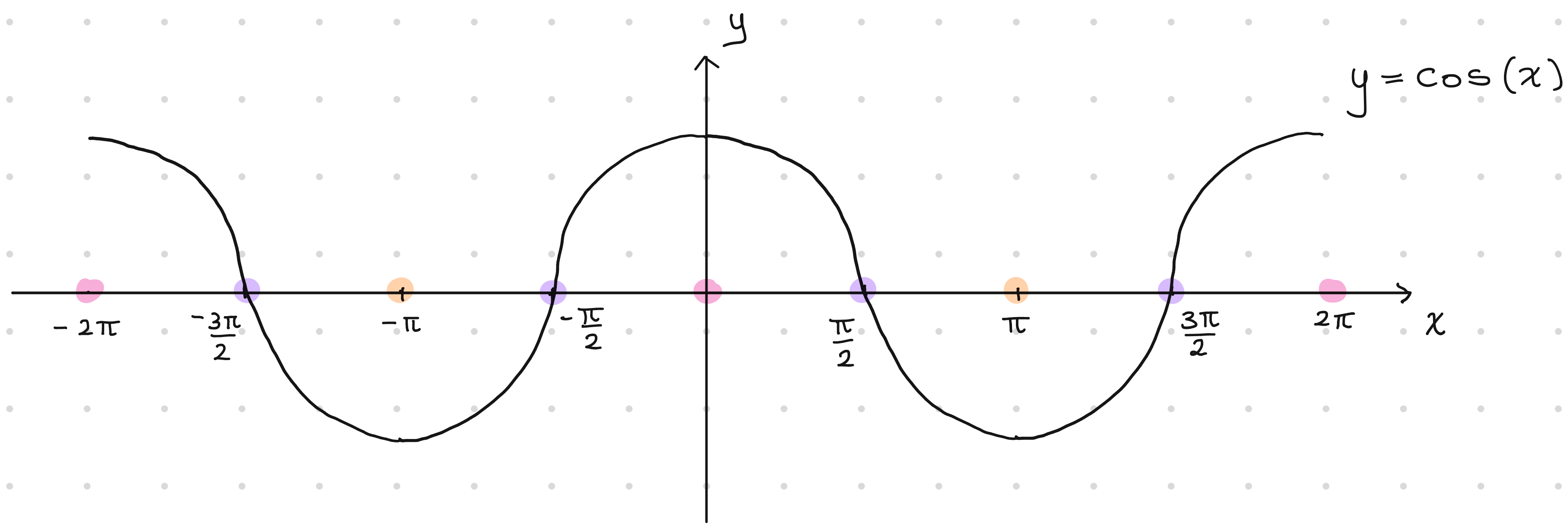
Ex: Let $a, b \in \mathbb{R}$. Say $a \sim b$ if $\cos a = \cos b$

$\sim$ is an equivalence relation. Let's describe $[0]$ and $[\pi/2]$

$[0] = \{ x \in \mathbb{R} \mid x \sim 0 \} = \{ x \in \mathbb{R} \mid \cos x = \cos 0 \} = \{ x \in \mathbb{R} \mid \cos x = 1 \} = \{ 2k\pi \mid k \in \mathbb{Z} \}$

$[\pi/2] = \{ x \in \mathbb{R} \mid x \sim \pi/2 \} = \{ x \in \mathbb{R} \mid \cos x = \cos \pi/2 \} = \{ x \in \mathbb{R} \mid \cos x = 0 \} = \{ (2k+1)\frac{\pi}{2} \mid k \in \mathbb{Z} \}$

If $a \in \mathbb{R}$, let $b := \cos a \in [-1,1]$. Then $[a] = \{ x \in \mathbb{R} \mid \cos x = b \}$.



$\{ [a] \mid 0 \le a \le \pi \}$ is the set of all equivalence classes.

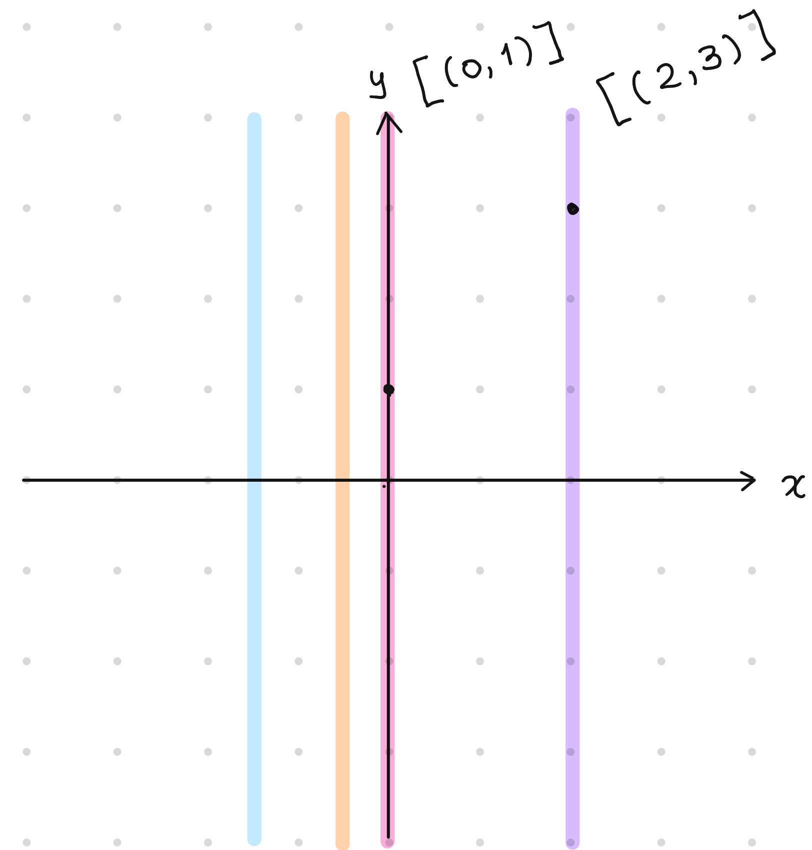
Ex: Let $(x,y), (a,b) \in \mathbb{R}\times\mathbb{R}$. Say $(x,y) \simeq (a,b)$ if $x = a$.

$\simeq$ is an equivalence relation. Let's describe $[(a,b)]$.

$[(a,b)] = \{(x,y) \in \mathbb{R}\times\mathbb{R} \mid x = a\} = \{(a,y) \mid y \in \mathbb{R}\}$ ← constant line $x = a$

$[(0,1)] = \{(0,y) \mid y \in \mathbb{R}\}$ y-axis

$[(2,3)] = \{(2,y) \mid y \in \mathbb{R}\}$ constant line $x = 2$



$\{[(a,0)] \mid a \in \mathbb{R}\}$ is the set of all equivalence classes

!!! Ex: The relation induced by a partition is an equivalence relation.

$A \neq \phi$, $P = \{A_i \mid i \in I\}$ a partition of A.

(1) R is reflexive: Let $a \in A$. Since $A = \bigcup_{i \in I} A_i$, there exists A_i s.t. $a \in A_i$.

Then aRa.

(2) R is symmetric: $aRb \Rightarrow \exists i \in I$ s.t. $a, b \in A_i \Rightarrow bRa$

(3) R is transitive: aRb and $bRc \Rightarrow \exists i \in I$, $a, b \in A_i$ and $\exists j \in I$, $b, c \in A_j$.

We must have $i = j$. Otherwise, $b \in A_i \cap A_j = \phi$. Contradiction!!!

Thus, $a, c \in A_i = A_j$, i.e. aRc.