Lecture 21

Module 3 - Rings

Def: (1) A ring is a triple $(R, +, \cdot)$ with $+$ and $\cdot$ binary operations such that

(i) $(R, +)$ is an abelian group.

(ii) $\cdot$ is associative: $a(bc) = (ab)c \quad \forall a, b, c \in R$.

(iii) The distributive laws hold: $a(b+c) = ab + ac$, $(a+b)c = ac + bc$ $\quad \forall a, b, c \in R$.

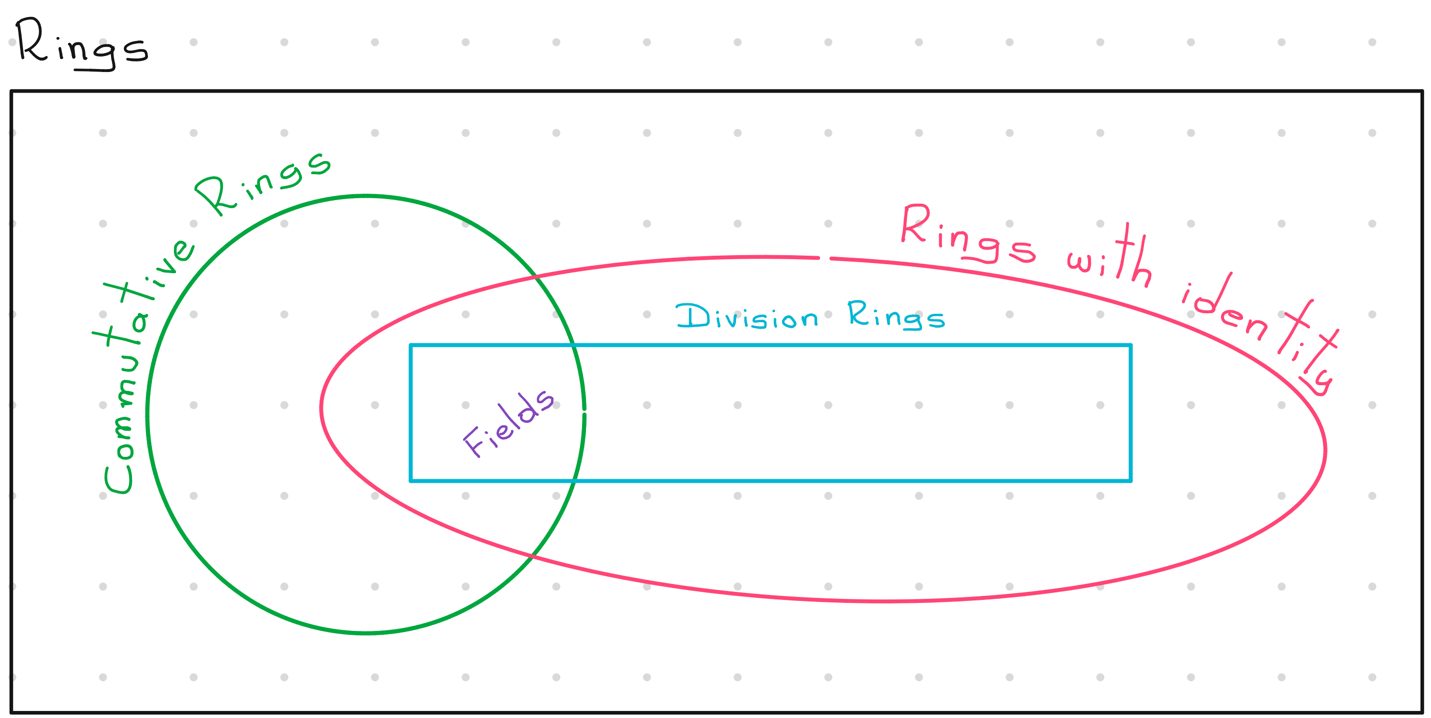
(2) A ring $(R, +, \cdot)$ is commutative if $ab = ba \quad \forall a, b \in R$.

(3) A ring $(R, +, \cdot)$ is a ring with identity if there exists $1 \in R$ such that

$a1 = a = 1a \quad \forall a \in R$.

(4) A ring with identity $(R, +, \cdot)$, where $1 \neq 0$, is a division ring if every nonzero element of R has a multiplicative inverse: $\forall a \in R \setminus \{0\}$, $\exists b \in R$, $ab = ba = 1$.

(5) A division ring $(R, +, \cdot)$ is a field if R is commutative.



Notation:

- Additive identity $0 \in R$
- Additive inverse $-r \in R$
- Multiplicative identity $1 \in R$
- Multiplicative inverse $r^{-1} \in R$

Examples:

1. $\{0\}$ is a commutative ring with identity $1 = 0$. We call $\{0\}$ the zero ring.

2. $(\mathbb{Z}, +, \cdot)$ is a commutative ring with identity.

   Addite identity: 0
   Multiplicative identity: 1

   Additive inverse: $-a$
   Only 1 and -1 have multiplicative inverses

3. Let $n \in \mathbb{Z}$, $n \geq 2$. Then $(n\mathbb{Z}, +, \cdot)$ is a commutative ring without identity.

4. $(\mathbb{Q}, +, \cdot)$, $(\mathbb{R}, +, \cdot)$ and $(\mathbb{C}, +, \cdot)$ are fields.

5) Let $N \in \mathbb{Q}$ be a number that is not a perfect square in $\mathbb{Q}$, then

$\mathbb{Q}[\sqrt{N}] := \{ a + b\sqrt{N} \mid a, b \in \mathbb{Q} \}$ is a field called the quadratic field.

$$(a+b\sqrt{N})^{-1} = \frac{a - b\sqrt{N}}{a^2 - Nb^2} \quad \text{where} \quad a \neq 0 \ \text{or} \ b \neq 0$$

6) $(\mathbb{Z}/n, \oplus, \odot)$ is a commutative ring with identity.

Addite identity : $[0]$

Multiplicative identity : $[1]$

Additive inverse : $[-a]$

$[a]$ has a multiplicative inverse $\iff$ $\gcd(a,n) = 1$.

7) p prime $(\mathbb{Z}/p, \oplus, \odot)$ is a field. We denote it by $\mathbb{F}_p$.

$\mathbb{Z}/n$ is a field iff n is a prime.

8) The real Quaternions $\mathbb{H} := \{ a + bi + cj + dk \mid a, b, c, d \in \mathbb{R} \}$ is a division ring. See Example (5), RA7.

9) $(M_{n\times n}(\mathbb{C}), +, \cdot)$ is a noncommutative ring with identity.

Addite identity: Zero matrix $0_{n\times n}$ Additive inverse: $-A$

Multiplicative identity: I_n Some matrices are not invertible

10) If R and S are rings, then the direct product $R \times S$ is a ring under componentwise addition and multiplication.

- $R \times S$ is commutative $\iff$ R and S are commutative
- $R \times S$ has an identity $\iff$ R and S have identities

11) Let $X \neq \phi$ be a set. Let $(A, +, \cdot)$ be a ring.

Define $\mathcal{F}(X, A) := \{ f : X \longrightarrow A \mid f \text{ is a function} \}$. $\mathcal{F}(X,A)$ is a ring under addition and multiplication of function: Let $f, g \in \mathcal{F}(X,A)$ and $x \in X$

$$(f+g)(x) := f(x) + g(x) \in A \quad \text{and} \quad (fg)(x) := f(x) \cdot g(x) \in A.$$

$$0(x) = 0_A \qquad\qquad 1(x) = 1_A \text{ (if } A \text{ has an identity).}$$

- $\mathcal{F}(X,A)$ is commutative $\Longleftrightarrow$ A is commutative
- $\mathcal{F}(X,A)$ has an identity $\Longleftrightarrow$ A has an identity
- If A is a division ring, then $\mathcal{F}(X,A)$ is not a division ring necessarily.
  See Example 4 of Zero divisors / Units.

Proposition 1: Let R be a ring. Then

(1) $0a = a0 = 0, \quad \forall a \in R$

(2) $(-a)b = a(-b) = -(ab), \quad \forall a, b \in R$

(3) $(-a)(-b) = ab, \quad \forall a, b \in R$

(4) If R has an identity 1, then the identity is unique and

$-a = (-1)a.$

Proof: Exercise.

Def: Let $(R, +, \cdot)$ be a ring.

(1) A nonzero element $a \in R$ is called a zero divisor if there is a nonzero element $b \in R$ s.t. either $ab = 0$ or $ba = 0$.

(2) Let R have an identity $1 \neq 0$. A nonzero element $a \in R$ is called a unit if there is $b \in R$ s.t. $ab = 1 = ba$.

The set of units in R is denoted $R^{\times}$ and it is a group called the group of units of R. Prove $(R^{\times}, \cdot)$ is a group.

⚠ A zero divisor cannot be a unit.

A noncommutative ring can have $ab = 0$ and $ba \neq 0$.

Ex:

1. A field F is a commutative ring with identity $1 \neq 0$ in which every nonzero element is a unit, i.e. $F^\times = F \setminus \{0\}$.

   $\mathbb{F}_p$ p prime, $\mathbb{Q}$, $\mathbb{Q}[\sqrt{N}]$, $\mathbb{R}$, $\mathbb{C}$

2. $\mathbb{Z}$ Units: $\mathbb{Z}^\times = \{\pm 1\}$ Zero divisors: None

3. $\mathbb{Z}/n$, $n \geqslant 2$

   In Exam 1 you proved that $[a] \in \mathbb{Z}/n$ is either a unit or a zero divisor.

   Units: $(\mathbb{Z}/n)^\times = \{ [a] \mid \gcd(a,n) = 1 \}$

   Zero divisors: $(\mathbb{Z}/n) \setminus \left((\mathbb{Z}/n)^\times \cup \{[0]\} \right)$

4. Let R denote $\mathcal{F}([0,1],\mathbb{R})$.

Units: $R^\times = \{ f \in R \mid f(x) \neq 0 \text{ for all } x \in [0,1] \}$

If $f \in R$, then $f^{-1} := \frac{1}{f}$ where $\left(\frac{1}{f}\right)(x) = \frac{1}{f(x)}$ for all $x \in [0,1]$.

Zero divisors: $R \setminus (R^\times \cup \{0\})$

If $f \in R$, $f \notin R^\times$ and $f \neq 0$, then $fg = 0$ where

$$g(x) := \begin{cases} 0, & \text{if } f(x) \neq 0 \\ 1, & \text{if } f(x) = 0 \end{cases}$$ is not the zero function.

Def: A commutative ring with identity $1 \neq 0$ is called an integral domain (ID) if it has no zero divisors.

Explicitly $\leadsto$ $\forall a,b,c \in R$ $\quad (ab = 0 \Rightarrow a = 0$ or $b = 0)$

$(a \neq 0$ and $b \neq 0 \Rightarrow ab \neq 0)$

Ex: $\mathbb{Z}$ - Division rings - Fields

Proposition 2: (1) Let $a,b,c \in R$ with a not a zero divisor. If $ab = ac$, then $a = 0$ or $b = c$.

"Cancellation Law"

(2) If R is an integral domain, then for all $a,b,c \in R$ $ab = ac$ implies $a = 0$ or $b = c$.

Proof: Exercise.

Corollary 3: Any finite integral domain is a field.

Proof: Let R be a finite ID.

We need ⇝
- R has identity ✓
- $1 \neq 0$ ✓
- R is commutative ✓
- $\forall a \in R \setminus \{0\}, \exists b \in R, ab = 1$

Let $a \in R \setminus \{0\}$. Define a map $i_a : R \longrightarrow R$ by $i_a(r) = ar$ for all $r \in R$.

WTS: i_a is surjective.

> Observe that i_a is injective. Let $r, s \in R$ s.t. $i_a(r) = i_a(s)$, i.e. $ar = as$.
>
> By Prop 2(b), $a = 0$ or $r = s$. Since $a \neq 0$, then $r = s$.
>
> Now, $i_a : R \longrightarrow R$ is injective and R is finite. This implies i_a must be surjective.

Thus, $1 \in R$ is so that $\exists b \in R$ s.t. $ab = i_a(b) = 1$.

Def: Let $(R, +, \cdot)$ be a ring. A subset $S \subseteq R$ is called a subring of R if

(1) $0_R \in S$

(2) S is closed under addition.

(3) S is closed under additive inverses.

(4) S is closed under multiplication.

A subring of the ring R is a subgroup of R that is closed under multiplication.

Examples:

1) $\mathbb{Z}$: $\mathcal{E} := \{\text{even integers}\}$ $\mathcal{O} := \{\text{odd integers}\}$

$\mathcal{E}$ is a subring

$\mathcal{O}$ is not a subring because $0 \notin \mathcal{O}$

2) $\mathbb{Z}$ is a subring of $\mathbb{Q}$ is a subring of $\mathbb{R}$ is a subring of $\mathbb{C}$.

3) $n\mathbb{Z}$ is a subring of $\mathbb{Z}$ for $n \in \mathbb{Z}$.

4) $\{f : \mathbb{R} \longrightarrow \mathbb{R} \mid f \text{ is continuous}\}$ is a subring of $\mathcal{F}(\mathbb{R}, \mathbb{R})$.

5) Let $S := \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{Z}\}$ (integral quaternions) is a subring of the real quaternions $\mathbb{H}$.

6) Let $M \in \mathbb{Z}$ be a number that is not a perfect square in $\mathbb{Z}$.

$\mathbb{Z}[\sqrt{M}] := \{a + b\sqrt{M} \mid a, b \in \mathbb{Z}\}$ is a subring of $\mathbb{Q}[\sqrt{M}]$.

commutative with an identity

7) Let F be a field and S a subring of F.

- S is not a field necessarily. See previous example: $\mathbb{Z}[\sqrt{M}]$ is not a division ring.

- If $1_F \in S$, then S is an ID. Prove it!

Subring Tests

Theorem 4: Let $(R,+,\cdot)$ be a ring and $S \subseteq R$.

S is a subring $\iff$ (1) $S \neq \emptyset$

(2) $a-b \in S \quad \forall a,b \in S$

(3) $ab \in S \quad \forall a,b \in S$

Proof: Exercise

Theorem 5: Let $(R,+,\cdot)$ be a finite ring and $S \subseteq R$.

S is a subring $\iff$ (1) $S \neq \emptyset$

(2) $a+b \in S \quad \forall a,b \in S$

(3) $ab \in S \quad \forall a,b \in S$

Proof: Exercise